Conics and Loci

Studies in Precalculus

Using

Geometry Expressions
Conics and Loci

Unit Overview

Many traditional studies of conic sections center on the implicit formulas and how they can be used to quickly sketch graphs. These skills are quickly becoming outdated as graphing technologies evolve. Geometric definitions of conic sections are often treated as a separate topic, sometimes even in a separate course.

The intent of this unit is to look at conic sections through three lenses:
- Definitions of conic sections as a locus of points
- Parametric equations of conic sections, in terms of trigonometric functions
- Implicit equations of conic sections

The main goal of this unit is to make connections between these three representations of the conic sections. At the end of the unit, students will have geometric definitions, parametric equations, and implicit equations for circles, ellipses and hyperbolas, as well as a geometric definition and implicit equation for parabolas.

Lesson 1: Introducing Loci

- The concept of a locus of points is introduced. Circles, angle bisectors, and perpendicular bisectors are among the loci that are studied. While interesting in its own right, most of this lesson gets students acquainted with the features of Geometry Expressions that they will need throughout the unit.

Lesson 2: Loci: Circles

- The definition of a circle as a locus of points is presented. Connections are made to the unit circle, the Pythagorean Theorem, and to the distance formula.

Lesson 3: The Ellipse

- An ellipse is constructed using the geometric definition. Students use Geometry Expressions’ symbolic capabilities to generate the parametric and implicit equations.

Lesson 4 Envelope Curves

- The concept of a locus of lines and the resulting envelope curve is used to re-define the ellipse. This definition is extended to introduce hyperbolas.

Lesson 5 Inside-Out Ellipses

- Lesson 4 is used as a jumping-off point for this lesson on hyperbolas. The geometric definition, parametric equation, and implicit equation are developed through comparison with ellipses.
Lesson 6  Eccentricity and Parabolas

- Parabolas are introduced as the locus of points equidistant from a line and a point. The concept of eccentricity is presented to expand this definition to include hyperbolas and ellipses.

Topics outside the scope of this unit

Topics outside the scope of this unit include:

- A discussion on why circles, ellipses, hyperbolas, and parabolas are called “conic sections.”
- Using implicit equation to sketch graphs of conics.
- Using the discriminant to determine the type of conic section.
- Rotation of axis to eliminate the $xy$ term
- An exhaustive discussion on the special properties of parabolas.
Learning Objectives

This lesson is the introduction to conic sections. Conic sections (parabolas, hyperbolas, ellipses, circles) can all be described as a locus of points. In this lesson, students will become re-acquainted with some familiar loci, and with how Geometry Expressions can be used to explore loci.

Math Objectives

• Learn the concept of a “locus.”

Technology Objectives

• Create loci with Geometry Expressions

Math Prerequisites

• Parallel lines, perpendicular bisectors and angle bisectors
• Parametric Functions

Technology Prerequisites

• None

Materials

• Computers with Geometry Expressions
Overview for the Teacher

1. Part one introduces the students to the software, and to the idea of a locus.

   Watch to see if students are changing the value of $a$ – it should remain constant. Otherwise, they might be led to believe that the locus is a family of concentric circles. The Variables Tool Panel can be used to lock variable $a$. Click on the entry for $a$, and then click on the lock icon.

   Note that $\theta$ measures the rotation of the segment counter-clockwise from the horizontal. However, if the student draws the segment from A to B, the A is placed at the origin. If the segment is drawn from B to A, then B is placed at the origin.

   The result is a circle with center A and radius $a$, as seen in Diagram 1.

2. The difficult part of question 2 is making a case for creating point A. To create a locus of points, Geometry Expressions requires a Parametric Variable, which seems artificial in this case.

   Geometry Expressions will not look for the other parallel line. It uses the position of P as a cue to determine which points to draw. The result is shown in Diagram 2.

   Students can get the other half of the locus by duplicating their work on the other side of the line, or by using the Reflection tool in the Construct Tool Panel.

   The desired result is two lines parallel to the original line, each a distance of $d$ from the original line.

3. Students are encouraged to deduce how to use Geometry Expressions more autonomously as the lessons progress.
If students put the point outside of the parallel lines, it will move one of the lines so that they coincide.

It is especially important to lock variable \( d \) for this part, since that will keep the lines a constant distance apart – they won’t wiggle around.

The locus of points equidistant from two parallel lines is a parallel line exactly half-way between the two lines.

For intersecting lines, students may try to create an additional constraint for their parametric variable. Most likely, they will get a message box telling them that they have too many constraints. Remind them that they can use distance \( d \) as the parametric variable for their locus.

The locus of points equidistant from intersecting lines bisects the angle formed by the lines.

4. Students are expected to create this locus from start to finish. The result is the perpendicular bisector of the line segment.

5. In summary, a locus is a collection of points meeting a given set of criteria. The locus can be described geometrically, for example though distances from fixed points or shapes.

The next lesson in this unit will examine the circle locus more closely. It will create a link between the geometric construction and algebraic equations of conics. Both parametric and implicit forms will be examined.
**Locci**

A Locus is a set of all the points that meet a particular description.

“What are all the points a certain distance from a fixed point?”

“If I tie my dog to a stake, what are the boundaries that he can’t cross?”

1. Find the locus of points equidistant from a fixed point (like all the places a dog can go if it is tied to a stake).

   Open a new drawing with Geometry Expressions
   
   **Draw** point A.  **Constrain** its location to \((x_0, y_0)\).
   
   Click on the point, and then click on
   
   **Draw** point B.  **Constrain** the distance between the points to \(a\).
   
   Hold the shift key and select each point. Then click on
   
   **Draw** a line segment connecting the two points.
   
   **Constrain** the direction of the line segment to \(\theta\).

   Choose \(\theta\) in the Variables Tool Panel. Use the slider bar to see the locations that point B can occupy.

   **Construct** the locus of the points, using \(\theta\) for the parametric variable.

   Select point B

   Click on **Construct locus**

   Choose \(\theta\) for the Parametric Variable.

   How would you describe this locus?

2. What is the locus of points equidistant from a line (perhaps the path a dog creates along the edge of a fence)?

   Open a new drawing with Geometry Expressions.

   **Draw** a line.

   **Constrain** the line’s implicit equation. You can leave the equation at its default setting.

   **Draw** point P so that it is not on the line. **Constrain** its distance from the line to be \(d\).
You can lock variables by selecting them in the Variables Tool Panel, and then clicking the lock icon. Lock variable \(d\).

Drag point \(P\) to see its locus – all of the points that are \(d\) units from the line.

Geometry Expressions will draw the locus, but it wants a point of reference that changes. We’ll ask it to draw each point that is \(d\) units from the line, but at all the different distances from some fixed point. So, we need to draw a fixed point.

**Draw** point \(A\) anywhere.

**Constrain** its coordinates.

Constrain the distance from \(A\) to \(P\) to \(t\). This constraint is called a **parametric variable**.

