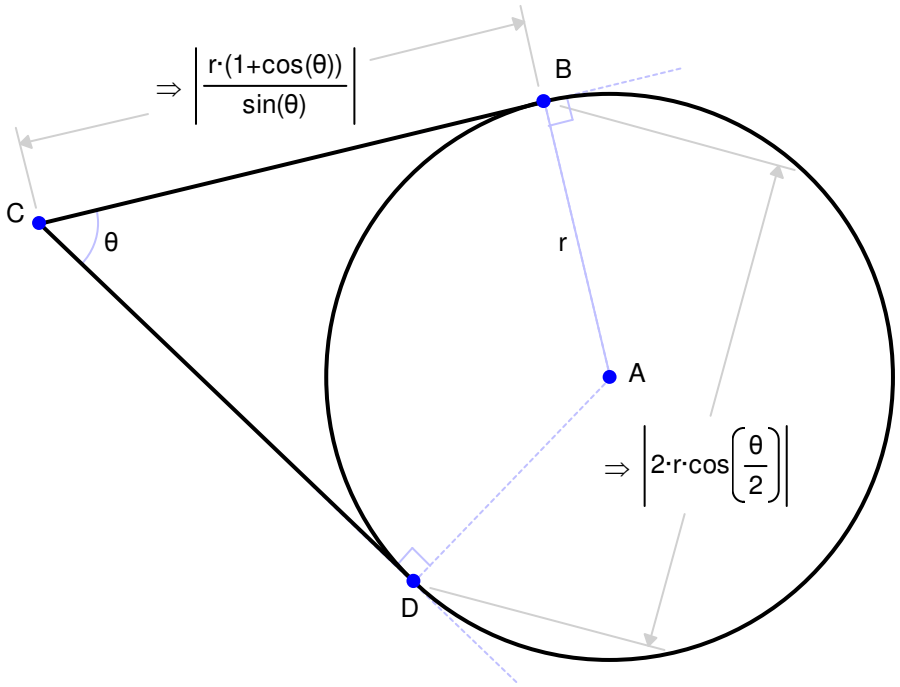
Circles

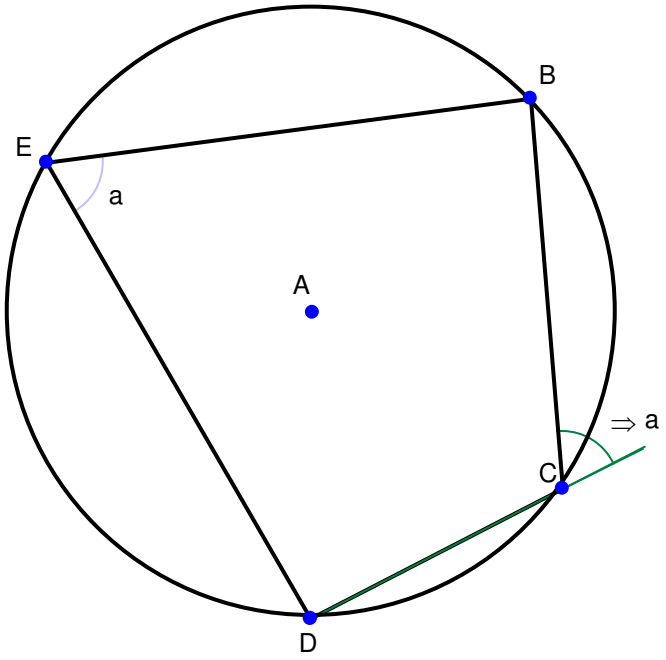
Example 66: Rounding a Corner

Assume we have a corner of angle θ, and we round it off with a curve of radius r, how far away from the corner does the round start? And what is the chord length?



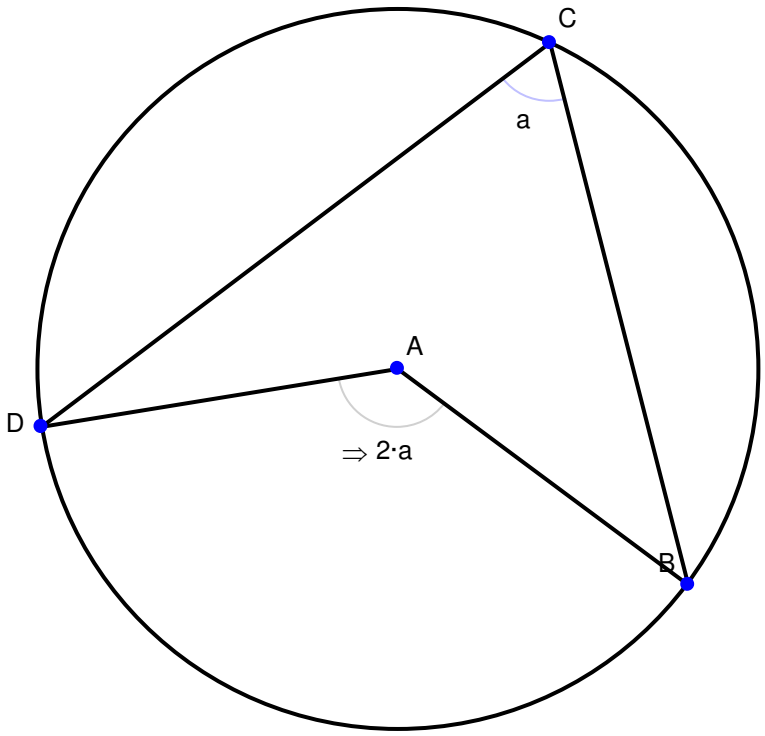
Example 67: Cyclic Quadrilaterals

Opposite angles of Cyclic quadrilaterals add up to 180, so the exterior angle equals the opposite interior angle:

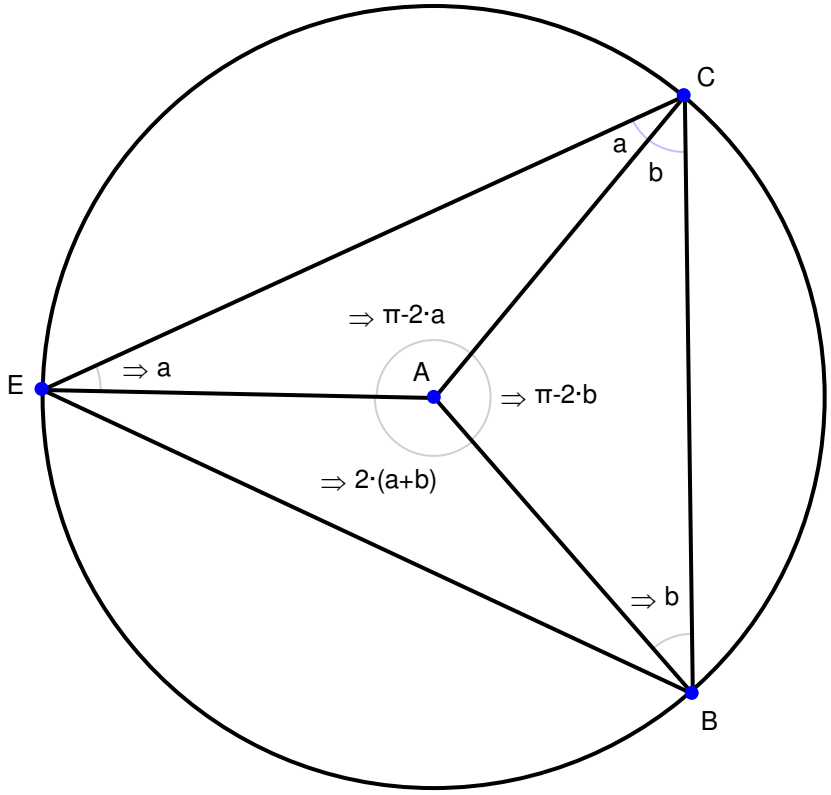


Example 68: Angles subtended by a chord

The angle at the center subtended by a chord is twice the angle at the circumference subtended by that chord:

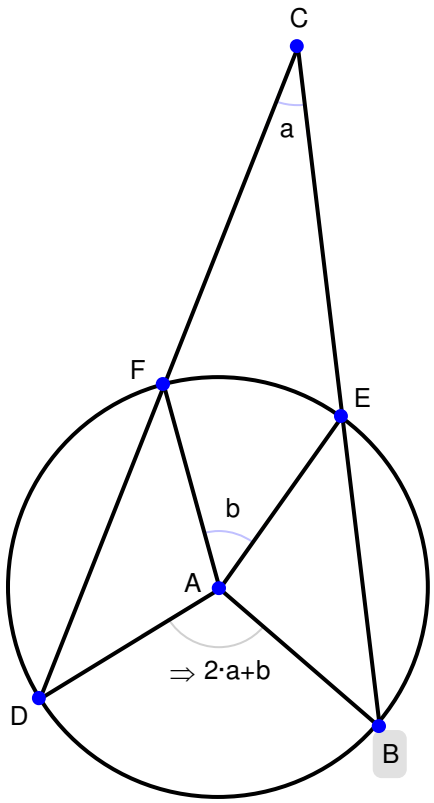


Why?



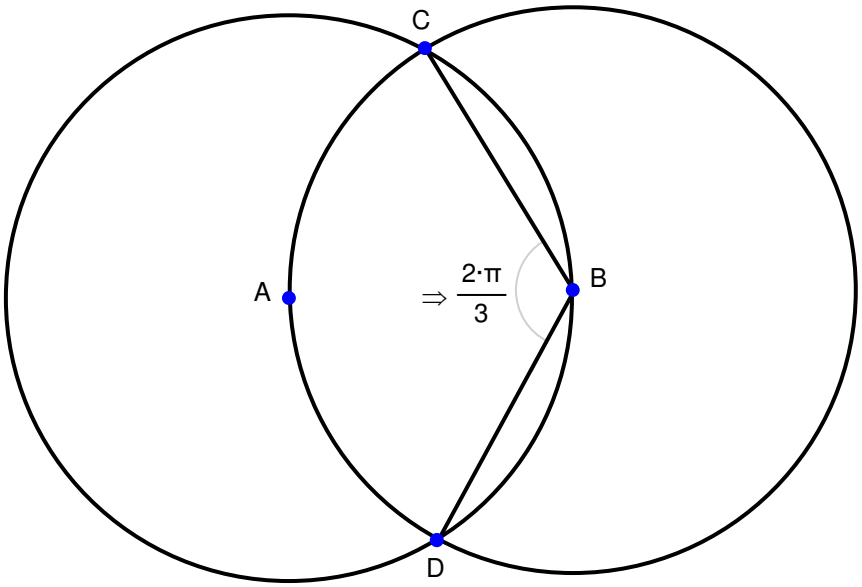
Example 69: Angle subtended by a point outside the circle

We generalize the above result for the angle subtended by a point outside the circle:

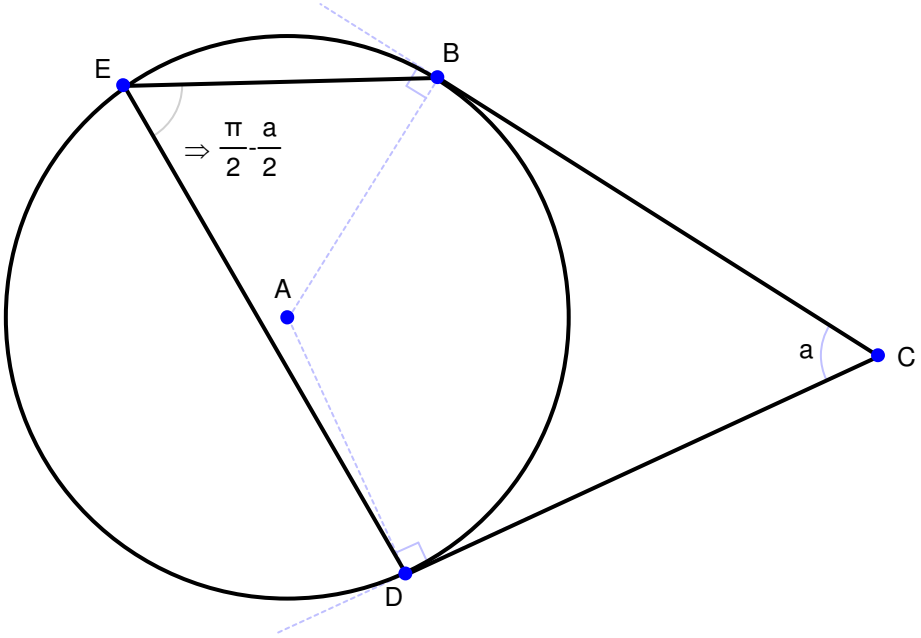


Example 70: Angle at intersection of two circles:

Here is a familiar method of making the 120 degree angle needed to draw a hexagon:

