The Symbolic Geometry System

This document is intended to grow with the symbolic geometry system, and provide a set of examples which we can use as a starting point for testing, documentation and demonstration.
Contents

THE SYMBOLIC GEOMETRY SYSTEM............................................................................................................................ 1

A First Example.................................................................................................................................................................. 6

Triangles........................................................................................................................................................................... 10
Example 1: Altitude of a right angled triangle ................................................................................................................ 10
Example 2: Altitude of the symmetrical altitude of an isosceles triangle ................................................................. 11
Example 3: Altitude of the non-symmetrical altitude of an isosceles triangle ..................................................... 12
Example 4: Altitude of a triangle defined by side lengths ......................................................................................... 13
Example 5: Altitude of an Isosceles Triangle .................................................................................................................. 14
Example 6: Triangle areas ............................................................................................................................................. 15
Example 7: Intersection of the Medians of a Right angled Triangle .................................................................. 16
Example 8: Centroid of a general triangle ..................................................................................................................... 19
Example 9: Angle of a Median ...................................................................................................................................... 20
Example 10: Orthocenter ............................................................................................................................................... 23
Example 11: Orthocenter Coordinates ........................................................................................................................ 24
Example 12: Circumcircle radius ................................................................................................................................. 25
Example 13: Circumcircle radius in terms of vectors ............................................................................................... 27
Example 14: Incircle Radius .......................................................................................................................................... 28
Example 15: Vector joining the incircle with the vertex of a triangle defined by vectors ................................... 29
Example 16: Incircle Center in Barycentric Coordinates ............................................................................................ 30
Example 17: How does the point of contact with the incircle split a line .............................................................. 31
Example 18: Length of line joining a vertex with the point of contact between the incircle and the opposite side ..32
Example 19: A Pythagoras-Like Diagram .................................................................................................................... 33
Example 20: A Penequilateral Triangle ........................................................................................................................ 34
Example 21: Another Penequilateral Triangle ........................................................................................................... 38
Example 22: Folding a right angled triangle ............................................................................................................... 40
Example 23: Interior point of an Equilateral Triangle ................................................................................................. 41
Example 24: Right angled triangle in semi-circle ..................................................................................................... 43
Example 25: Line Joining the apex to base of an Isosceles triangle ........................................................................ 44
Example 26: Area of Triangle defined by 2 vectors ................................................................................................. 45
Example 27: Similar Triangle ....................................................................................................................................... 46
Example 28: Areas of triangles bounded by Cevians ............................................................................................... 47
Example 29: Decomposing a vector into components parallel and perpendicular to a second vector ............ 48

Quadrilaterals................................................................................................................................................................... 49
Example 30: Diagonals of a rhombus ............................................................................................................................ 50
Example 31: Diagonals of a parallelogram ................................................................................................................... 51
Example 32: Diagonals of a Kite ................................................................................................................................. 52
Example 33: Cyclic Quadrilateral .................................................................................................................................. 55
Example 34: A Right Trapezoid ................................................................................................................................... 56
Example 35: Areas of Quadrilaterals ........................................................................................................................... 57
Example 36: Areas of triangles in a trapezoid ............................................................................................................... 60
Example 37: Diameter of the circumcircle .................................................................................................................. 62
Example 38: Quadrilateral with perpendicular diagonals and one right angle .................................................... 63
Example 39: Finding the diameter of an arc given the perpendicular offset from the chord ................................ 64
Example 40: Napoleon’s Theorem .............................................................................................................................. 65
Example 41: An Isosceles Triangle Theorem ........................................................................................................... 66
Example 42: A Quadrilateral with Perpendicular Diagonals .................................................................................... 67
Example 43: Intersection of Common Tangent with Axis of Symmetry of Two Circles ........................................ 68
Example 44: Slope of the Angle Bisector ........................................................................................................... 70
Example 45: Location of intersection of common tangents ................................................................................. 71
Example 46: Altitude of Cyclic Trapezium defined by common tangents of 2 circles ........................................... 73
Example 47: Areas Cyclic Trapezia defined by common tangents of 2 circles .................................................. 74
Example 48: Triangle formed by the intersection of the interior common tangents of three circles ...................... 75
Example 49: Distance between sides of a rhombus ................................................................................................. 76
Example 50: Angles of Specific Triangles ........................................................................................................... 77
Example 51: Sides of Specific Triangles ................................................................................................................ 79
Example 52: Angles in the general triangle ........................................................................................................... 81
Example 53: Some implied right angles ................................................................................................................ 85
Example 54: Triangle defined by 2 angles and a side ............................................................................................. 90
Example 55: Triangle defined by two sides and the included angle ..................................................................... 91
Example 56: Triangle defined by 2 sides and the non-included angle ................................................................. 92
Example 57: Incenter ........................................................................................................................................... 93
Example 58: Quadrilateral Formed by Joining the Midpoints of the sides of a Quadrilateral ......................... 94
Example 59: Some measurements on the Pythagoras Diagram ......................................................................... 95
Example 60: An unexpected triangle from a Pythagoras-like diagram ............................................................... 96
Example 61: A Theorem on Quadrilaterals .......................................................................................................... 97
Example 62: Rectangle Circumscribing an Equilateral Triangle ....................................................................... 100