To see the graph of the locus, 
Select point \(P\).

**Construct** its locus. 
Use \(t\) for its Parametric Variable.

Are there any points that are distance \(d\) away from the line, but that weren’t drawn? Computers aren’t always able to do a complete job, based on your description. Unlock variable \(d\), and try dragging \(P\) again and see what happens. Are there any other locations possible for \(P\)? Sketch them in your diagram.

3. **What is the locus of points equidistant from two lines?**

   There are two cases.

   Case number 1: two parallel lines: **Draw** a line. **Constrain** one of the line’s equations.

   **Draw** a second line, and **constrain** the two lines to be parallel.

   **Draw** point \(P\) between the two lines. **Constrain** the distance from \(P\) to each line to be \(d\).

   Think about how you created a parametric variable for your locus in part 2. Create a parametric variable, and create the locus. What is the locus of points equidistant from two lines, if the lines are parallel?
Case number 2: Two intersecting lines:

**Draw** intersecting lines, and **constrain** both of their equations.
**Draw** point P, constrained to be distance \(d\) from each line.

For parallel lines, the distance from the point to the lines was always the same. For intersecting lines, the distance changes. That means you can use \(d\) as your parametric variable when you create the locus.

Describe the locus in your words.

What is the special name for this locus?

4. **What is the locus of points equidistant from the endpoints of a line segment?**

Use the line segment tool to **draw** a line segment. 
Explore the locus of points equidistant from the endpoints of the line segment.

Describe the locus in your words.

What is the special name for this locus?

5. **Summarize what you have learned in this lesson.**

What is the definition of a locus?

What are some different ways that a locus can be described?
Learning Objectives

Students are now acquainted with the idea of “locus,” and how Geometry Expressions can be used to explore loci. Now, they will look more closely at the circle, defined as the locus of points on a plane equidistant from a fixed point. In particular, they will find the parametric and implicit equations of circles.

The approach is to make sense of the parametric equation in terms of the unit circle, and the implicit equation in terms of the distance formula. Review of these two concepts may be beneficial before the start of the lesson.

Math Objectives

• Learn or re-enforce the general parametric and implicit equations of a circle.
• Connect translations and dilations to the general equations.
• Connect center and radius to the general equations.

Technology Objectives

• Use Geometry Expressions to find equations of curves.
• Use Geometry Expressions to translate and dilate figures.

Math Prerequisites

• Distance formula
• Algebra, including factoring parts of an expression and completing the square.
• Unit Circle
• Pythagorean identity
• Translations and dilations
• Parametric functions vs. implicit equations

Technology Prerequisites

• Knowledge of Geometry Expressions from previous lessons.

Materials

• Computers with Geometry Expressions.
Overview for the Teacher

1. Question one gets the students back into using Geometry Expressions to find the locus of a point. Diagram 1 exhibits typical results.

   A common error is to use \( r \) (the default) as the parametric variable instead of changing it to \( \theta \).

   Students are asked why they didn’t just use the circle tool. Expected results are that the definition of a circle as a locus of points was reinforced.

2. In Diagram 2 you can see the general parametric equation of a circle, where \( d \) is the radius and (h, k) is the center. If students are getting numerical constants, they are calculating the real equation instead of the symbolic equation. Help them to select the symbolic tab and try again.

   After changing the constraints so that the center is (0,0) and the radius is 1, the parametric equation will be

   \[
   \begin{cases}
   X = \cos(\theta) \\
   Y = \sin(\theta)
   \end{cases}
   \]

   If students are not getting this, they may be changing the value of the \( d \) in the Variable Tool Panel, rather than changing the constraint itself to 1.

3. Diagram 3 shows the results of finding the implicit equation for the circle. The simplification process involves some grouping and simple factoring. The end result is the distance formula:

   \[
   r^2 = (X - h)^2 + (Y - k)^2
   \]

   which is frequently written \( r^2 = (x - h)^2 + (y - k)^2 \)
Changing the center to (0,0) and the radius to 1 results in \(-1 + X^2 + Y^2 = 0\) or \(X^2 + Y^2 = 1\).

Exceptional students will begin making connections between the equation of a circle, the distance formula, and the Pythagorean Theorem at this point. Most will likely have been exposed to these connections in a previous course. You may wish to highlight these relationships at this time.

4. The result is the Pythagorean Identity: \(\cos^2 \theta + \sin^2 \theta = 1\)

5. Translating and dilating the unit circle will restore us to the general form. Note that the center will be \((u_0, v_0)\) instead of \((h, k)\). If students find this confusing, they can change the variables used in the vector to \((h, k)\).

Encourage students to change the vector by dragging D, especially if they cannot see the whole picture on the screen.

If your students are not comfortable with completing the square, then the generalization of the implicit equation may be a demonstration.

6. If students get incorrect results for their dilations, check to see that they have chosen the center of the translated circle as their center of dilation. If they choose some other point, their dilated circle will not be concentric with the translated circle, and their equations will be wrong. See diagram 6 for typical correct results.

7. In summary:
   The general parametric form of the equation of a circle is
   \[
   \begin{align*}
   X &= h + r \cos \theta \\
   Y &= h + r \sin \theta
   \end{align*}
   \]
   \((x_0, y_0)\) is the center of the circle and \(d\) is the radius.

   The general implicit formula of a circle is
   \[r^2 = (X - h)^2 + (Y - k)^2\]

   Again, \((h, k)\) is the center of the circle and \(d\) is the radius.
It is important that students have these three concepts, as they will be repeated for the rest of the conic sections:

1. Description of the curve as a locus of points.
2. Parametric equation of the curve.
3. Implicit equation of the curve.

Subsequent lessons will search for further attributes of the curve being studied. For example, the study of ellipses will include the focus, major axis and minor axis.
Loci: Circles

In the last lesson, you were introduced to the idea of a “locus.” A locus is a collection of points that meet a particular description. For example, the locus of points equidistant from a fixed point is a circle.

1. Reconstruct the locus of points equidistant from a fixed point.
   
   Open Geometry Expressions.
   Create point A, and constrain its location to (h, k).
   Create point B, and constrain its distance from point A to be r.
   Create line segment $\overline{AB}$ – be sure to draw starting at point A and ending at point B – and constrain its direction to be $\theta$.
   
   Create the locus of points $d$ units from point A. Use $\theta$ as your parameter.
   
   You could have just used the circle tool if all you wanted to do was to draw a circle. What did you learn about circles by doing it this way?

2. Often, a locus can be described with equations describing its x and y coordinates. This type of algebraic description is called a Parametric Equation. X and Y are described as functions of a third variable, called a parameter.

   Click on the circle you created in step one.
   Click on Calculate Symbolic Parametric equation. Geometry Expressions has just given you the General Parametric Equation of a circle. Write it in the box to the right.
   
   Recall the Unit Circle. How are the coordinates of points on the unit circle defined in terms of cosine and sine?
   