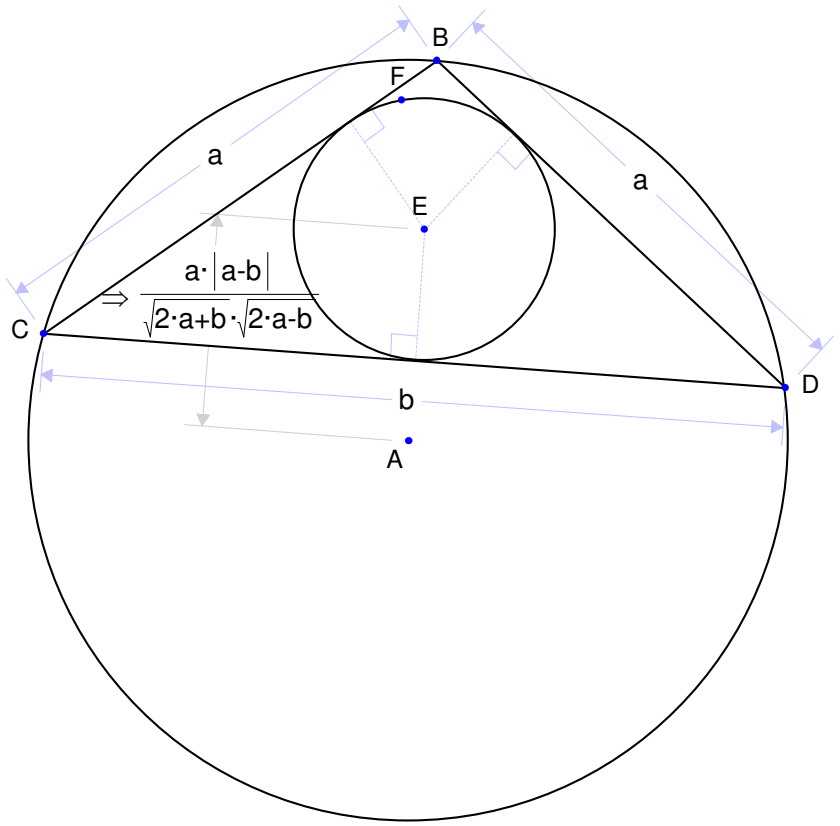


Example 71: **Angle subtended by two tangents**



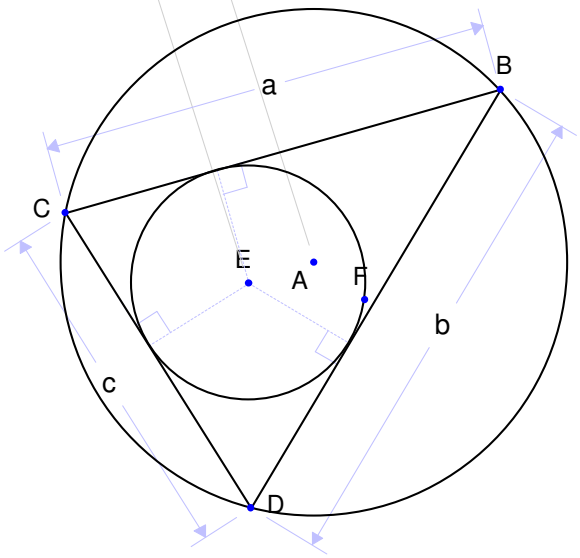
Example 72: Distance between the incenter and circumcenter

First, an isosceles triangle

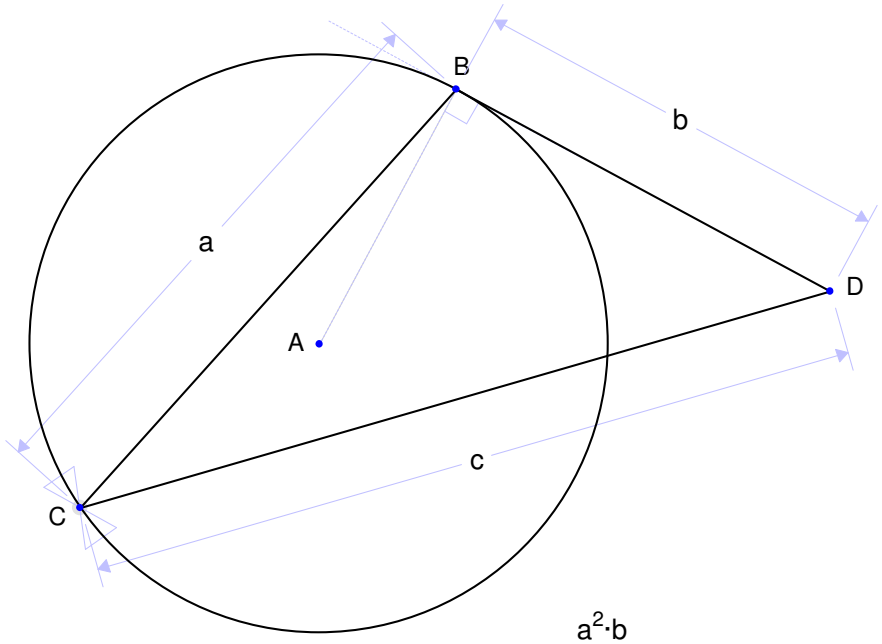


Then, more generally:

$$\Rightarrow \frac{\sqrt{a}\cdot\sqrt{b}\cdot\sqrt{c^4+c^3\cdot(-a-b)+c^2\cdot\left(-a^2+3\cdot a\cdot b-b^2\right)+c\cdot\left(a^3-a^2\cdot b-a\cdot b^2+b^3\right)}}{\sqrt{a+b+c}\cdot\sqrt{(a+b-c)\cdot(a-b+c)\cdot(-a+b+c)}}$$



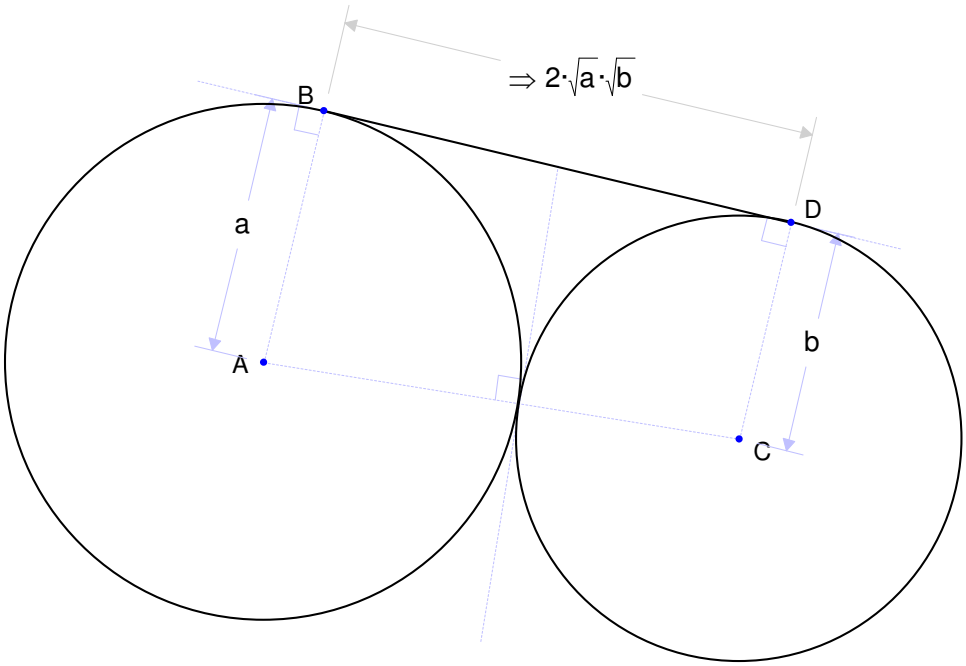
Example 73: Radius of Circle through 2 vertices of a triangle tangent to one side



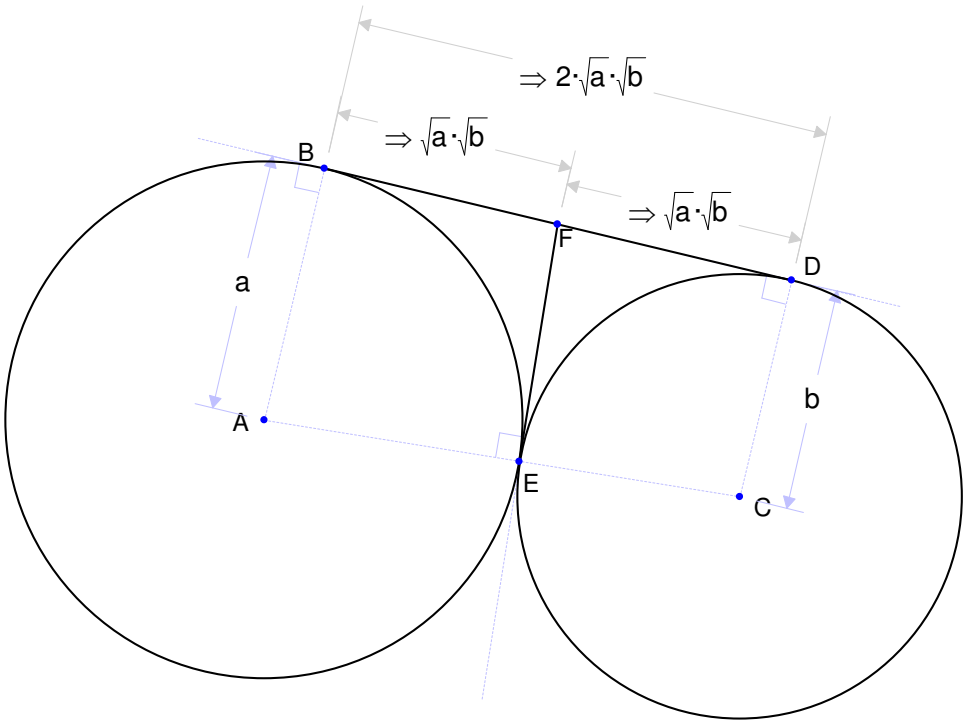
$$\Rightarrow \frac{a^2 \cdot b}{\sqrt{a+b+c} \cdot \sqrt{(a+b-c) \cdot (a-b+c) \cdot (-a+b+c)}}$$

Example 74: Length of the common tangent to two tangential circles

A succinct formula:

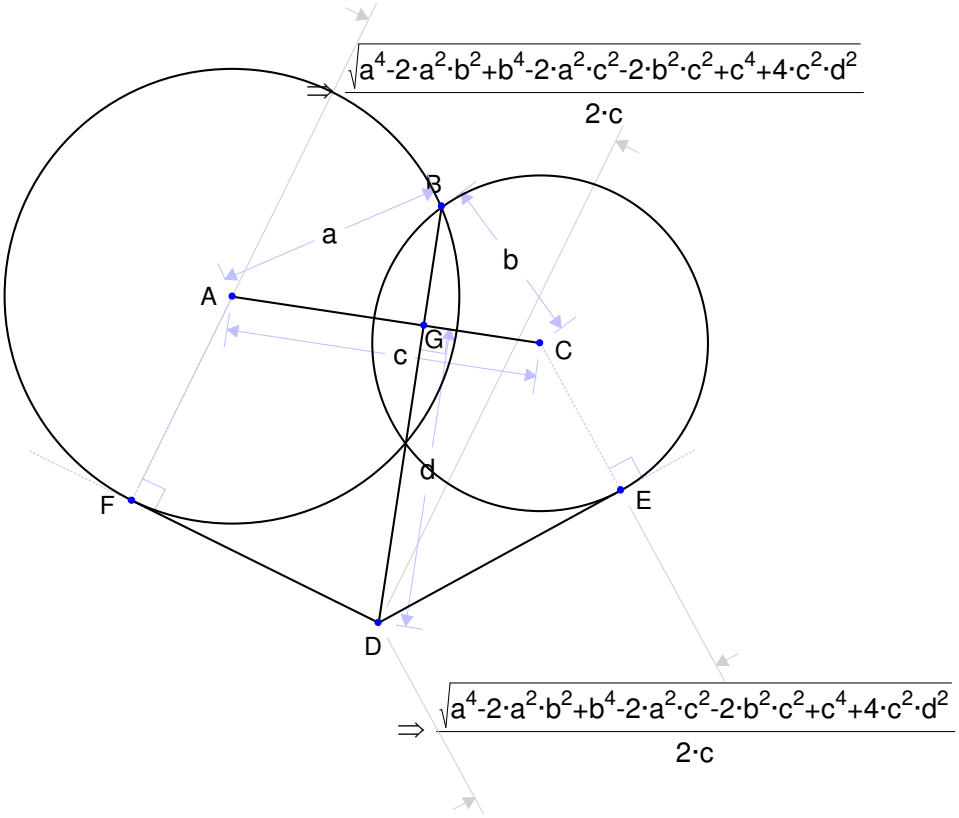


Also the internal common tangent bisects this:


⇒ 2·√a·√b
⇒ √a·√b
⇒ √a·√b
B
F
D
a
A
E
C
b

Example 75: Tangents to the Radical Axis of a Pair of Circles

The radical axis of a pair of circles is the line joining the points of intersection. The lengths of tangents from a given point on this axis to the two circles are the same.

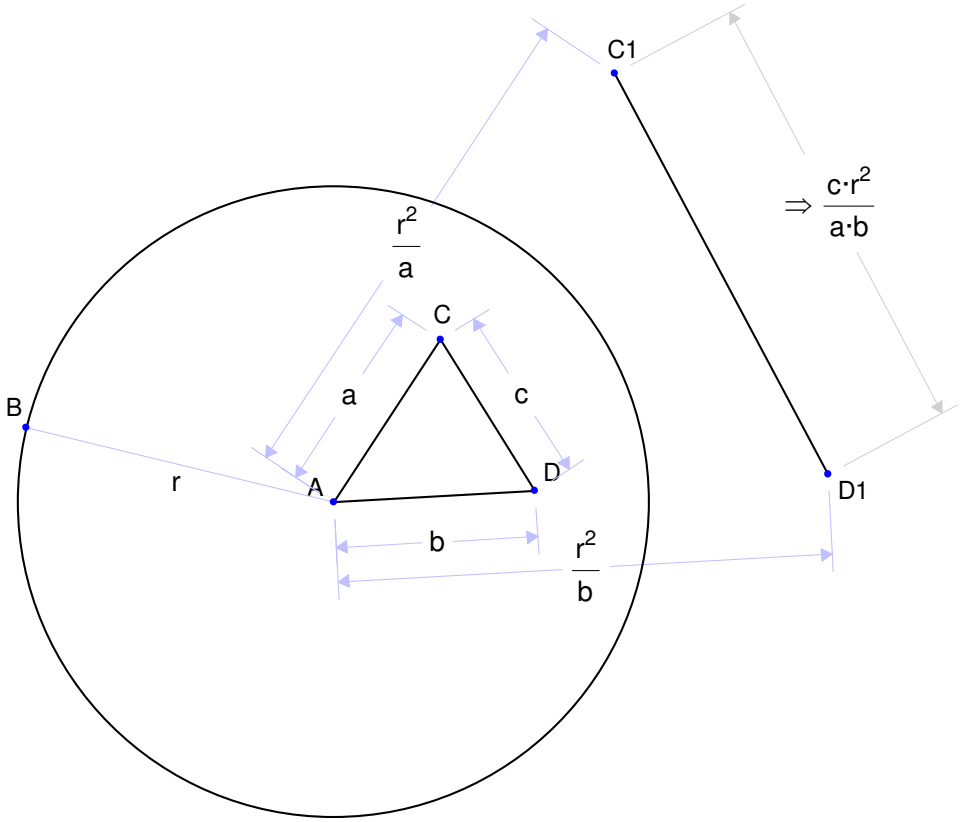


Example 76: Inverting a segment in a circle

To invert a point C in a circle AB radius r, you find a point E on the line AC such that