   Change the constraint on the center to (0,0) and change the radius to 1. What does this do to the Parametric equation?
3. You can use CTRL-Z to undo your changes to the constraints on your drawing. Do so repeatedly, until the radius is \( r \) and the center is \((h, k)\). Then use Geometry Expressions to create the implicit equation. Select the circle.

Click on Calculate Implicit equation.

The result looks confusing at first, but you can clean it up:

Add \( d^2 \) to both sides.
Group the \( X \) and \( x_0 \) terms.
Group the \( Y \) and \( y_0 \) terms.
What formula is beginning to emerge?

Factor the \( X \) and \( x_0 \) terms.
Factor the \( Y \) and \( y_0 \) terms.
The result is the General equation of a circle.

Change the constraints as you did in part 2. The result is the implicit equation for the unit circle. Write it here:

4. Using the equations for the unit circle, substitute the two parametric equations into the implicit equation. What relationship do you get?

5. We are going to subject our unit circle to some transformations. First, translate the unit circle.

Create a vector.

Constrain the components of the vector to \( \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \).
Select the circle and its center (use shift and click on each).

Click Construct Translation

Click the vector.

Simplify the Implicit Equation here, to get the General Equation of a circle.

Translation of a circle
What are the coordinates of the center of the translated circle?

Click on the center.

Click on Calculate Symbolic Coordinates.

Find the parametric equation of the translated circle.

Where do the coordinates of the center appear in the equations?

Find the implicit equation of the translated circle.

Use the method of *completing the square* to simplify the implicit equation.

6. Now, dilate the circle that you translated.

   Click on the circle.

   Click on Construct Dilation.

   Click on the center of the translated circle.

   Type $r$ for your scale factor.

   What is the radius of the dilated circle?

   Find the parametric equation of the translated/dilated circle.

   Where does $a$, the length of the radius appear in the equations?

   Find the implicit equation of the translated/dilated circle.

   Simplify with completing the square, as you did in part 5.
7. Summary

The general parametric form of the equation of a circle is:

Where ______ is the center of the circle and ____ is the radius of the circle.

The general implicit form of the equation of a circle is:

Where ______ is the center of the circle and ____ is the radius of the circle.
Learning Objectives

In this lesson, students will generalize their knowledge of the circle to the ellipse. The parametric and implicit equations of an ellipse will be generated, as will two important properties of the ellipse: that the sum of the distances from any point to the foci is equal to the major diameter, and the Pythagorean Property of Ellipses.

Math Objectives

- Understand the geometric definition of an ellipse.
- Generate the parametric and implicit equations for an ellipse.
- Locate the foci, given the equation of an ellipse.
- Discover relationships between the parameters of an ellipse.

Technology Objectives

- Use Geometry Expressions to create a more complex locus of points.
- Find evidence for equivalence using Geometry Expressions.

Math Prerequisites

- Pythagorean Theorem
- Translations
- Parametric functions and implicit equations
- Sine and Cosine

Technology Prerequisites

- Knowledge of Geometry Expressions from previous lessons.

Materials

- Computers, with Geometry Expressions.
Overview for the Teacher

1. Diagram 1 represents a typical result for question 1. If students are getting semicircles with center at point F1, they are creating a locus in terms of \( t \) instead of \( d \). Only half an ellipse is shown because the entire ellipse cannot be generated as a function of \( d \). Points below the foci will have the same distances as points above. An ellipse is like a circle in that it is a curve based on distances from fixed points. It is different in that it “has different radii.”

2. An appropriate result would be 
\[
\begin{align*}
X &= a \cos \theta \\
Y &= b \sin \theta
\end{align*}
\]
Transposing the sine and cosine functions will have no effect on the final curve. Note that the transposed version is also the “sample” parametric function given by the software. Using the transposed version yields the same results, graphically. It just changes the starting place and direction that the ellipse is graphed.

Encourage students to change values for \( a \) and \( b \) to get an ellipse that is not just a circle. A sample is shown in Diagram 2.

3. Some assumptions are made here about the symmetrical nature of an ellipse. You may wish to explore these assumptions with the class at this time.

Desired solutions:
\begin{enumerate}
\item The distance from F1 to P is \( 2a - m \) or \( 2c + m \).
\item The distance from F2 to P is \( m \).
\item The sum is therefore \( 2a \) or \( 2c + 2m \).
\item The width of the ellipse is \( 2a \).
\item \( t = 2a \). Remind students that they need to type \( 2*a \).
\end{enumerate}

4. The constraint from F2 to P is \( 2a - d \). Diagram 3 shows expected results.
5. Desired solutions:
   a. The sum of the distances is 2a.
   b. The distance from F2 to P is equal to a, since the three points form an isosceles triangle.
   c. Using the Pythagorean theorem, $a^2 = b^2 + c^2$, so $c = \sqrt{a^2 - b^2}$
   d. Point P will now appear to be on the ellipse, as shown in Diagram 4.

6. In both instances, the implicit equation will be $Y^2 \cdot a^2 + X^2 \cdot b^2 - a^2 \cdot b^2 = 0$

7. The general parametric function that is generated is
   \[
   \begin{cases}
   X = u_0 + a \cos(T) \\
   Y = v_0 + b \sin(T)
   \end{cases}
   \]

   The implicit formula is
   \[
   Y^2 a^2 + X^2 b^2 - a^2 b^2 - 2Xb^2 u_0 + b^2 u_0^2 - 2Ya^2 v_0 + a^2 v_0^2 = 0
   \]

8. Steps are as follows:
   \[
   \begin{align*}
   Y^2 a^2 + X^2 b^2 - a^2 b^2 - 2Xb^2 u_0 + b^2 u_0^2 - 2Ya^2 v_0 + a^2 v_0^2 &= 0 \\
   (X^2 b^2 - 2Xb^2 u_0 + b^2 u_0^2) + (Y^2 a^2 - 2Ya^2 v_0 + a^2 v_0^2) &= a^2 b^2 \\
   b^2 (X^2 - 2Xu_0 + u_0^2) + a^2 (Y^2 - 2Yv_0 + v_0^2) &= a^2 b^2 \\
   b^2 (X - u_0)^2 + a^2 (Y - v_0)^2 &= a^2 b^2 \\
   \frac{b^2 (X - u_0)^2}{a^2 b^2} + \frac{a^2 (Y - v_0)^2}{a^2 b^2} &= \frac{a^2 b^2}{a^2 b^2} \\
   \frac{(X - u_0)^2}{a^2} + \frac{(Y - v_0)^2}{b^2} &= 1
   \end{align*}
   \]

9. Results as follows:
   a. If $a = b$, then the result is a circle.
   b. If $a < b$, then the ellipse is taller than it is wide. The foci lie on a vertical line rather than a horizontal line. The Pythagorean Property would then be $b^2 = a^2 + c^2$
10. Summary:

The general parametric form of the equation of an ellipse is

\[
\begin{align*}
X &= u_0 + a \cos(T) \\
Y &= v_0 + b \sin(T)
\end{align*}
\]

where \((u_0, v_0)\) is the center of the ellipse and \(a\) and \(b\) are the radii of the ellipse.