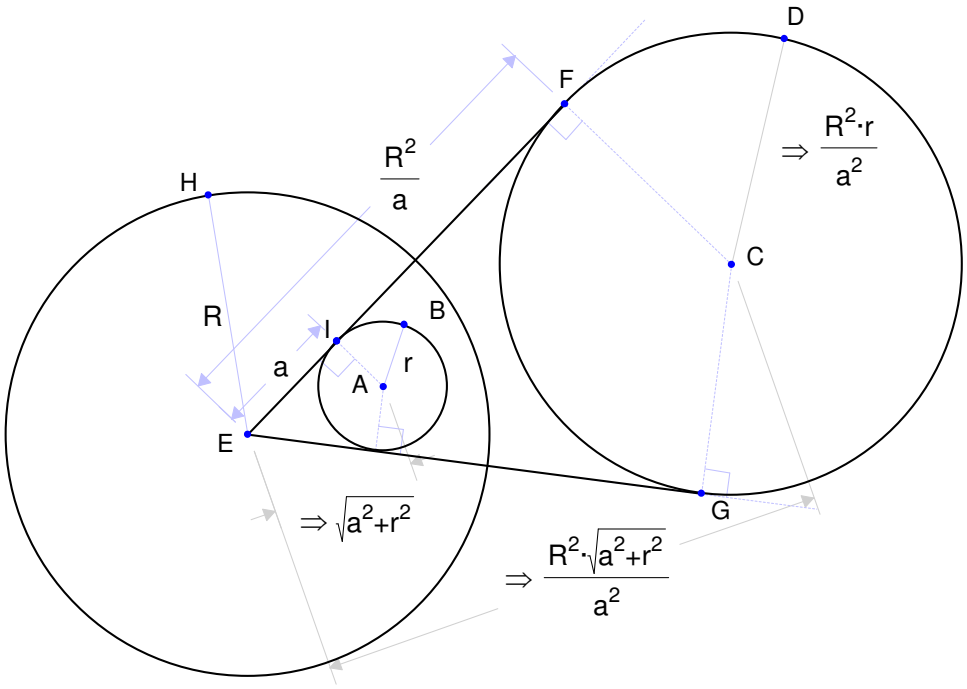
$$AC \cdot AE = r^2$$

If |AC| = a and |AD| = b and |CD| = c, then we find the length of the inverted segment C1D1



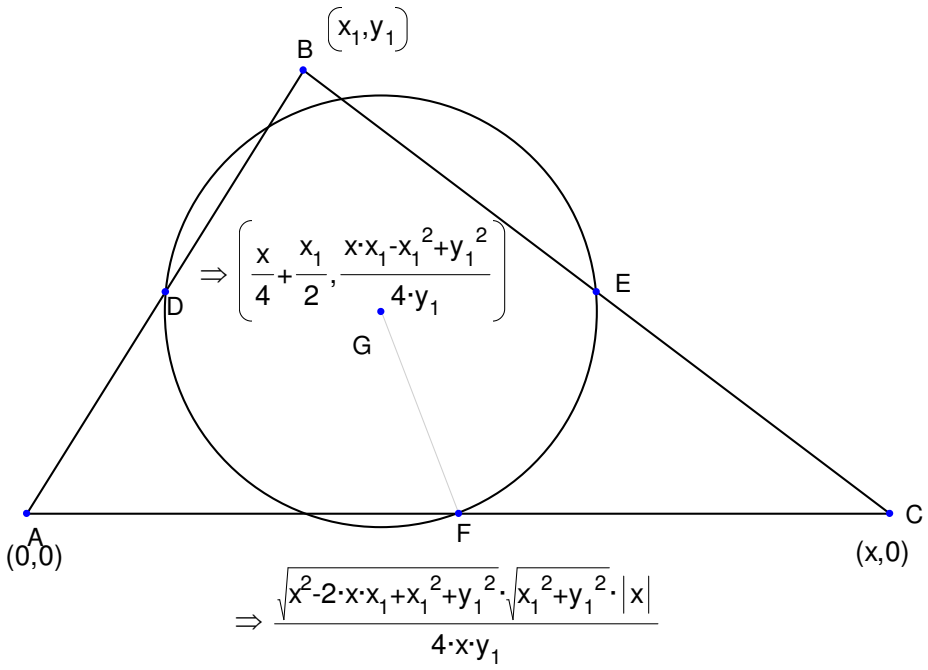
Example 77: Inverting a circle

We can invert a circle, by inverting its points of tangency:



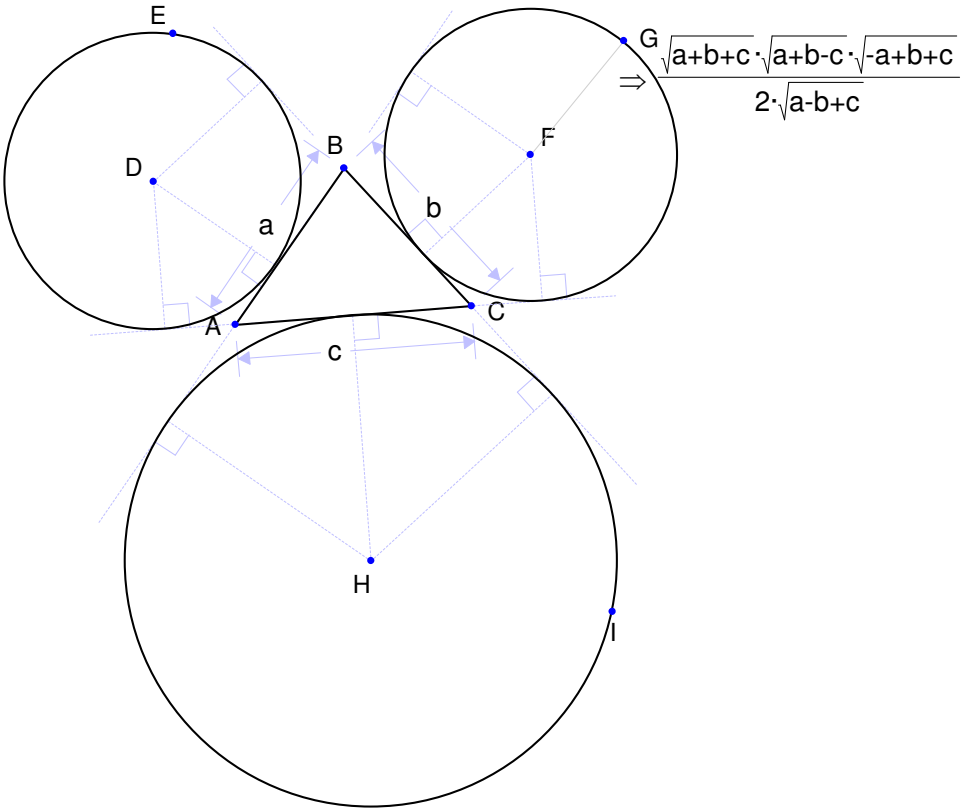
Example 78: The nine point circle

The nine point circle is the circle through the midpoint of each side of a triangle. Taking a triangle with one vertex at the origin, another at (x,0) and another at (x1,y1), we look at the center and radius of the nine point circle:

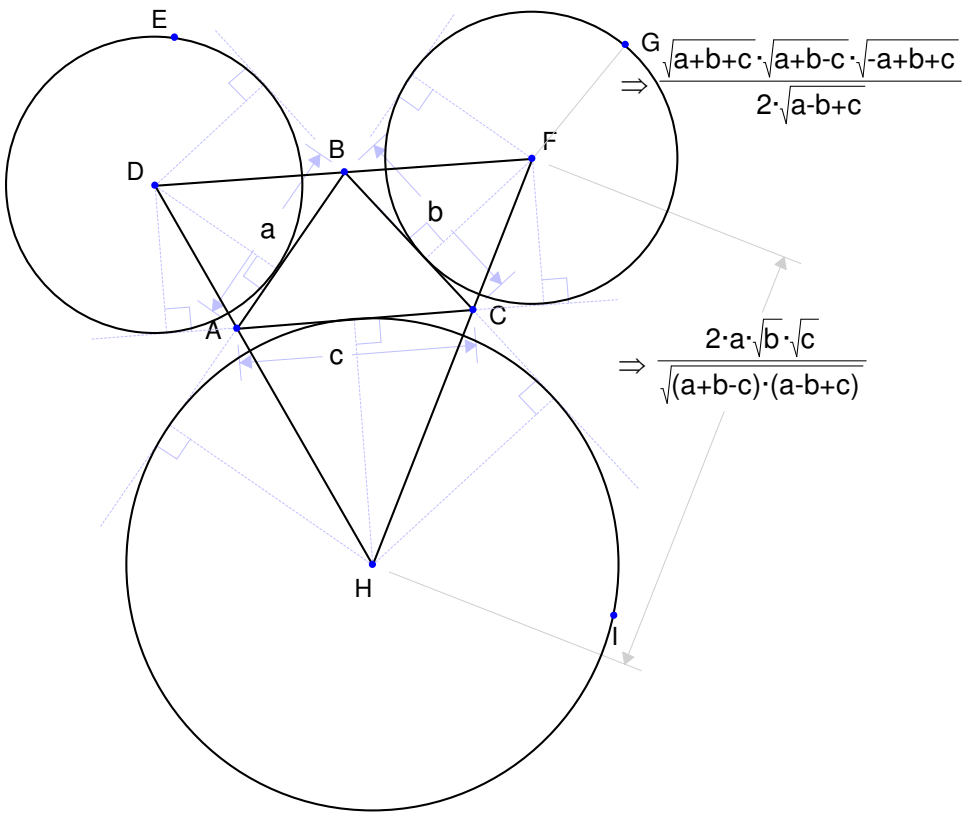


Example 79: Excircles

The three excircles of a triangle are tangent to the three sides but exterior to the circle:

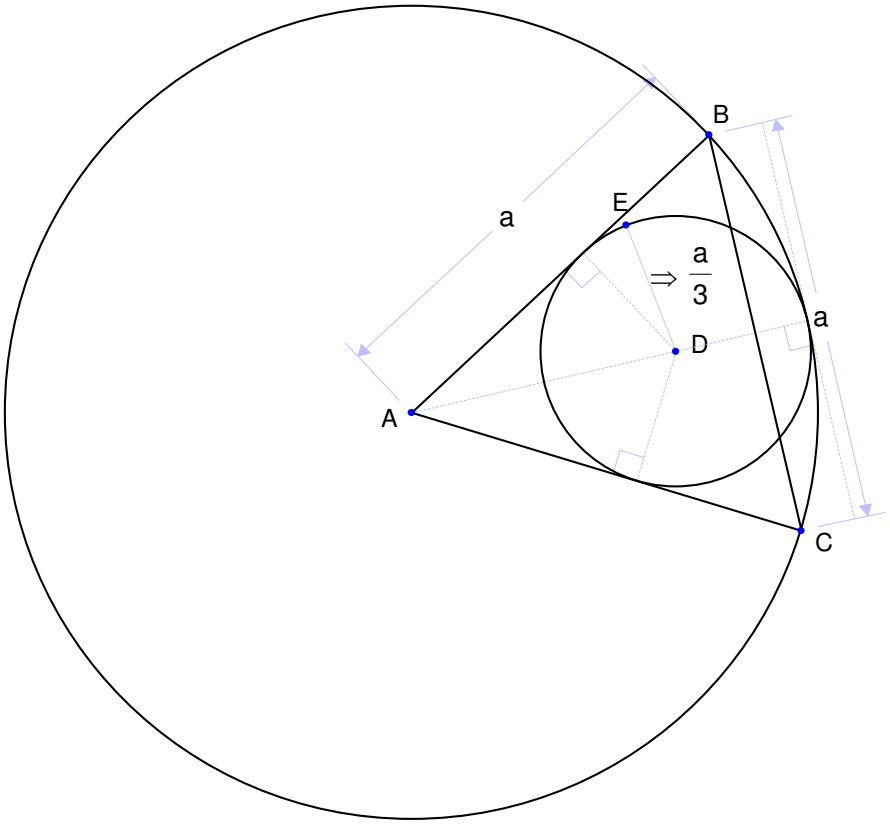


We examine the triangle joining the centers of the excircles:



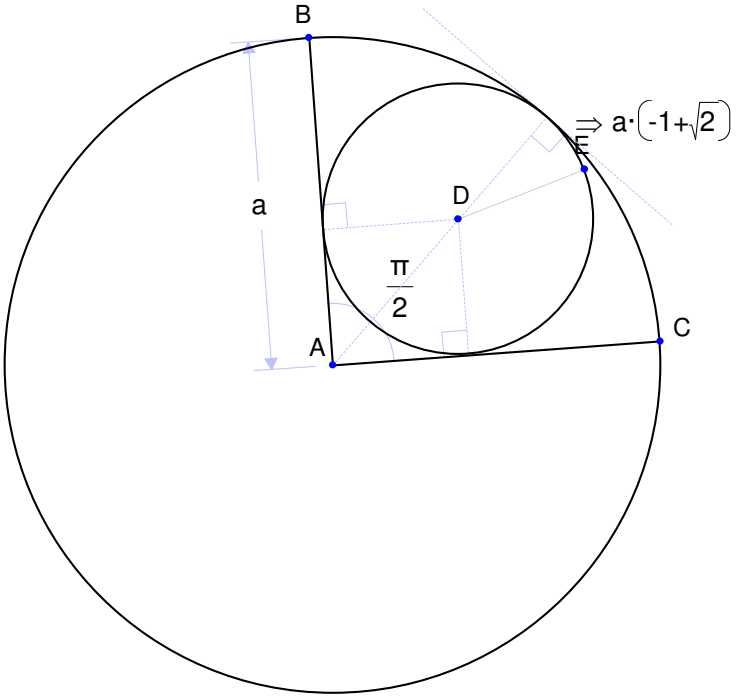
Example 80: Circle tangential to two sides of an equilateral triangle and the circle centered at their intersection through the other vertices