The general implicit form of the equation of an ellipse is

\[
\frac{(X - u_0)^2}{a^2} + \frac{(Y - v_0)^2}{b^2} = 1
\]

where \((u_0, v_0)\) is the center of the ellipse.

If \(a > b\), then \(2a\) is the major diameter and \(2b\) is the minor diameter.

If \(b < a\), then \(2b\) is the major diameter and \(2a\) is the minor diameter.

The sum of the distances from any point on the ellipse to the two foci is \(2a\).

The distance from the center of the ellipse to either focus follows the equation \(c^2 = a^2 + b^2\).
The Ellipse

In the last lesson, you found equations for a circle: the locus of points equidistant from a fixed point.

An ellipse is defined as all the points such that the \textbf{sum} of the distance from \textbf{two} fixed points is a constant. Each of the two points is called a \textbf{focus} (the plural of “focus” is “\textbf{foci}”).

1. Create an ellipse using the definition above.
   
   Open a new Geometry Expressions drawing.

   Create two points, and name them F1 and F2. Constrain the coordinates of these points.

   Create a third point, and name it P. Constrain the distance from P to F1 to be \(d\). Constrain the distance from P to F2 to be \(t – d\) (see Diagram 1 to understand why!)

   Lock the value of \(t\) in the Variable Panel, but keep \(d\) unlocked.

   Find the locus of P with parameter \(d\).

   If you drag P around, you can see that the locus forms part of an ellipse.

   To get more of the ellipse:
   - Double click on the curve.
   - Change the Start Value and End Value for \(d\)

   The most you can get is half of the ellipse, because the software assumes you know which side P is on – you drew it there! To get the rest of the ellipse:
   - Draw a line segment from F1 to F2.
   - Select the locus curve.
   - Click on construct reflection \(\textbf{4}\) and then click on the segment.

   Some of the ellipse may still be missing.

   How is an ellipse like a circle? How is it different?
Before continuing, make sure Geometry Express is set to Radians.
In the Edit Menu
Select preferences.
Click on the Math icon at the left.
Under Math, change Angle Mode to radians.

2. Recall that the general parametric equation for a circle with the center at the origin is
\[
\begin{align*}
X &= r \cos(T) \\
Y &= r \sin(T)
\end{align*}
\]
Predict the general parametric equation for an ellipse.

Test your prediction with Geometry Expressions
Open a new Geometry Expressions drawing.
Click on the Function tool in the Draw tool panel.
Type in your prediction to see if you are right. Make additional guesses if you need to.

3. The curve you created in part 2 looks like an ellipse, but is it really an ellipse? If it is, we will be able to find its foci, and the constant sum of distances from the foci.
   a. In Diagram 2, how far is it from focus F1 to point P?
   b. How far is it from focus F2 to point P?
   c. What is the sum of distances from P to the two foci?
   d. What is the horizontal width of the ellipse?
   e. Write an expression for your answer to part c, in terms of distance \( a \).

4. Open a new Geometry Expressions drawing, and create this parametric function:
\[
\begin{align*}
X &= a \cos(T) \\
Y &= b \sin(T)
\end{align*}
\]
Use the Variable Tool Panel to change the values of \( a \) and \( b \) so that \( a \) is greater than \( b \). Lock variable \( a \) and \( b \).
Add three points to your Geometry Expressions drawing.
First, turn on the axes.
Draw F1 and F2 on the x-axis.
Draw P so that it is not on either axis, nor is it on the curve.
Constrain the distance from F1 to P to be \( d \).

What should the constraint from F2 to P be? Review the results from part 3 to help you decide.

5. Refer to Diagram 3 to find the positions of F1 and F2. The triangle shown is an isosceles triangle, with P at the vertex.

a. If P is on the ellipse, what is the sum of the distances from F1 to P and from F2 to P (your solution to 3c)?

b. Given that the triangle is isosceles, what is the distance from F1 to point P?

c. Use the Pythagorean Theorem to write an expression for the distance from the origin to point F2.

d. In your Geometry Expressions drawing, constrain the distance from F1 to the origin to your answer to 5c. Do the same for the distance from F2 to the origin. (You will need to draw a point at the origin first). NOTE: If you want to type a square root in to Geometry Expressions, use \( \sqrt{} \), and if you want to type in an exponent, use \(^\)\). For example, \( \sqrt{a^2 + b^2} \) can be typed: \( \text{sqrt}(a^2 + b^2) \)

e. Does P appear to fall on the ellipse? Drag it around. Does it stay on the ellipse?
6. If the curve in your Geometry Expressions drawing is truly an ellipse, then its implicit equation will match the locus of point P.

Select the curve, and click on the Calculate Implicit Equation icon. Now, hide the curve.

Select the curve.
Right click on the curve.
Choose hide.

Create the locus of point P with respect to \(d\), and calculate its implicit equation.

Restore the original curve by clicking on Show All in the View menu.

Are the two implicit equations the same? How do they differ, if at all?

7. How does a translation affect the equation of an ellipse?

Open a new drawing and use Draw Function to create a new ellipse.

Choose Parametric for the type and enter \(\begin{cases} X = a \cos(T) \\ Y = b \sin(T) \end{cases}\).

Create a vector, and constrain it to its default values, \(\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}\).

Translate the ellipse.
Select the ellipse
Click on Construct Translation
Click on the vector.

Calculate the parametric equation of the new ellipse, and record it in the box.

Calculate the implicit equation of the new ellipse.
8. The General form for the implicit equation of an ellipse is
\[\frac{(x-u_0)^2}{a^2} + \frac{(y-v_0)^2}{b^2} = 1.\] Verify that your implicit equation is equivalent to the general form.

9. In part 6, you found a relationship known as “The Pythagorean Property for Ellipses”

\[a^2 = b^2 + c^2\]

\(a\) is half the horizontal axis of the ellipse
\(b\) is half the vertical axis of the ellipse
\(c\) is the distance from the center of the ellipse to each focus

a. What happens if \(a = b\)?

b. Is it possible for \(a < b\)? How would you need to modify the Pythagorean Property for the ellipse in Diagram 4?

In any ellipse, the larger of 2a and 2b is called the **major axis**. The smaller of 2a and 2b is called the **minor axis**. If \(a > b\), then the foci lie on a horizontal line. If \(a < b\), then the foci lie on a vertical line. If \(a = b\), then the ellipse is actually a circle.
10. Summary:

The general parametric form of the equation of an ellipse is:

where ______ is the center of the ellipse, is the horizontal radius of the ellipse, and ___ and ____ are the radii of the ellipse.

The general implicit form of the equation of an ellipse is:

where ______ is the center of the ellipse.

If _______, then ___ is the major diameter and ___ is the minor diameter.

If _______, then ___ is the major diameter and ___ is the minor diameter.

The sum of the distances from any point on the ellipse to the two foci is _______.

The distance from the center of the ellipse to either focus follows the equation _______.

\[ \text{( , )} \]
Learning Objectives

Students begin by looking at an envelope curve that generates an ellipse. The curve is modified to become a hyperbola, thereby introducing that concept.

Math Objectives

- Extend the idea of a locus of points to a locus of lines.
- Find a definition of the ellipse as an envelope curve.
- Find definitions for hyperbola, first as a locus of lines, and then as an envelope curve.

Technology Objectives

- Use Geometry Expressions to create an envelope curve
- Use Geometry Expressions as an aide to creating geometric proof.

Math Prerequisites

- Two-column triangle congruence proofs

Technology Prerequisites

- Skills with Geometry Expressions, as developed in previous lessons.

Materials

- Computer with Geometry Expressions
Overview for the Teacher

1. The purpose of the first part of the lesson is to become familiar with the envelope curve created by a locus of lines, and the related Geometry Expressions capabilities. It is important that students follow directions closely for part 1. Some will attempt to draw in all of the lines, copying the diagram in the student master. This is not really helpful for the rest of the lesson. An error that is more difficult to diagnose may occur if students have their axis turned on, use the origin for point A, and place points B and C on the axes. When they try to set points M and N proportional along the curve, they will run into an error. That is because the software knows M and N are on the axis, but does not know they are bounded to the segments. The easy fix is probably to turn off the axis and start over.

2. Diagram 2 shows the expected result. If the envelope curve does not show up, it is most likely because the student has chosen the wrong parametric variable for the locus. The parameter should be $\theta$. Another possibility is that the domain for $\theta$ is incorrect. By default, it is assigned values between 0 and 6.28 ($2\pi$).

Generating the locus of point E with parameter $\theta$ will result in the same curve.

3. You may choose whether to do this proof as a class, guide them through it, or let individuals or groups complete the proof, depending on what is right for your class.

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ED \perp BC$</td>
<td>Given</td>
</tr>
<tr>
<td>$\angle EDC \equiv \angle EDB$</td>
<td>Right angles are congruent</td>
</tr>
<tr>
<td>$BD \equiv CD$</td>
<td>Definition of bisector</td>
</tr>
<tr>
<td>$\overline{ED} \equiv \overline{ED}$</td>
<td>Reflexive property of congruence</td>
</tr>
<tr>
<td>$\triangle EBD \equiv \triangle ECD$</td>
<td>Side-Angle-Side</td>
</tr>
<tr>
<td>$\overline{EB} \equiv \overline{EC}$</td>
<td>CPCTC</td>
</tr>
<tr>
<td>$AE + EB = AE + EC$</td>
<td>Additive property of equality</td>
</tr>
</tbody>
</table>
AE + EB = r, the radius of the circle  Given
AE + EC = r, the radius of the circle  Substitution
AE + AC is a constant  Definition of a radius
The locus is an ellipse  Definition of an ellipse

4. Most of the assumptions that Geometry Expression makes when it is asked to “use assumptions” have to do with whether one value is greater than another. In this case, the drawing indicates \( a < r \), and that is the basic assumption that is made. If this is not assumed, Geometry Expressions uses absolute values in place of the assumption, and the symbolics are not as fully simplified.

Watch students to make sure they type the expression in correctly, and with the right subscripts. When testing the lesson with current software, the expression resolved to \( r \). It is possible that, depending on how the drawing is constructed, the expression will not be completely simplified. In that case, help students to finish the simplification process themselves.

Diagram 3 shows typical student results.
5. The reason that students need to delete the calculated distances first is that Geometry Expressions is assuming that $r > a$, and we are about to change that.

It is best if point E appears on the branch of the hyperbola that is near point C. Students can toggle the value of $\theta$ to make this happen. Diagram 4 includes the expression for the sum of the two lengths.

The successful student will try calculating the difference of the two lengths. Some will reverse the order, resulting in $-r$ instead of $r$. This corresponds to E being on the other branch of the hyperbola.

6. The result for part 6 is a circle with center $(0,0)$ and radius $r/2$. While this is not a dramatic or important result, it does satisfy the need mathematicians have for “completeness.”

The next lesson in this series will develop parametric and implicit functions for hyperbolas. The implicit function is quite similar to the ellipse, but the parametric formulas are a little more surprising.
Conics and Envelope Curves

If you’ve ever done a “string art” project, then you are familiar with the concept of an envelope curve. You can see a curve begin to form where two adjacent lines intersect. The envelope curve is the locus of these intersection points.

1. Create the locus of points that are intersections of the lines shown in Diagram 1. Note that Geometry Expressions will not draw all of the lines. It will only draw the curve formed by all the lines.

   Open Geometry Expressions.
   Create line segment \( \overline{AB} \), and place point M on the line segment.

   Select the point and the segment, and click on the Constrain Point proportional along curve tool. Leave the constraint to be \( t \).

   Choose \( t \) from Variable tool panel, and animate it. Watch the motion of the point along the segment.

   Create a second line segment, \( \overline{CA} \). Make sure to start the line segment with point C and to end it with point A, or your point will move in the wrong direction.

   Place point N on segment \( \overline{CA} \). Select the point and the segment and click on Constrain point proportional along curve. Change the constraint to be \( t \).

   Press play again and watch the points. Make sure that one point is moving towards A while the other is moving away from A.

   Create segment \( \overline{MN} \). Play the animation again.

   Now, select \( \overline{MN} \) and click on the Construct Locus tool.

   NOTE: the resulting curve is the “envelope” of the lines. The actual locus of lines is the “string” in the string art.

   Animate your locus once more. Notice how the line just slides along the curve.
2. Follow these steps to create a second envelope curve.

Open a new drawing.
Create a circle with center A at the origin, and point B NOT on the x or y axis. Constrain the radius to be \( r \).
Create radius \( \overline{AB} \). Constrain the angle between the radius and the x-axis to be \( \theta \).

Create point C not on the x-axis, in the interior of the circle. Constrain the coordinates of point C to be \((a, 0)\).
Create segment \( \overline{BC} \).

Construct the perpendicular bisector of \( \overline{BC} \) by selecting the segment and then clicking on the Construct Perpendicular bisector tool. You may wish to change the color of this line to help you see it better. Select the line, right click, and choose properties from the menu.

Animate variable \( \theta \). Try to visualize the envelope curve created by the perpendicular bisector. It might be helpful to construct point E at the intersection of the radius and the perpendicular bisector.

Now, select the perpendicular bisector and click the locus tool. Use \( \theta \) as the parametric variable. What shape is created? Are you sure?

3. How can we prove that the shape is an ellipse?
Refer to diagram 2.

To show that the shape is an ellipse, prove that \( AE + CE \) is equal to a constant.