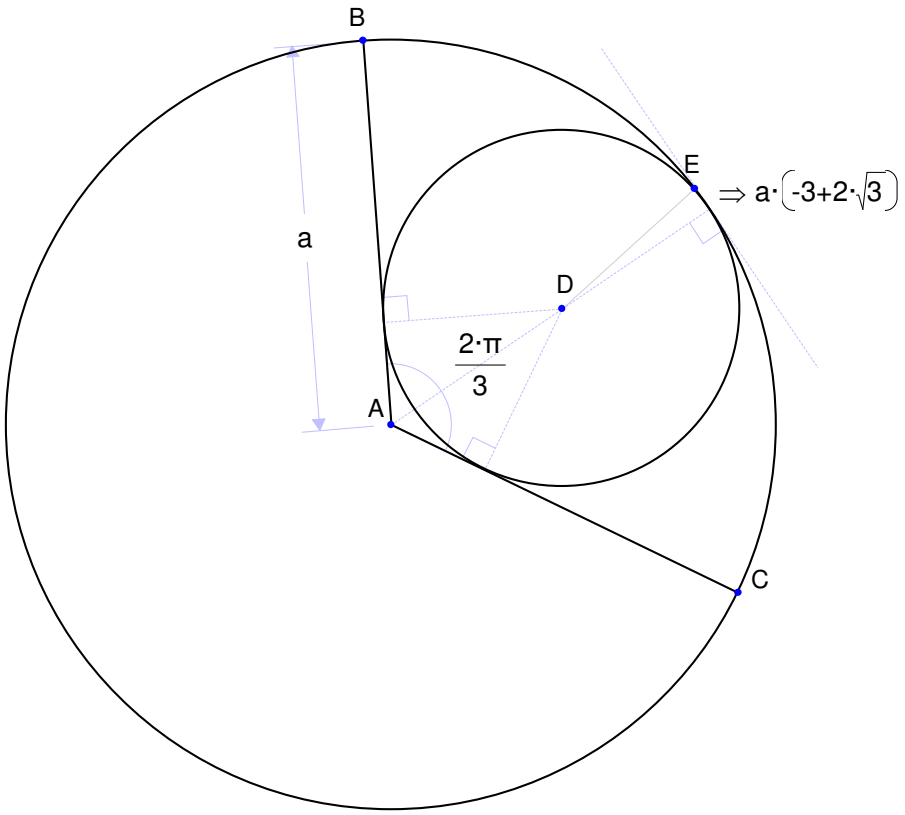
This circle has radius one third the side of the triangle…

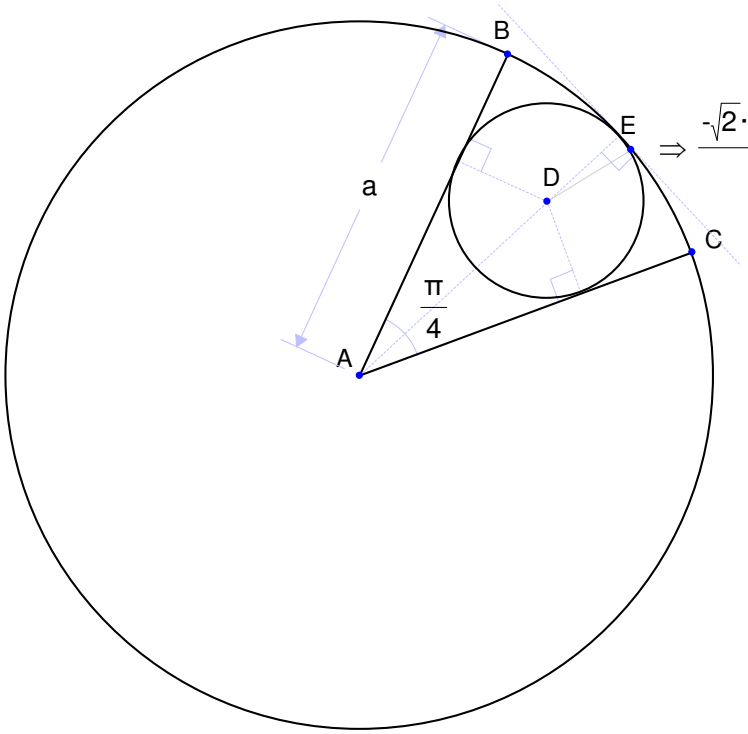


Example 81: Circle tangent to 3 sides of a circular segment

Here are the radii of the circle tangent to two sides and the circular arc of a circular segment, for one or two popular angles:



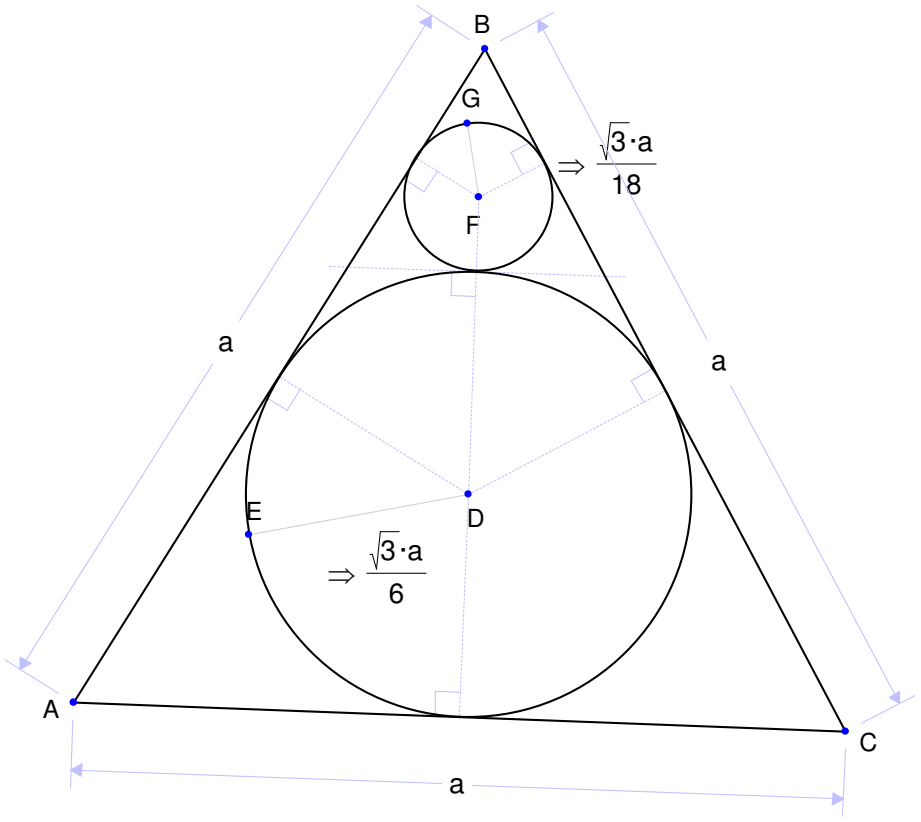

B
E
$\Rightarrow a\cdot\left(-3+2\cdot\sqrt{3}\right)$
a
D
$\frac{2\cdot\pi}{3}$
A
C

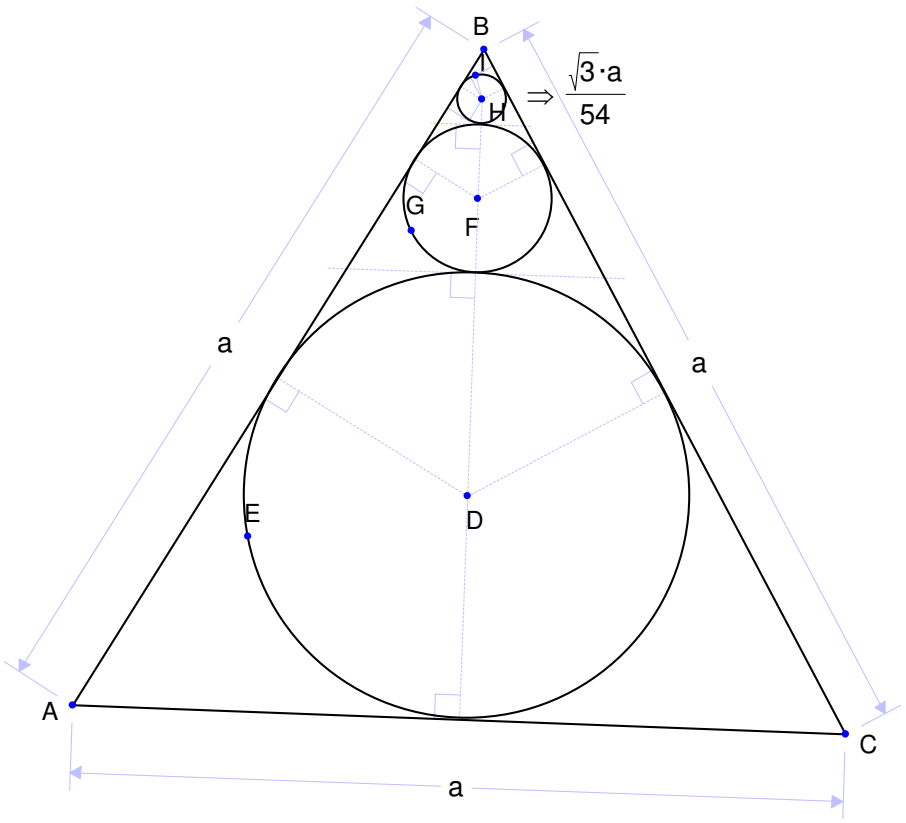


$$\Rightarrow \frac{-\sqrt{2}\cdot\left(\sqrt{2}\cdot a-a\cdot\sqrt{8+4\cdot\sqrt{2}}\right)}{2\cdot\left(3+2\cdot\sqrt{2}\right)}$$

Example 82: Various Circles in an Equilateral Triangle

We look at the radii of various circles in an equilateral triangle:

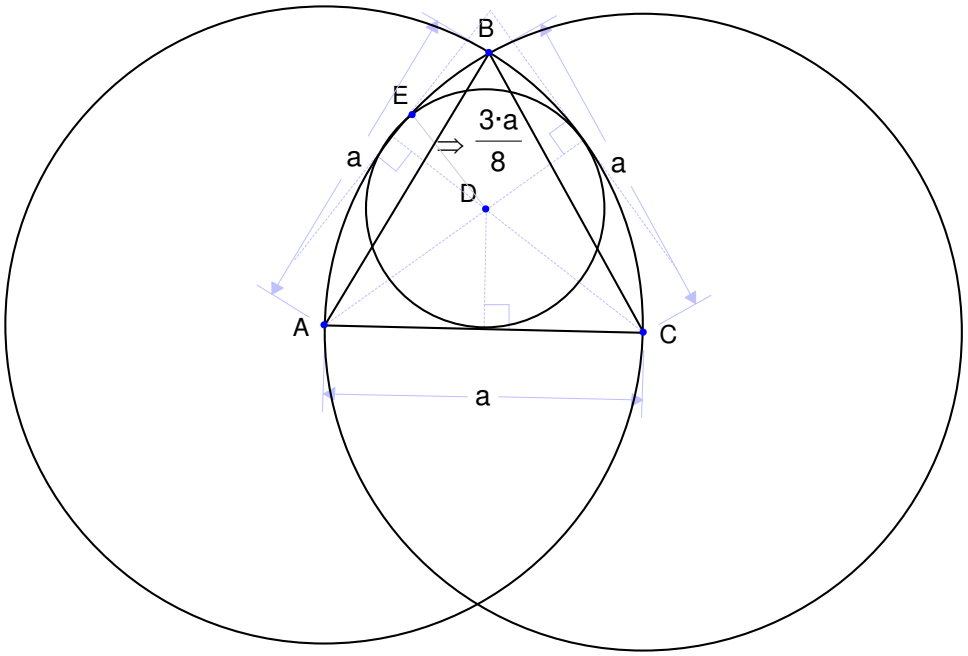




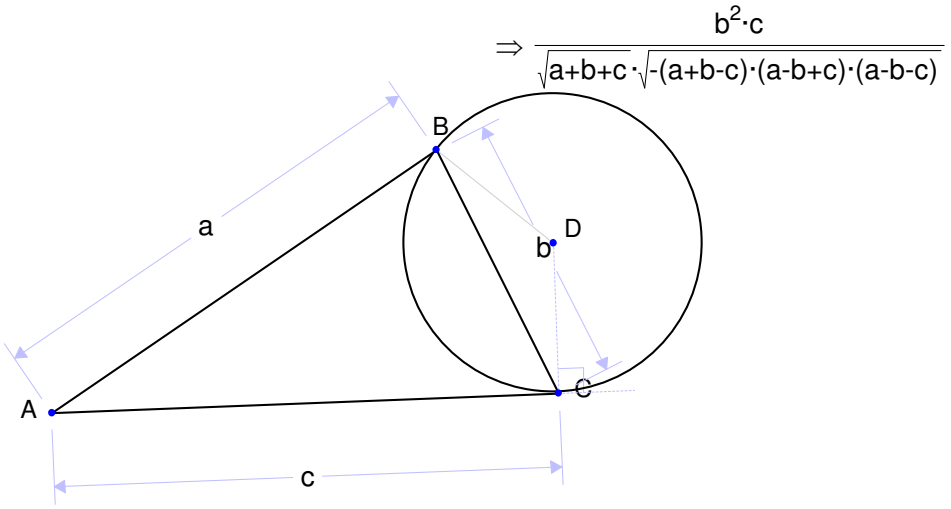
What would the next length in the sequence be?

Example 83: Circle tangential to base of equilateral triangle and constructing circles

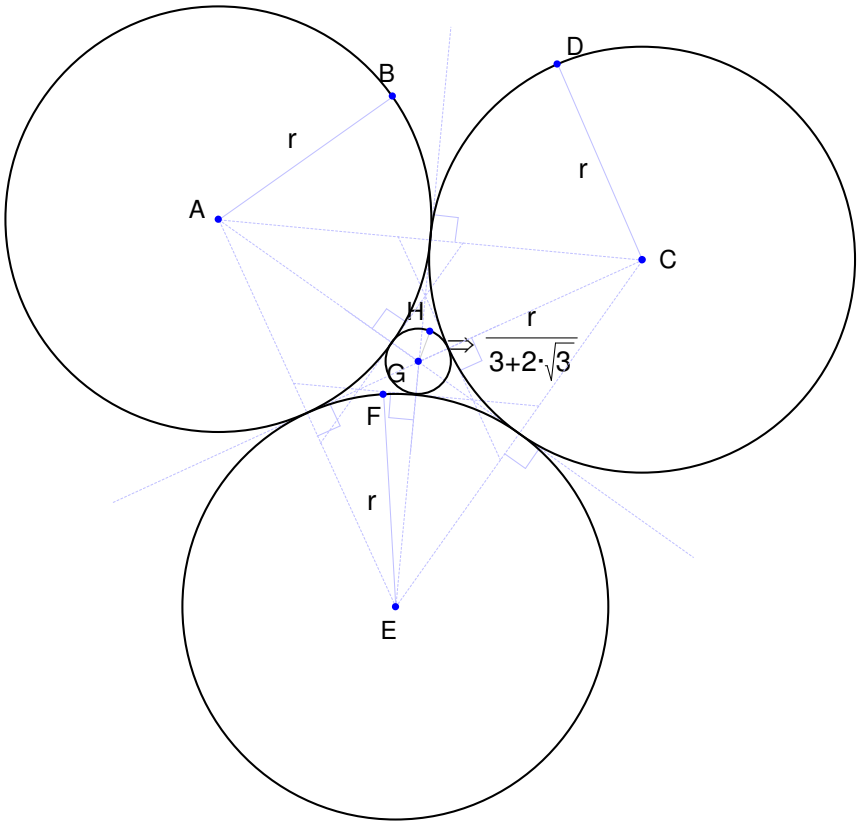
Now we create a circle centered on C through A and B, and make circle DE tangential to it rather than to line AB. We see the circle has radius 3a/8



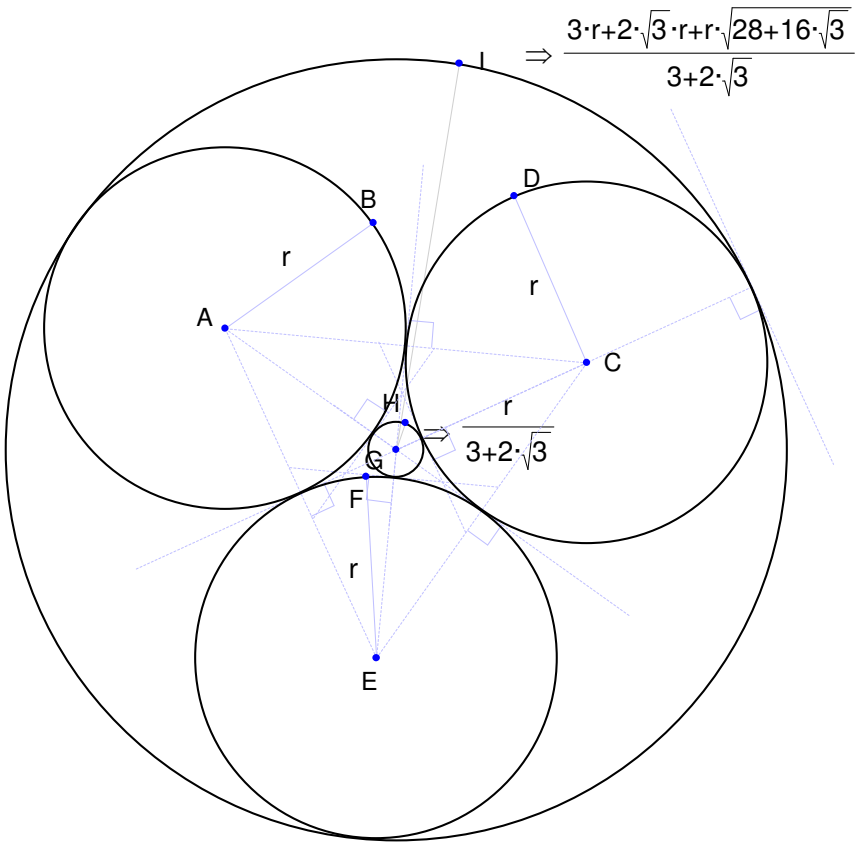
Example 84: Radius of the circle through two vertices of a triangle and tangent to one side



Example 85: Circle Tangent to 3 Circles with the same radii



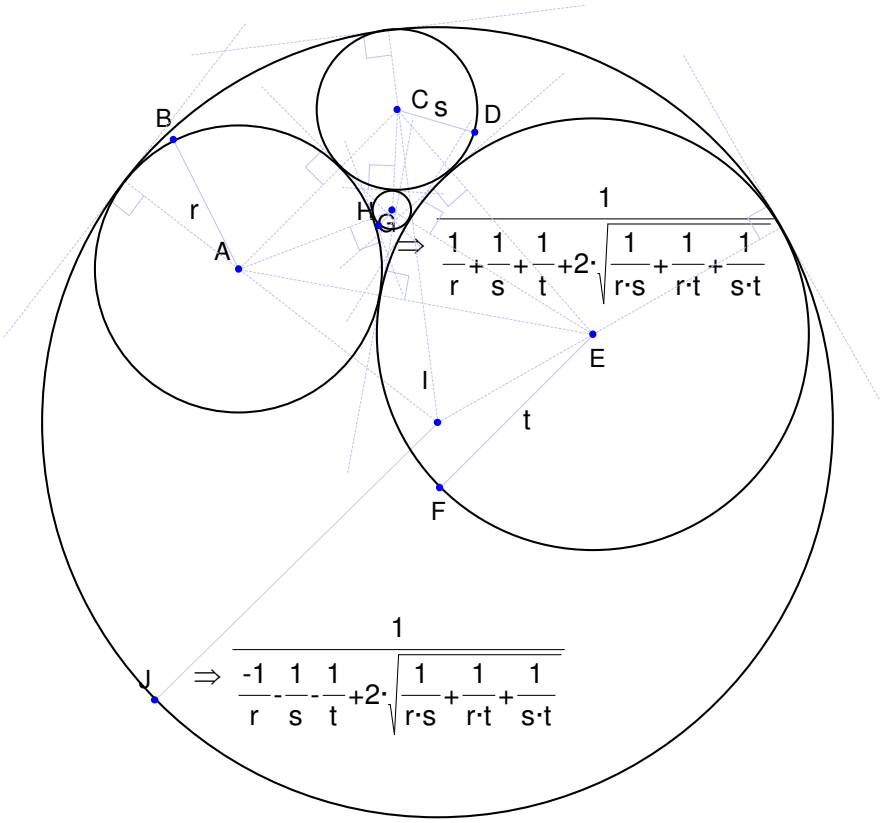
And the external one:



$$\Rightarrow \frac{3\cdot r+2\cdot\sqrt{3}\cdot r+r\cdot\sqrt{28+16\cdot\sqrt{3}}}{3+2\cdot\sqrt{3}}$$

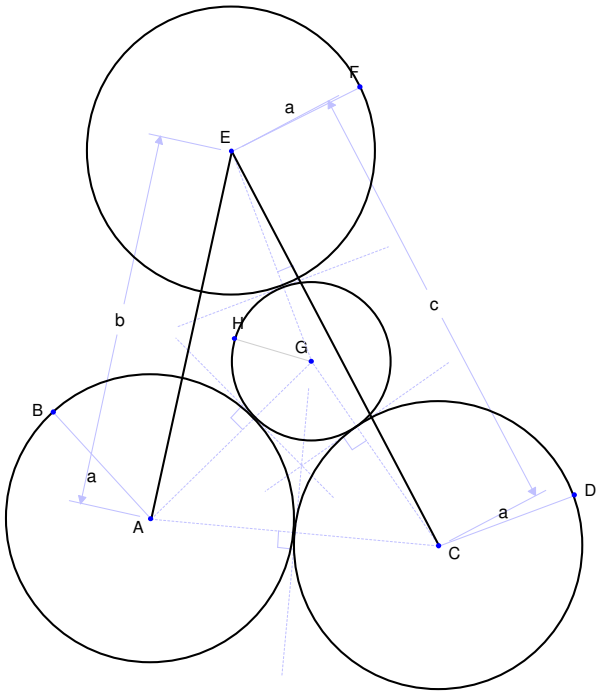
$$\Rightarrow \frac{r}{3+2\cdot\sqrt{3}}$$

Example 86: Circles Tangent to 3 Circles of Different Radii:



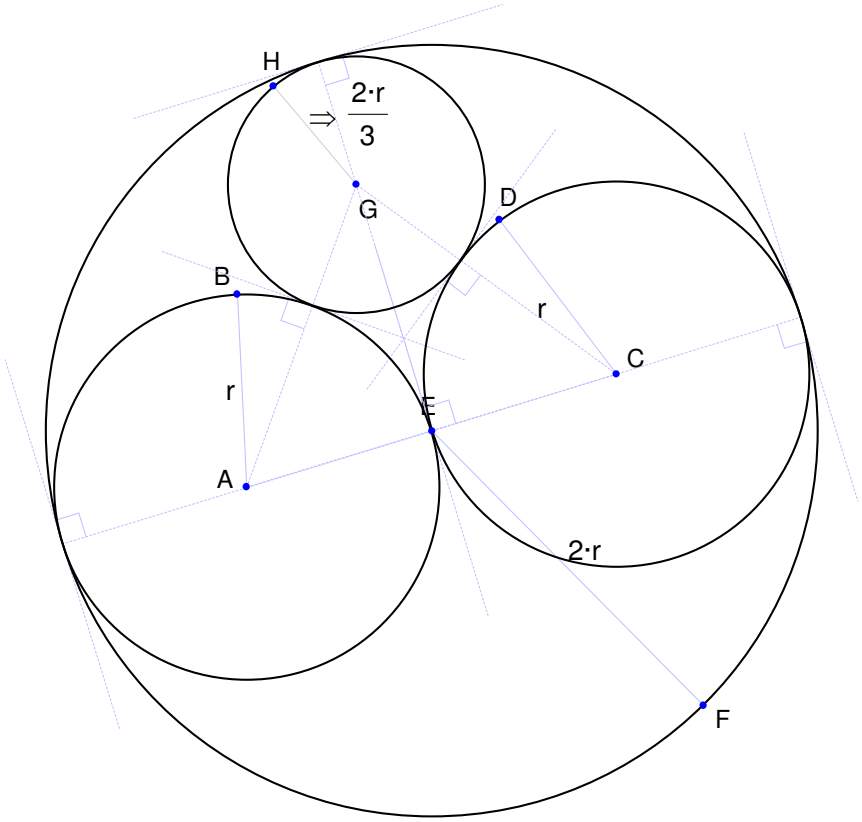
Example 87: Circle Tangential to 3 circles same radius

3 circles of the same radius, two are tangential and the third is distance b and c from those

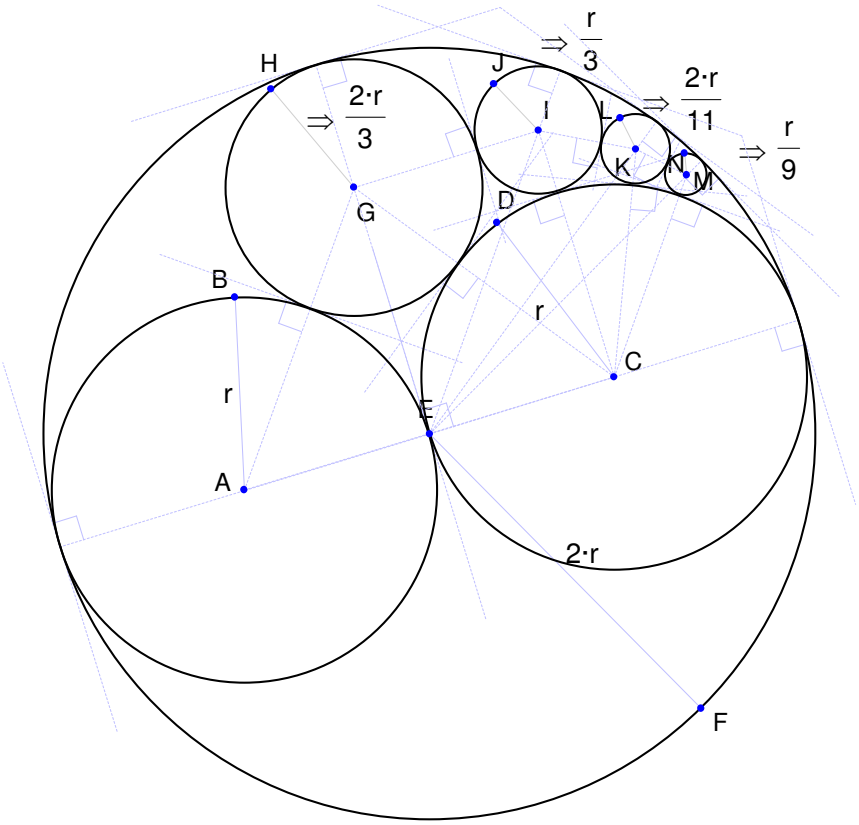


$$\Rightarrow \frac{\sqrt{-2\cdot a+b+c}\cdot\sqrt{2\cdot a+b+c}\cdot\sqrt{2\cdot a+b-c}\cdot\sqrt{2\cdot a-b+c}\cdot\left(-8\cdot a\cdot b\cdot c+4\cdot a\cdot\sqrt{-2\cdot a+b+c}\cdot\sqrt{2\cdot a+b+c}\cdot\sqrt{2\cdot a+b-c}\cdot\sqrt{2\cdot a-b+c}\right)}{4\cdot\left(16\cdot a^4-8\cdot a^2\cdot b^2+b^4+c^4+c^2\cdot\left(-8\cdot a^2-2\cdot b^2\right)\right)}$$