Given: \( \overline{ED} \) is the perpendicular bisector of \( \overline{BC} \)
4. Go back to the Geometry Expressions, and we’ll try to confirm our proof.
If you have not already done so, construct the intersection of the radius and the
perpendicular bisector. Select \( \overline{AB} \) and the perpendicular bisector, and click on the
Construct Intersection tool. Label the point E.

Before continuing, turn on “use assumptions.” We are going to allow Geometry
Expressions to assume that length \( r \) is greater than length \( a \). It will make some calculations
simpler, but we’ll have to be careful later.

Click the Edit menu and select Preferences.
Click on Math
Under the heading Output, set Use Assumptions to True.

Select A and E and calculate symbolic the distance between the points.
Select E and C and calculate symbolic the distance between those points.

Both symbolic representations are quite complex. But what, if the shape is an ellipse, what
will be true if we add them together?

Click on the Draw Expressions tool and then click on the drawing.
Type \( z[0]+z[1] \) and hit Enter.

The result is the first line shown. All of the lines after that are the assumptions being used.
What is the result? Does it help prove that the shape is an ellipse? Why?

5. What happens if point C is outside the circle?

Slowly drag point C until it is outside the circle.
If you lose track of point E, change the value of \( \theta \) with the slider bar in the Variables panel until it appears.

The locus curve that you see now is called a hyperbola.

Notice that the calculations for AE, CE, and the
sum are still using the same assumptions.
Select each calculation and:
Right click, and select Properties
Change Use Assumptions to False.

Then
Select each calculation and:
Right click, and select Properties
Change Use Assumptions to True.
Now Geometry Expressions will assume that \( a > r \) which agrees with point C being outside the circle.

Is the sum of the distances still a constant, or is the expression more complex?

Try changing the plus sign to something else. Double click on the expression \( z_2 \) to edit it. When the result is a constant, you can complete the definition of a hyperbola:

A hyperbola is the locus of points such that the _________________ of the distances from the foci is a constant.

6. What happens if point C is the center of the circle?

   Constrain the coordinates of point C to be \((0,0)\).

   Calculate AE as you did in parts 4 and 5 (CE and AE are the same segment now).

   How would you describe this locus of points?
Lesson 5: Inside Out Ellipses

Learning Objectives

What, exactly, is a hyperbola? What are its characteristics and defining equations? How does it relate to the unit circle, and to ellipses? These are the primary objectives of this lesson.

Math Objectives

• Find out what a hyperbola is, and how it is related to an ellipses.
• Learn about characteristics of a hyperbola, particularly its foci and its asymptotes.
• Discover General and parametric equations for hyperbolas.

Technology Objectives

• Use Geometry Expressions to draw hyperbolas, using the geometric definition, the parametric equation, and the implicit equation.
• Use Geometry Expressions to connect these three disparate representations of a hyperbola.

Math Prerequisites

• Definitions of secant and tangent, both in terms of right triangles and in terms of sine and cosine.
• Pythagorean identity.
• Solving simple algebra equations.
• Knowledge of circles and ellipses, as provided in earlier lessons of this unit.

Materials

• Computer with Geometry Expressions.
Overview for the Teacher

The student title for this lesson is “Inside-out Ellipses.” This presages the next lesson, which is on eccentricity.

The main goal for this lesson is to get a feel for hyperbolas and how they relate to ellipses and circles. Ultimately, the students will have a geometric definition of the hyperbola as a locus of points, and algebraic definitions in parametric form and in implicit form.

1. In part 1, students use the parametric and implicit equations of the unit circle to reproduce the Pythagorean Identity, \( \sin^2(\theta) + \cos^2(\theta) = 1 \). Diagram 1 shows what students should be seeing.

2. Here is the sequence for question 2.

\[
\cos^2(T) + \sin^2(T) = 1 \\
\cos^2(T) = 1 - \sin^2(T) \\
\cos^2(T) = \frac{1}{\cos^2(T)} - \sin^2(T) \\
1 = \sec^2(T) - \tan^2(T)
\]

Thus, \( x \) will be \( \sec(T) \) and \( y \) will be \( \tan(T) \)

Diagram 2 shows the hyperbola and its parametric function.

3. The asymptotes are going to be used to find the foci of this hyperbola. They are also interesting in their own right.

Some students will still have the hyperbola on the same drawing as the unit circle, and others will not. Makes sure they have both the hyperbola and the unit circle when they proceed to part 4.
4. A small leap of faith is needed in this step, since no proof is offered that the foci can be constructed in this manner. Rather, the foci are used to create a hyperbola that results in a match with this one. Diagram 4 shows student results before all the lines are hidden.

5. The answer to “What is AC – CD” is 2, so the distances from the foci to the point are \(d\) and \(d + 2\). The point falls on the existing hyperbola. The equation for the locus is \(1 - X^2 + Y^2 = 0\). Some students may need help recognizing this as \(X^2 - Y^2 = 1\), our starting point.

6. Changing \(a\) stretches the hyperbola horizontally, but since this also moves the foci towards or away from the origin, the hyperbola changes vertically too. Changing \(b\) only stretches the hyperbola vertically since the foci are not affected. The ellipse is tangent to the hyperbola at its vertices.

The line \(y = \frac{b}{a}x\) is one of the asymptotes for the hyperbola. The other asymptote is \(y = -\frac{b}{a}x\). You may wish to point out that \(\frac{b}{a}\) and \(-\frac{b}{a}\) have opposite signs, but are not reciprocals. Therefore, the asymptotes are not perpendicular. See Diagram 5 for results.

7. The parametric function for a hyperbola translated by \(\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}\) is shown in Diagram 6.
8. Inductive reasoning is used at this point because it reinforces the roles of the different parameters. You may wish to derive the general implicit equation in class as well.

Start with the trig identity derived earlier: \[ \sec^2 T - \tan^2 T = 1. \]
Solve each of the parametric equations for \( \sec T \) and \( \tan T \), respectively.
Substitute the results into the identity.
Make a few algebraic adjustments, and you are finished.

The general equation for a hyperbola
is
\[
\frac{(X-u_0)^2}{a^2} - \frac{(Y-v_0)^2}{b^2} = 1
\]

- \( a \) stretches the hyperbola horizontally
- \( b \) stretches the hyperbola vertically
- \( u_0 \) is the horizontal translation
- \( v_0 \) is the vertical translation

9. Students should graph the implicit function
\[ y = \sqrt{1+x^2} \].
They will see the top half of a hyperbola that opens up, as seen in Diagram 7. The other half of the hyperbola is
produced by \[ y = -\sqrt{1+x^2} \]. A drawing of the parametric form is shown in Diagram 8. Both parts of the hyperbola are drawn because no plus-or-minus is required in the parametric form.
10. Summary:

A hyperbola is the locus of points for which the difference of the distances from the foci is a constant.