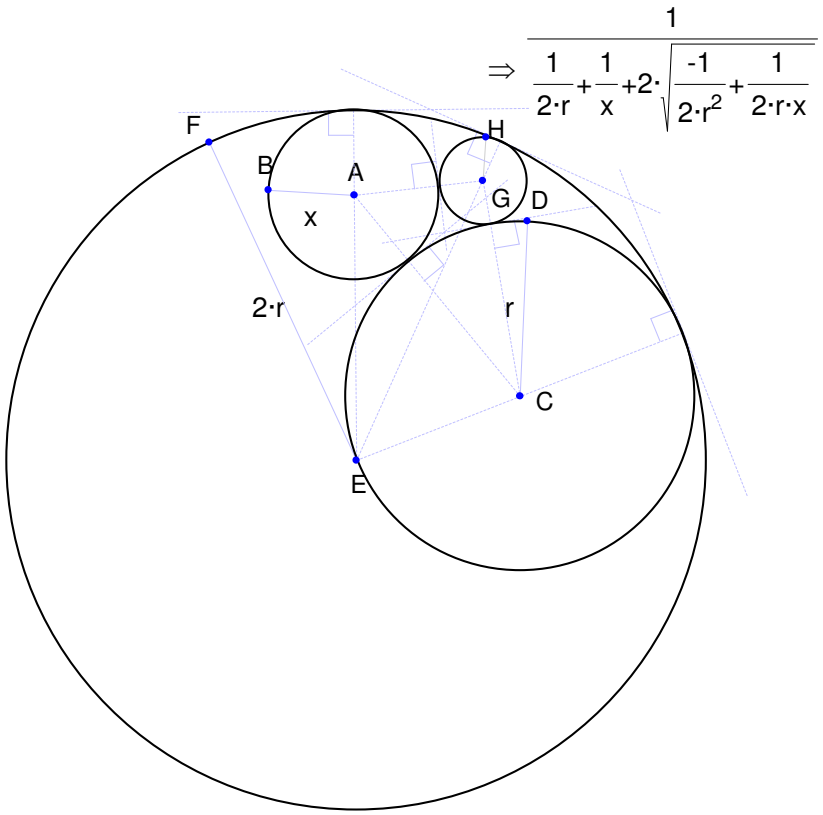
Example 88: Two circles inside a circle twice the radius, then a third



And if we keep on going:



The general case looks like this:



We can copy this expression into Maple to generate the above sequence:

```
> 1/(1/2*1/r+1/x+2*sqrt(-1/2*1/(r^2)+1/2/x/r));
```

$$\frac{1}{\frac{1}{2}\frac{1}{r}+\frac{1}{x}+\sqrt{-2\frac{1}{r^2}+\frac{2}{x\,r}}}$$

```
> subs(r=1,%);
```

$$\frac{1}{\frac{1}{2}+\frac{1}{x}+\sqrt{-2+\frac{2}{x}}}$$

```
> f:=x->1/(1/2+1/x+sqrt(-2+2/x));
```

$$f := x \rightarrow \frac{1}{\frac{1}{2}+\frac{1}{x}+\sqrt{-2+\frac{2}{x}}}$$

```
> f(1);
```

$$\frac{2}{3}$$

```
> f(2/3);
```

$$\frac{1}{3}$$

```
> f(1/3);
```

$$\frac{2}{11}$$

```
> f(2/11);
```

$$\frac{1}{9}$$

```
> f(1/9);
```

$$\frac{2}{27}$$

```
> f(2/27);
```

$$\frac{1}{19}$$

```
>
```

A little analysis of the series can lead us to postulate the formula 2/(n^2+2) for the n'th circle:

Let's feed the n-1th term into Maple:

```
> f(2/((n-1)^2+2));
```

$$\frac{1}{\frac{3}{2}+\frac{1}{2}(n-1)^2+\sqrt{(n-1)^2}}$$

In order to get the expression to simplify, we make the assumption that n>1:

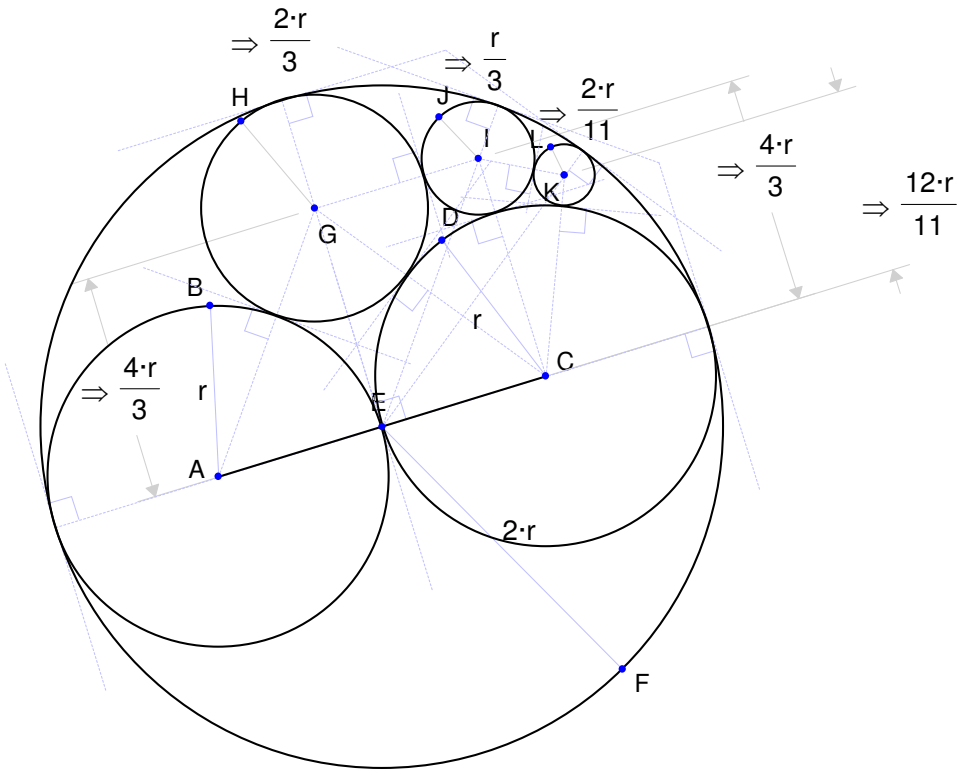
```
> assume(n>1);
> simplify(f(2/((n-1)^2+2)));
```

$$2\,\frac{1}{2+n\sim^2}$$

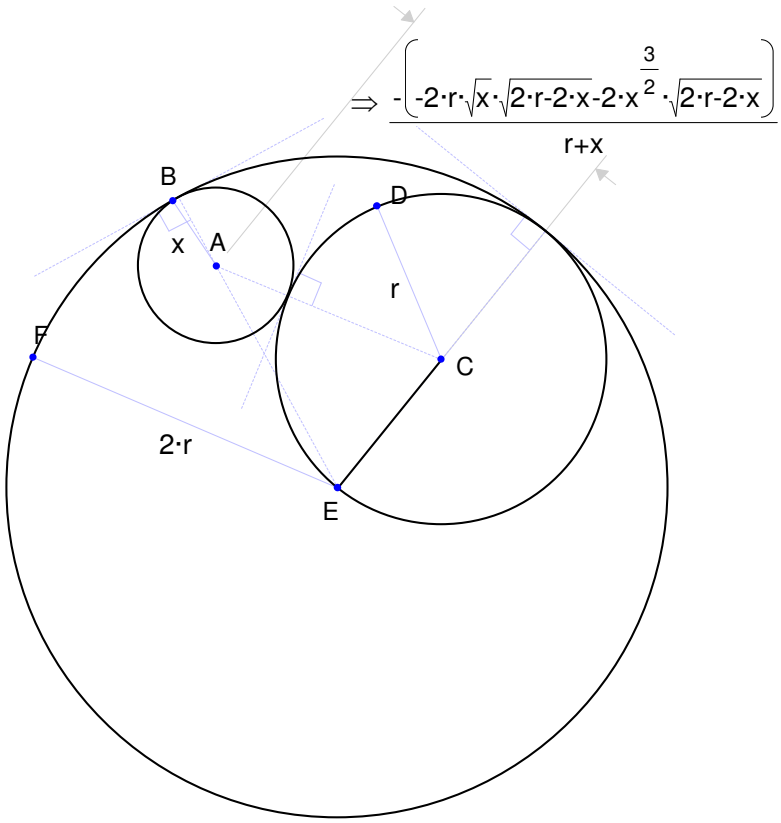
We see that this is the next term in the series. By induction, we have shown that the n'th circle has radius $\frac{2}{2+n^2}$

Example 89: A theorem old in Pappus' time

A theorem which was old in Pappus' days (3rd century AD) relates the radii to height of the circles in figures like the above:



Applying the general model, we get a formula:



Again, we can copy this into Maple for analysis:

```
> -(-2*sqrt(x)*r^3-4*x^(3/2)*r^2-2*x^(5/2)*r)*sqrt(2*r-2*x)/(r+x)^2/r;
```

$$-\frac{(-2\sqrt{x}\,r^3-4\,x^{(3/2)}\,r^2-2\,x^{(5/2)}\,r)\sqrt{2\,r-2\,x}}{(r+x)^2\,r}$$

```
> simplify(%);
```

$$2\sqrt{x}\,\sqrt{2\,r-2\,x}$$

```
> subs(x=2/(n^2+2),r=1,%);
```

$$2\sqrt{2}\,\sqrt{\frac{1}{2+n\sim^2}}\,\sqrt{2-\frac{4}{2+n\sim^2}}$$

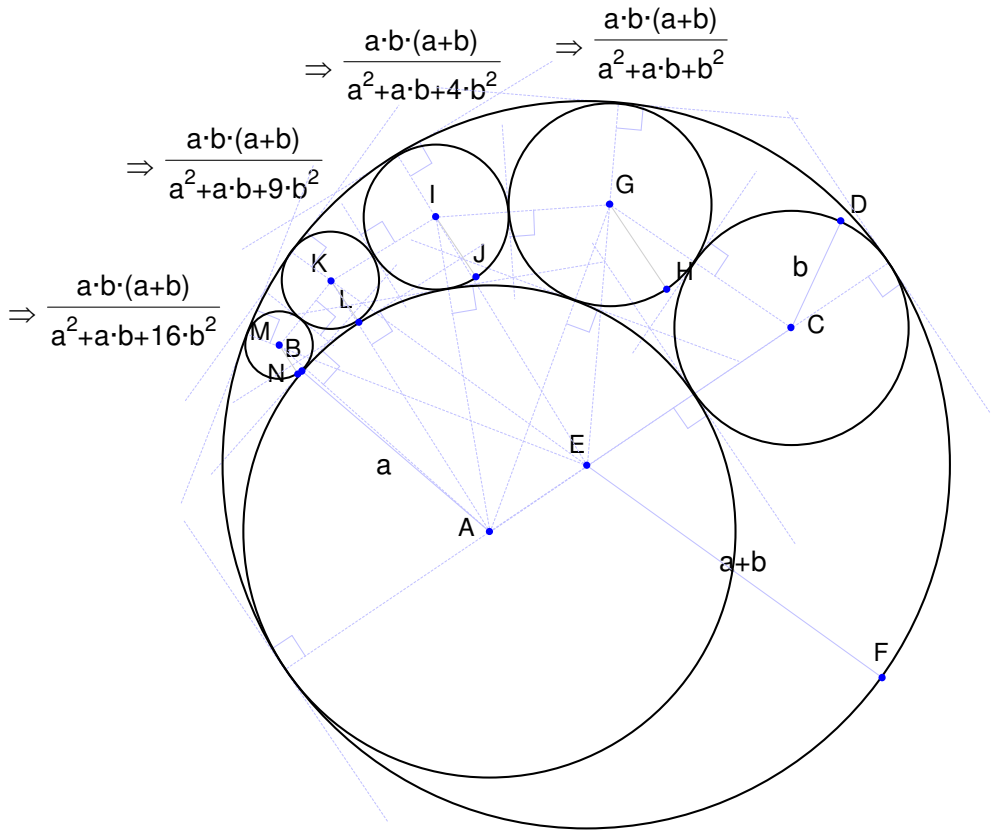
```
> simplify(%);
```

$$4\,\frac{n\sim}{2+n\sim^2}$$

We see that the height above the centerline for these circles is the radius multiplied by 2n.