If a hyperbola opens to the left and to the right:

Its parametric equation is:
\[
\begin{align*}
X &= a \sec(T) + u_0 \\
Y &= b \tan(T) + v_0
\end{align*}
\]

Its implicit equation is:
\[
\frac{(X-u_0)^2}{a^2} - \frac{(Y-v_0)^2}{b^2} = 1
\]

If a hyperbola opens up and down:

Its parametric equation is:
\[
\begin{align*}
X &= a \tan(T) + u_0 \\
Y &= b \sec(T) + v_0
\end{align*}
\]

Its implicit equation is:
\[
\frac{(Y-u_0)^2}{a^2} - \frac{(X-v_0)^2}{b^2} = 1
\]
Inside-out Ellipses

In the last lesson, we turned an ellipse inside-out and ended up with a hyperbola. In this lesson, we are going to find general equations for hyperbolas.

Make sure Geometry Expressions is in Radians before you proceed.

1. Recall the parametric and implicit formulas for the Unit Circle.
   Open Geometry Expressions.
   Create a circle with center at the origin, and with radius 1.
   Calculate the symbolic parametric equation for the circle.
   Calculate the symbolic implicit equation for the circle.

   Now, substitute the parametric equations into the implicit equation. What identity did you create?

2. What do you suppose the graph of $x^2 - y^2 = 1$ would look like?
   First, we need to find an identity in the form $<\text{trig function}>^2 - <\text{trig function}>^2 = 1$.

   Start with the identity you recalled in part one.
   Subtract $\sin^2(T)$ from both sides.
   Divide each term by $\cos^2(T)$

   Compare your result to the pattern: $x^2 - y^2 = 1$
   What is the trig expression that is standing in for $x$?

   What is the trig expression that is standing in for $y$?

   **Note: for the next step, if you need to enter**
   $\sec(x)$, use $1/\cos(x)$
   $\csc(x)$, use $1/\sin(x)$
   $\cot(x)$, use $1/\tan(x)$

   Draw a function in Geometry Expressions
   Select Parametric type
   Type your trig expression in for $x$ and $y$.
   Your result should appear to be a hyperbola. But is it?
The asymptotes of the “hyperbola” that you just created are the lines $y = x$ and $y = -x$.
Add them to your drawing

Create a line.
Select the line, and click on Constrain Implicit Equation. Type in $y = x$
Repeat for $y = -x$

3. Remember that a hyperbola is the locus of points such that the difference of the distances from the foci is a constant. To verify whether our “hyperbola” is actually a hyperbola, first we need to find its foci.

If you don’t have the unit circle on the same drawing as your “hyperbola,” construct it now.

Construct the points of intersection for the circle and each asymptote.
Select the circle and Select the line
Click on Construct Intersection

Construct perpendiculars at each intersection point.
Select the line and the point.
Click on Construct Perpendicular.

The points where the perpendiculars intersect the x-axis are the foci of the hyperbola.
Construct points at those intersections
Select the perpendicular and the x-axis.
Click on Construct Intersection.

Clean up your drawing by hiding all of the straight lines.
Click on the line
Right click
Select Hide from the pop-up menu.
4. We’ve found the foci, so now all we need to do is find the constant difference. Use diagram 1 to help you find it.

A and D are foci. Consider point C on the hyperbola. How far is it from C to D (length CD)? How far is it from A to C (length AC)? What is AC – CD?

On your Geometry Expressions drawing, draw a point that is not on the hyperbola or on the circle. Constrain the distance from one of the foci to your point to be $d$.

What did you get for AC – CD in Diagram 1? Add that value to $d$ and use it to constrain the distance from the other focus to your point.

What happens to your point?

Hide the hyperbola.

Create the locus for your point, using $d$ as a parameter. Adjust the start and end values if you think its necessary (you can change them later by double-clicking on the locus curve).

Find the symbolic implicit equation of the hyperbola, and record it here:

5. So far we have found the parametric and implicit equations of the unit hyperbola. But what are the general equations?

Start with a new Geometry Expressions drawing. Create a unit hyperbola using parametric equations.

Now, modify the parametric equations like this:

$$
\begin{align*}
X &= a \cdot \frac{1}{\cos(T)} \\
Y &= b \cdot \tan(T)
\end{align*}
$$
Change the value of \( a \) with the slider bar on the Variable Tool Panel. What happens?

Change the value of \( b \) with the slider bar on the Variable Tool Panel. What happens?

Create the ellipse with this parametric equation:
\[
\begin{align*}
X &= a \cos(T) \\
Y &= b \sin(T)
\end{align*}
\]
How are the hyperbola and the ellipse related?

Create a line and constrain its implicit equation to be \( y = \frac{b}{a} x \). This line is one of the asymptotes of the hyperbola. What do you think is the equation of the other asymptote?

Test your conjecture on your Geometry Expressions drawing until you get it right.

6. How does a translation affect the equation of the hyperbola?

Delete or hide the ellipse and the asymptotes from your drawing.

To translate the hyperbola:
- Create a vector and constrain it to the default settings.
- Select the hyperbola.
- Click on the Construct Translation tool.
- Click on the vector to translate the hyperbola.

Select the new hyperbola
- Click on Calculate Symbolic Parametric equation and record the results.
7. Use inductive reasoning to surmise the general implicit equation of a hyperbola, and complete the table

<table>
<thead>
<tr>
<th>Unit Circle</th>
<th>Unit Hyperbola</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X = \cos(T) )</td>
<td>( X = \sec(T) )</td>
</tr>
<tr>
<td>( Y = \sin(T) )</td>
<td>( Y = \tan(T) )</td>
</tr>
<tr>
<td>( x^2 + y^2 = 1 )</td>
<td>( x^2 - y^2 = 1 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>General Ellipse</th>
<th>General Hyperbola</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X = a \cos(T) + u_0 )</td>
<td>( X = a \sec(T) + u_0 )</td>
</tr>
<tr>
<td>( Y = b \sin(T) + v_0 )</td>
<td>( Y = b \tan(T) + v_0 )</td>
</tr>
<tr>
<td>( \frac{(X-u_0)^2}{a^2} + \frac{(Y-v_0)^2}{b^2} = 1 )</td>
<td></td>
</tr>
</tbody>
</table>

In the general equation of a hyperbola:

- \( a \) results in ________________________________________________________________
- \( b \) results in ________________________________________________________________
- \( u_0 \) results in ________________________________________________________________
- \( v_0 \) results in ________________________________________________________________
8. Subtraction is not commutative, so what is the graph of \( y^2 - x^2 = 1 \)?

Solve the equation for \( y \).
Open a new Geometry Express file and graph the result.
Remember to type \texttt{sqrt} for square root, and to put the radicand in parenthesis.

Is the entire graph shown, or is some missing?

How is this hyperbola different from the others?

We got this graph by exchanging \( x \) and \( y \). Create a hyperbola with parametric equations, but exchange the \( x \) and \( y \). What is the result?

9. Summary

A hyperbola is the locus of points _______________________________________________________

_____________________________________________________________________________________

If a hyperbola opens to the left and to the right:

Its parametric equation is:

Its implicit equation is:

If a hyperbola opens up and down:

Its parametric equation is:

Its implicit equation is:

The asymptotes of the hyperbola have slopes _____________ and ______________.

\[ y^2 - x^2 = 1 \]
Learning Objectives
So far, the unit has neglected parabolas. In this lesson, we begin with the definition of a parabola as the locus of points equidistant from a point (the focus) and a line (the directrix). Next, the definition is changed to include eccentricity, the ratio of the two distances, resulting in new definitions for ellipses and hyperbolas.

Math Objectives
- Define the parabola as the locus of points equidistant from a point (the focus) and a line (the directrix).
- Introduce the concept of eccentricity.
- Define ellipses and hyperbolas in terms of focus, directrix, and eccentricity.
- Find the limit of the locus as $e$ goes to 0, and compare with the limit of the equation as $e$ goes to 0.

Technology Objectives
- Use Geometry Expressions to create a locus of points and derive its equation.

Math Prerequisites
- Knowledge of conic sections presented earlier in this unit.
- Ratios
- Equations of parabolas
- Factoring by grouping

Technology Prerequisites
- Geometry Expressions, as learned in this unit.

Materials
- Computers with Geometry Expressions.
Overview for the Teacher

1. The geometric definition of a parabola is the locus of points equidistant from a line (the directrix) and a point (the focus). Diagram 1 shows sample student work.

2. Some students will try to generate the symbolic implicit equation for the curve before nailing down some of the constraints. Make sure they follow the steps to lead them to a simpler result.

Diagram 2 shows the parabola with the x-axis for the directrix, and the focus on the y-axis.

3. In part 3, the drawing will be essentially the same, but the equation will change to

\[ X^2 - 2Yc + c^2 + Y^2 \left(1 - e^2\right) = 0 \]

The process of substituting 1 for e will help students see the role that e plays in the equation as well as in the diagram.

Caution: Some students may think that that e represents Euler’s constant. It does not.

Diagram 3 shows a curve with eccentricity less than 1: an ellipse.

The \(X^2\) and \(Y^2\) coefficients will have the same sign in this case.
Diagram 4 shows a curve with eccentricity greater than 1, a hyperbola. Some students may think they have returned to a parabola—help them see that the $Y^2$ term in the equation contradicts this.

For a hyperbola, the $X^2$ and $Y^2$ terms have opposite signs. This corresponds with the general equations they have already discovered.

4. The focus-directrix model breaks down for $e = 0$; but the equation generated resembles a circle, but it is a circle with radius 0.

5. Conclusion:

If $e > 1$, then $X^2$ and $Y^2$ coefficients have opposite signs and the curve is a hyperbola.

If $e = 1$, then $Y^2$ term is eliminated and the curve is a parabola.

If $0 < e < 1$, then $X^2$ and $Y^2$ coefficients have the same sign and the curve is an ellipse.

As $e$ approaches 0, the $X^2$ and $Y^2$ coefficients become the same the curve becomes closer and closer to a circle.
Eccentricity

The final locus that we will look at in this unit is the locus of points equidistant from a fixed line and a fixed point. From there, we’ll see what happens if the ratio of those distances is something other than one.

1. What is the locus of points equidistant from a fixed line and a fixed point?

   Open a new Geometry Expressions file.
   Create a line, and constrain its implicit equation to the default.
   Select the line
   Click on Constrain Implicit Function
   This line is called “the directrix.”

   Create a point that isn’t on the line, and name it F.
   Constrain its coordinates to the default.
   This point is called “the focus.”

   Create another point that isn’t on the line. Label it P
   Constrain the distance from P to F to be \( d \).
   Constrain the distance from P to the directrix to also be \( d \).

   Create the locus of point \( P \) with \( d \) as the parameter. Set the start value to 0 and the end value to 20.

   To see the whole curve
   Construct a perpendicular from F to the directrix.
   Select the locus.
   Reflect it across the perpendicular.

   What type of curve do you think this is?

2. Calculate the Symbolic implicit equation for the locus

   To make this simpler, we are first going to rotate and translate the locus curve. Remember that rotation and translation preserve the size and shape of a figure.

   Constrain the implicit equation of the directrix to \( y = 0 \).
   Constrain the coordinates of \( F \) to \((0, c)\).
Solve the equation for $y$

Do you recognize the form of the equation? What type of curve is this?

Does your answer agree with your guess in part 1?

3. What if the two distances you used to create the locus were different?

Change the distance from $P$ to $F$ to be $d*e$. The value $e$, which is the ratio of the two distances, is called “the eccentricity.”

Set the value of $e$ to be 1, and you have the same curve as in part 2.

Write down the implicit equation for the locus.

Substitute $e = 1$.

What is the result?

If $e = 1$, then the curve is __________________________________________________________
Use the Variable Tool panel to change the eccentricity to a value that is less than 1.

Remember that the general implicit form for an ellipse is
\[
\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1
\]
and the general implicit form for a hyperbola is
\[
\frac{(x-x_0)^2}{a^2} - \frac{(y-y_0)^2}{b^2} = 1
\]

Is the \(Y^2\) coefficient positive or negative?

Based on this, what type of curve do you think this is? Why?

If \(e < 1\), then the curve is _______________________________________________

Now, set the value of \(e\) to be greater than 1.

Is the \(Y^2\) coefficient positive or negative?

What type of curve do you think this is?

If \(e > 0\), then the curve is _______________________________________________
4. What curve do you expect if the eccentricity is equal to 0?

Use the slider bar on the Variable Tool Panel to move $e$ towards 0. If your curve disappears, make values for $d$ and $c$ closer together.

As $e$ gets closer to 0, what happens to PF?

As $e$ gets closer to 0, what happens to the shape of the curve?

Substitute 0 in for $e$ in the equation. $X^2 - 2Yc + c^2 + Y^2 \left(1 - e^2\right) = 0$

Put the equation into $(x - u_0)^2 + (y - v_0)^2 = r^2$ form.

What is the center of the circle?

What is the radius of the circle?

As the eccentricity gets closer to circle, the radius also gets closer to zero. Therefore the eccentricity model cannot be used to define a circle. It only defined ellipses, hyperbolas, and parabolas.
5. Conclusion

The fixed line is called the *directrix*.

The fixed point is called the *focus*.

The curves we studied are the locus of points for which the distance from the directrix and the distance from the focus had the same ratio. That ratio is called the *eccentricity, e*.

The curves are all of the form $X^2 - 2Yc + c^2 + Y^2 (1 - e^2) = 0$

If $e > 1$, then $X^2$ and $Y^2$ coefficients have opposite signs and the curve is a __________________________.

If $e = 1$, then the $Y^2$ term is eliminated and the curve is a __________________________.

If $0 < e < 1$, then $X^2$ and $Y^2$ coefficients have the same sign and the curve is an __________________________.

As $e$ approaches 0, the $X^2$ and $Y^2$ coefficients become the same the curve becomes closer and closer to ________________