Example 90: Yet Another Family of Circles

We generalize the situation from a couple of examples ago. We look at the family generated by two circles of radius a and b inside a circle of radius a+b:

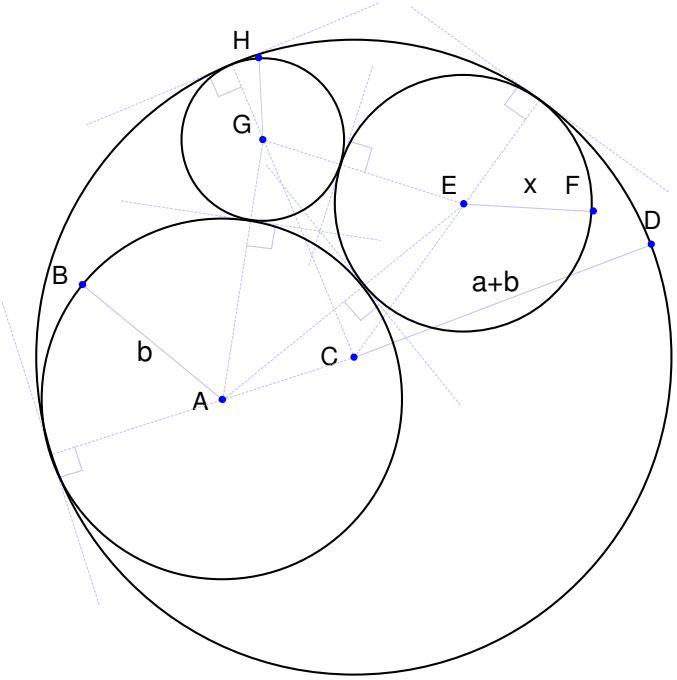


The pattern is pretty obvious this time: the radius of the nth circle is:

$$\frac{ab(a+b)}{n^2a^2+ab+b^2}$$

To prove this, we derive the formula for the general circle radius x, and analyze in Maple:

$$\Rightarrow \frac{1}{\frac{1}{b}+\frac{1}{x}-\frac{1}{a+b}+2\cdot\sqrt{\frac{1}{b\cdot x}-\frac{1}{b\cdot(a+b)}-\frac{1}{x\cdot(a+b)}}}$$



Now we try feeding in one of the circle radii into this formula in maple (first making the assumption that the radii are positive (along with n>1 for later use):

```
> assume(a>0,b>0,n>1);
> f:=x->1/(1/b+1/x-1/(a+b)+2*sqrt(1/(x*b)-1/((a+b)*b)-
1/((a+b)*x)));
```

$$f := x \to \frac{1}{\frac{1}{b}+\frac{1}{x}-\frac{1}{a+b}+2\sqrt{\frac{1}{x\,b}-\frac{1}{(a+b)\,b}-\frac{1}{(a+b)\,x}}}$$

```
> f((a+b)*b*a/(9*a^2+b*a+b^2));
```

$$1/\left(\frac{1}{b\sim}+\frac{9\,a\sim^2+b\sim\,a\sim+b\sim^2}{(a\sim+b\sim)\,b\sim\,a\sim}-\frac{1}{a\sim+b\sim}+2\sqrt{\frac{9\,a\sim^2+b\sim\,a\sim+b\sim^2}{(a\sim+b\sim)\,b\sim^2\,a\sim}-\frac{1}{(a\sim+b\sim)\,b\sim}-\frac{9\,a\sim^2+b\sim\,a\sim+b\sim^2}{(a\sim+b\sim)^2\,b\sim\,a\sim}}\right)$$

```
> simplify(%);
```

$$\frac{(a\sim + b\sim)\, b\sim\, a\sim}{16\, a\sim^2 + b\sim\, a\sim + b\sim^2}$$

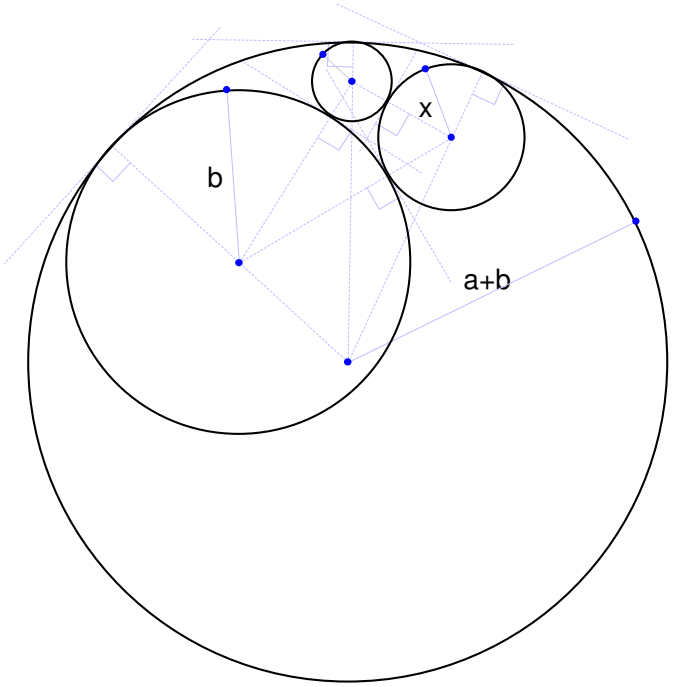
Let's try the general case, feeding in the formula for the n-1st radius:

```
> simplify(f((a+b)*b*a/((n-1)^2*a^2+b*a+b^2)));
```

$$\frac{b\sim\, a\sim\, (a\sim + b\sim)}{b\sim\, a\sim + a\sim^2\, n\sim^2 + b\sim^2}$$

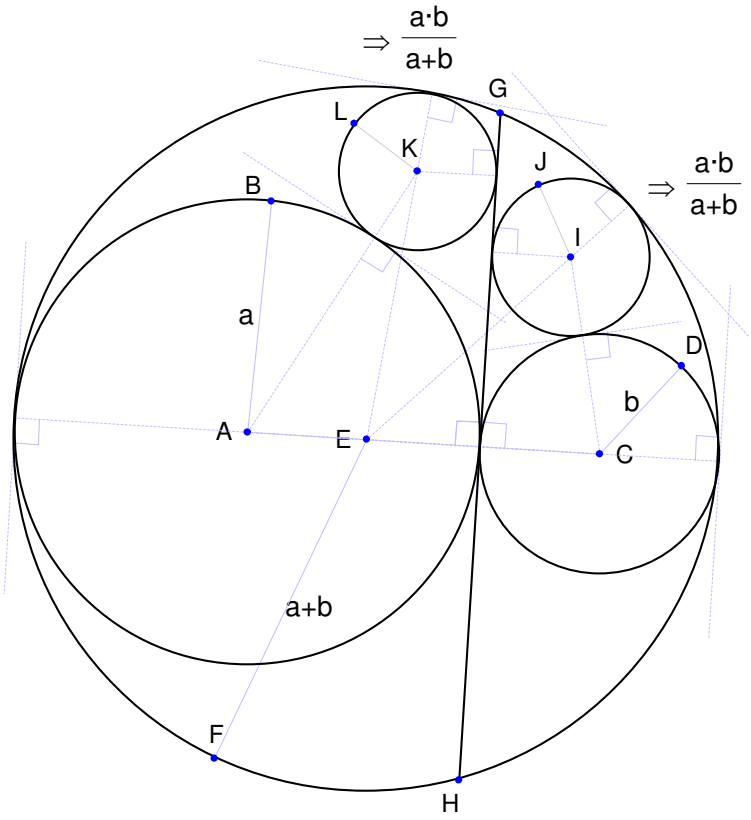
By induction, we have proved the general result.

$$\Rightarrow \frac{1}{\frac{1}{b}+\frac{1}{x}-\frac{1}{a+b}+2\cdot\sqrt{\frac{1}{b\cdot x}-\frac{1}{b\cdot(a+b)}-\frac{1}{x\cdot(a+b)}}}$$



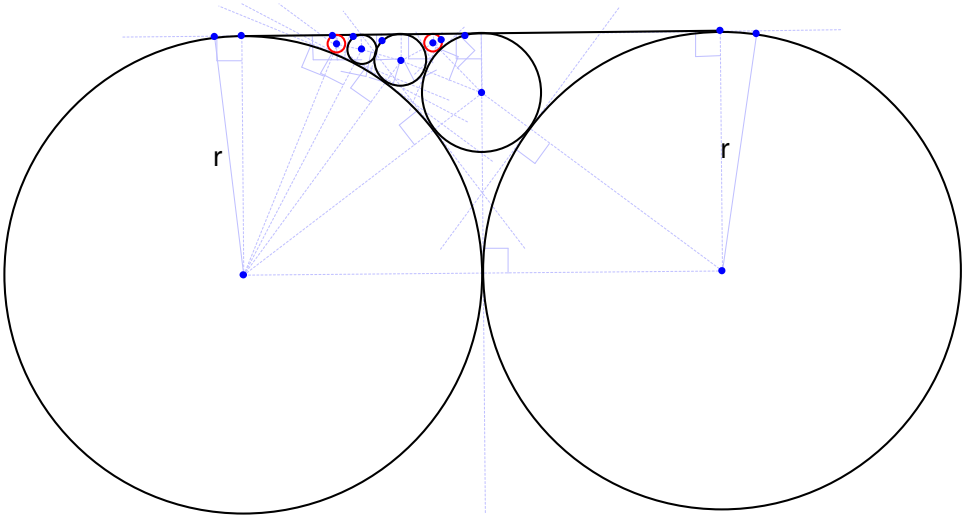
Example 91: Archimedes Twins

The given circles are mutually tangential with radius a, b and a+b. Archimedes twins are the circles tangential to the common tangent of the inner circles. We see from the symmetry of the radius expression that they are congruent.

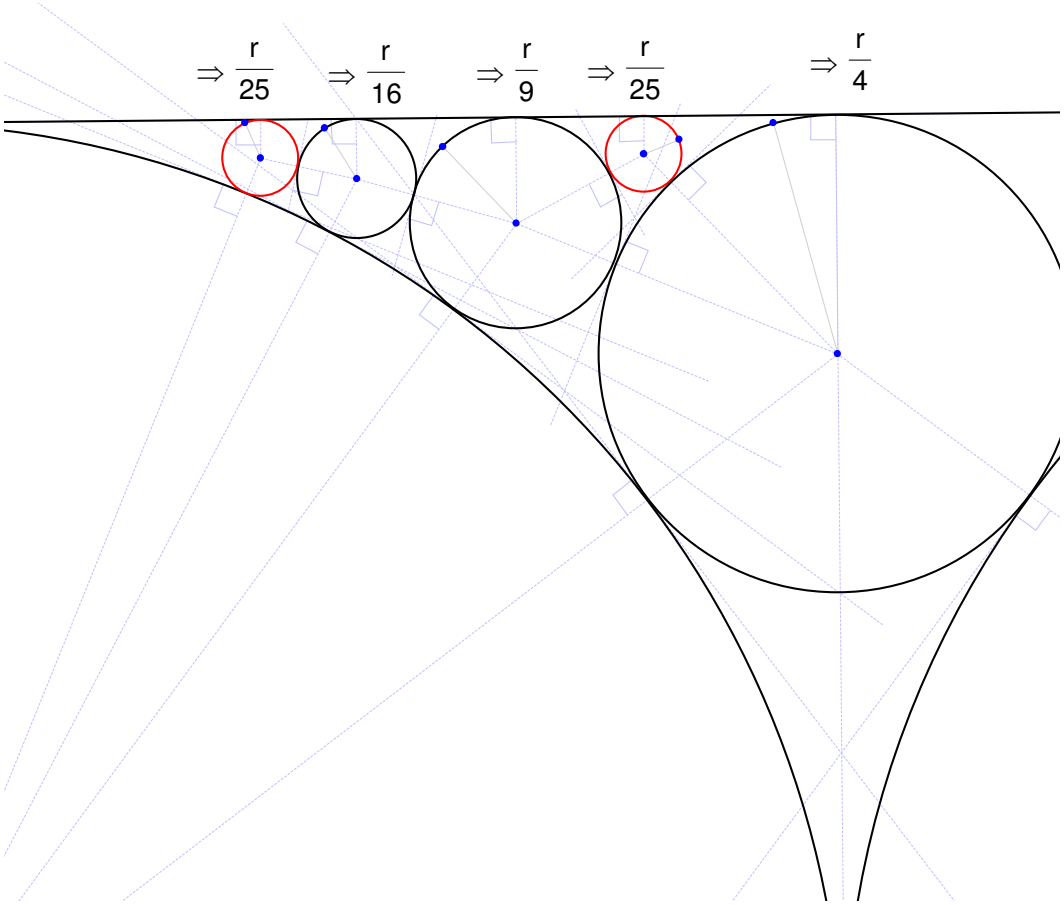


Example 92: Circles tangential to two Touching circles and their common tangent

Here is another family of circles., based on original circles of radius r

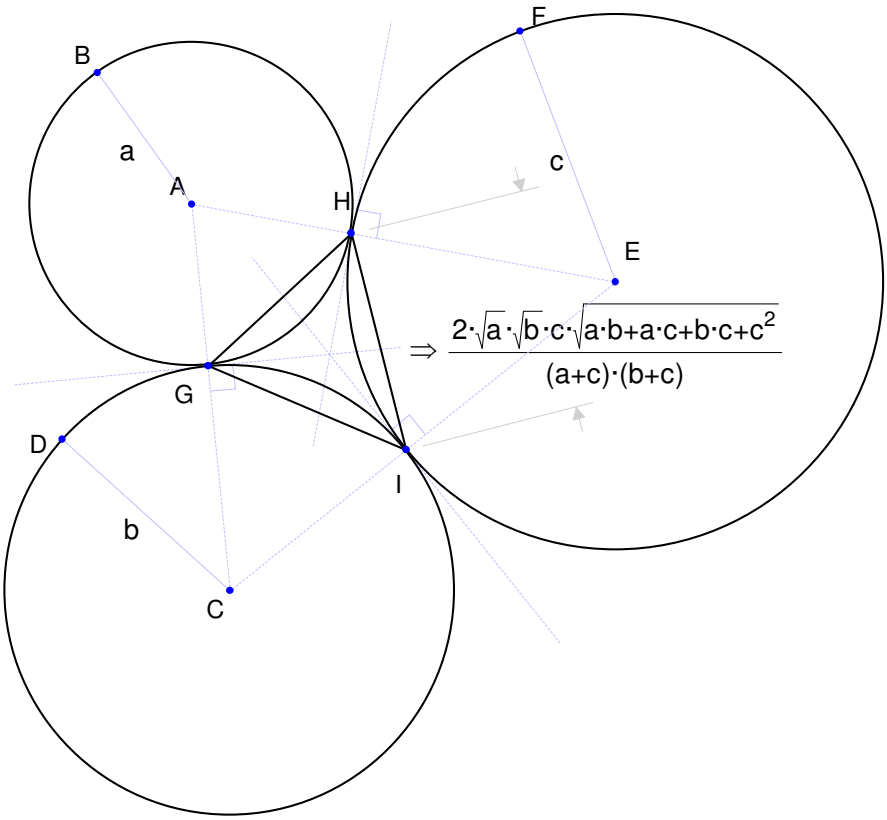


Notice that the two red ones have the same radii:



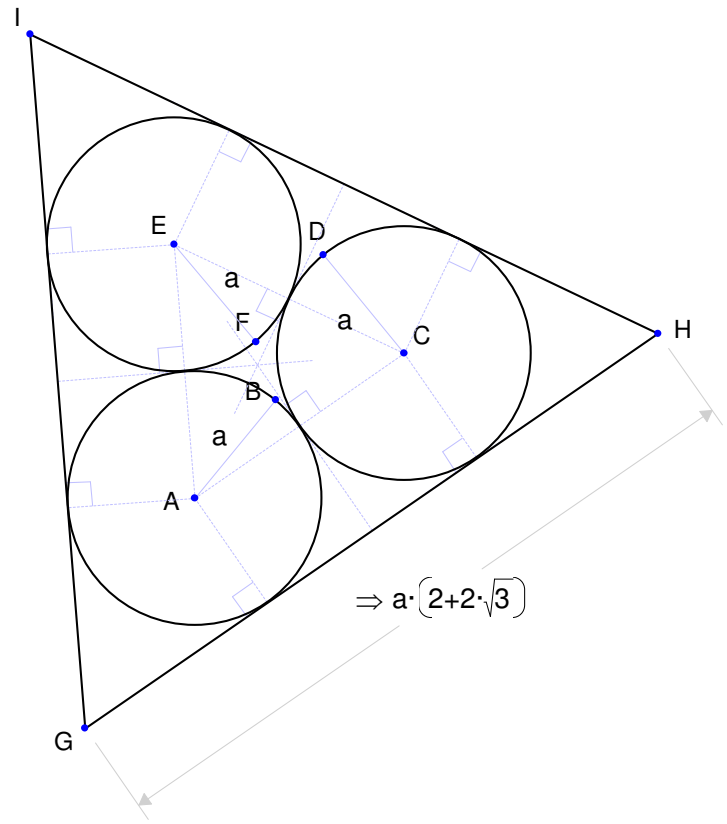
Example 93: The triangle joining the points of tangency of 3 circles

Given tangential circles radii a,b,c, we find the side length of the triangle joining the points of tangency:



Example 94: The triangle tangential to 3 tangential circles

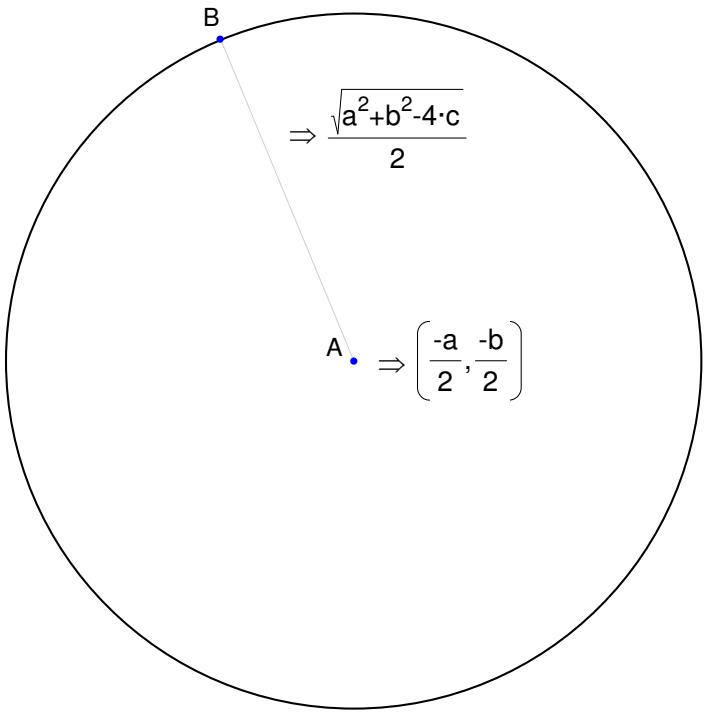
We look at the triangle formed by the common tangents to these 3 circles – in the case where the circles all have radius a, this is an equilateral triangle, with length as shown below:



Example 95: Center and Radius of a Circle Given Equation

Find the center and radius of the circle whose equation is

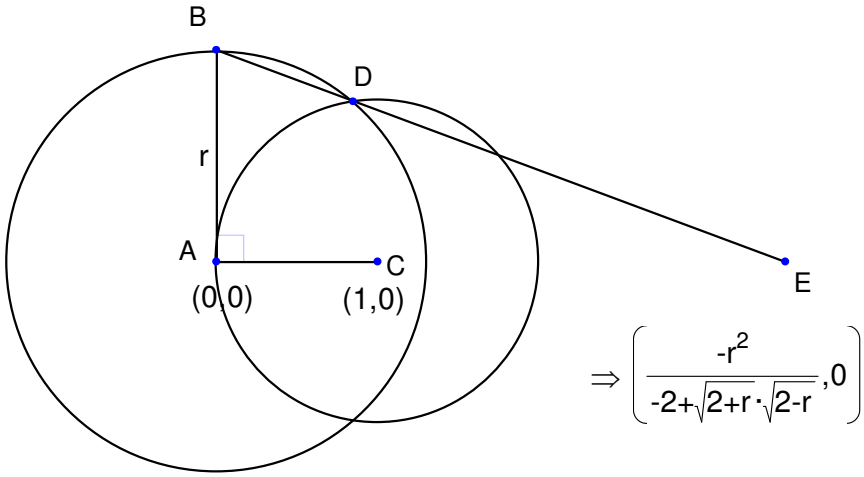
$$x^2 + y^2 + ax + by + c = 0$$



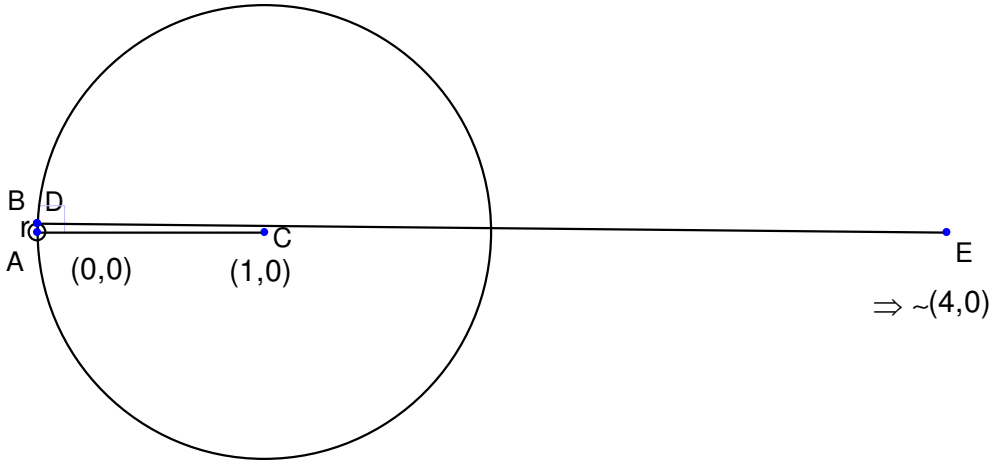
$X^2+Y^2+X\cdot a+Y\cdot b+c=0$

Example 96: A limit point

In the example, E is the intersection of BD with the x axis. We examine the x coordinate of E as r tends to 0



This can be done by means of a picture:

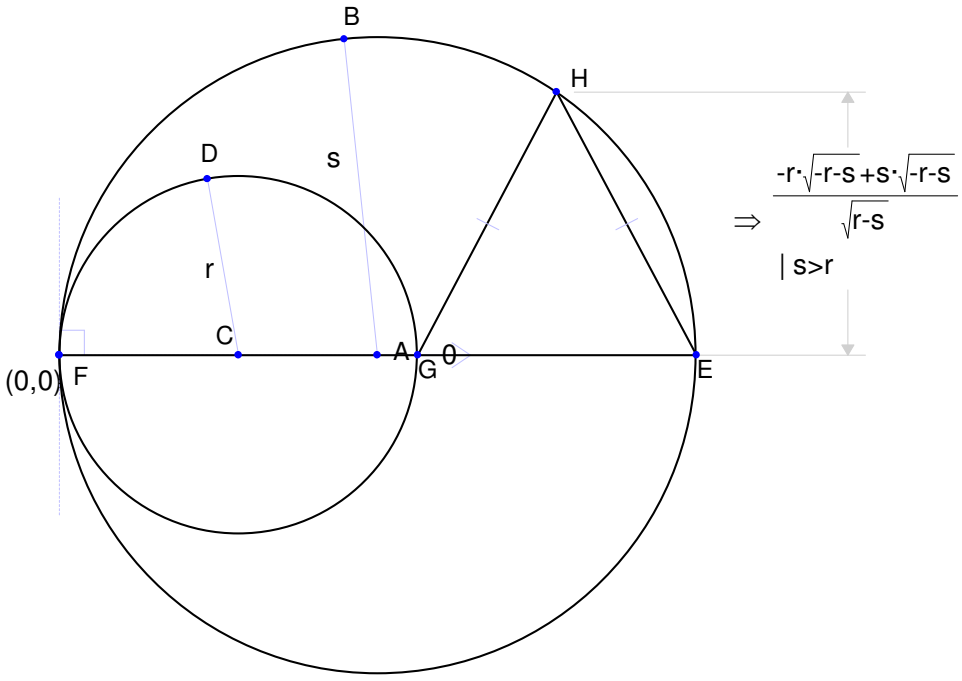


Or analytically in Maple:

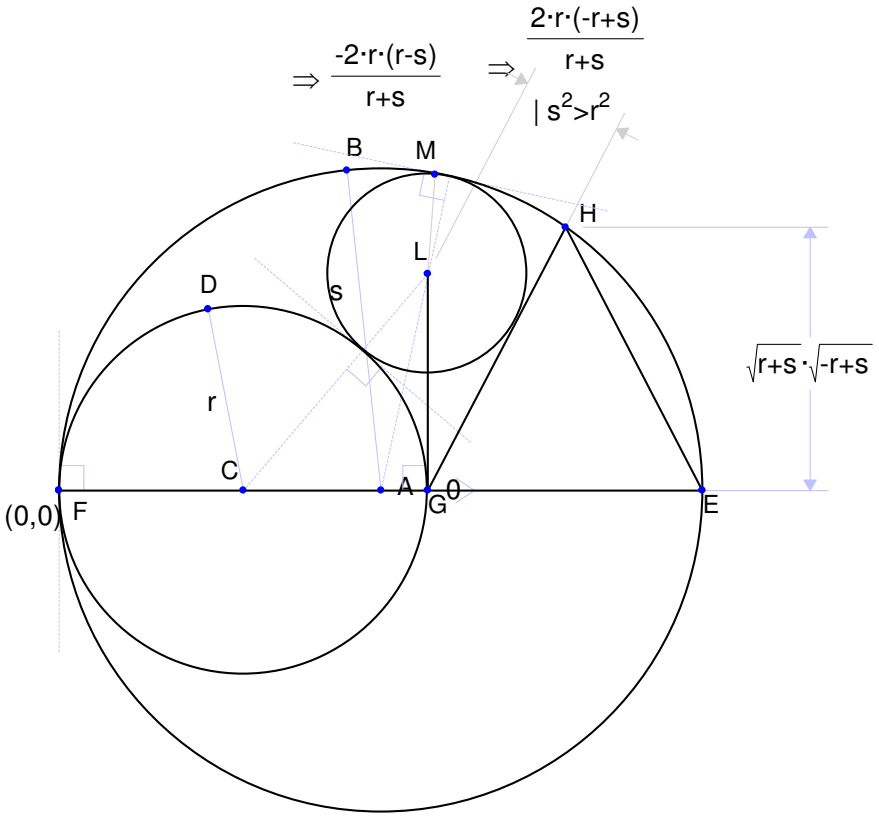
```
> limit(-r^2/(sqrt(-r+2)*sqrt(r+2)-2),r=0);
```

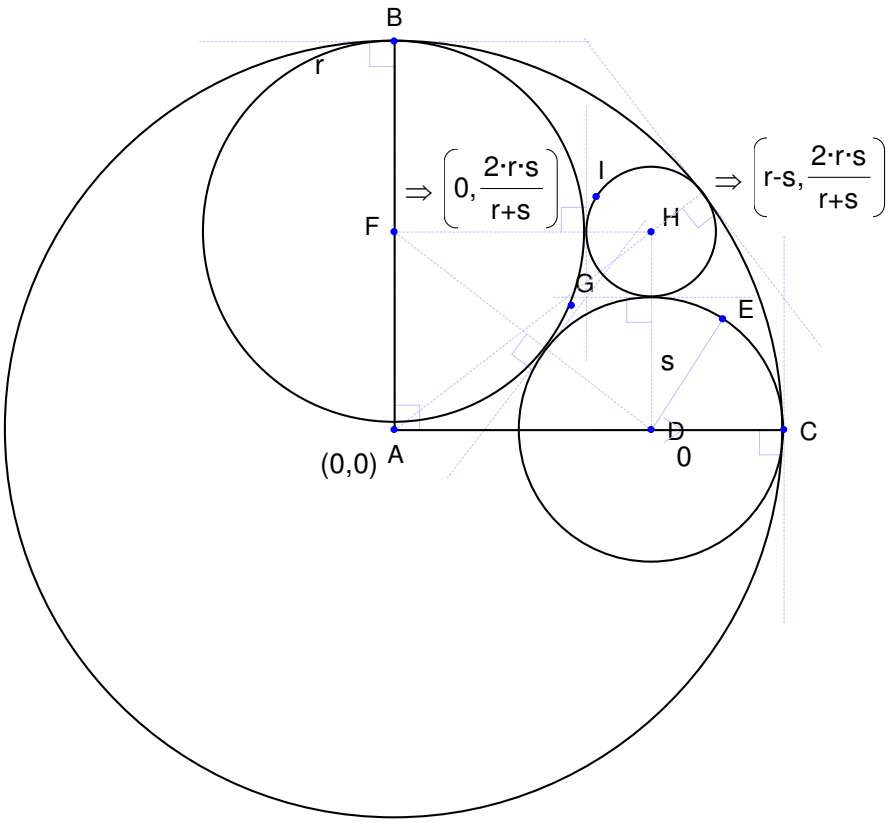
Example 97: Buehler's Circle

Given a circle inside and tangent to another circle, we create an isosceles triangle whose base connects the intersection between the axis of symmetry and the two circles, and whose apex lies on the outer circle. We get the height of this triangle:



Using this as input, we construct the circle tangent to both circles, whose center is on the line through G perpendicualr to AC. We examine its radius and the distance from its center to the line. We infer that it is tangential to this line.



Example 98: Circle to two circles on orthogonal radii of a third

In the diagram, Circles DE and FG are centered on perpendicular radius lines AB and AC. Circle HI is tangential to all three. We observe that AFHD is a rectangle.