Polygons ........................................................................................................................................................... 101
Example 63: Regular Pentagons and more ........................................................................................................... 102
Example 64: A Regular Pentagon Construction ................................................................................................. 106
Example 65: Area of a Hexagon bounded by Triangle side trisectors ............................................................... 107

Angles and Circles .......................................................................................................................................... 108
Example 66: Rounding a Corner ......................................................................................................................... 109
Example 67: Cyclic Quadrilaterals ....................................................................................................................... 110
Example 68: Angles subtended by a chord ......................................................................................................... 111
Example 69: Angle subtended by a point outside the circle ................................................................................. 113
Example 70: Angle at intersection of two circles ................................................................................................. 114
Example 71: Angle subtended by two tangents .................................................................................................. 115

Circles .............................................................................................................................................................. 116
Example 72: Distance between the incenter and circumcenter ............................................................................. 117
Example 73: Radius of Circle through 2 vertices of a triangle tangent to one side ............................................. 119
Example 74: Length of the common tangent to two tangential circles ............................................................. 120
Example 75: Tangents to the Radical Axis of a Pair of Circles ........................................................................... 122
Example 76: Inverting a segment in a circle ......................................................................................................... 123
Example 77: Inverting a circle ............................................................................................................................ 124
Example 78: The nine point circle ....................................................................................................................... 125
Example 79: Excircles ........................................................................................................................................ 126
Example 80: Circle tangential to two sides of an equilateral triangle and the circle centered at their intersection through the other vertices ......................................................................................... 128
Example 81: Circle tangent to 3 sides of a circular segment ............................................................................... 129
Example 82: Various Circles in an Equilateral Triangle ....................................................................................... 132
Example 83: Circle tangential to base of equilateral triangle and constructing circles .................................... 134
Example 84: Radius of the circle through two vertices of a triangle and tangent to one side .......................... 135
Example 85: Circle Tangent to 3 Circles with the same radii ............................................................................. 136
Example 86: Circles Tangent to 3 Circles of Different Radii ............................................................................. 138
Example 87: Circle Tangential to 3 circles same radius ..................................................................................... 139
Example 88: Two circles inside a circle twice the radius, then a third ............................................................... 140
Example 89: A theorem old in Pappus’ time ..................................................................................................... 145
Example 90: Yet Another Family of Circles ........................................................................................................ 148
Example 91: Archimedes Twins ......................................................................................................................... 152
USING SYMBOLIC GEOMETRY

Example 92: Circles tangential to two Touching circles and their common tangent ........................................ 153
Example 93: The triangle joining the points of tangency of 3 circles................................................................. 155
Example 94: The triangle tangential to 3 tangential circles.................................................................................. 156
Example 95: Center and Radius of a Circle Given Equation .................................................................................. 157
Example 96: A limit point ........................................................................................................................................ 158
Example 97: Buehler’s Circle ............................................................................................................................... 160
Example 98: Circle to two circles on orthogonal radii of a third ...................................................................... 162

Equations of Lines and Circles ......................................................................................................................... 163
Example 99: Intersection of Two Lines .............................................................................................................. 164
Example 100: Equation of Line Through Two Points ...................................................................................... 165
Example 101: Intersection of a Line with a Circle .............................................................................................. 166
Example 102: Equation of the Line Joining the Intersection of Two Circles...................................................... 167
Example 103: Projecting a Point onto a Line ........................................................................................................ 168
Example 104: Parabola of the Triangle formed by 3 Lines Whose Equations are Known......................... 169
Example 105: Equation of the Altitude of a Triangle Defined by lines with Given Equations .................... 170
Example 106: Equation of the Line through a Given Point at 45 degrees to a Line of Given Equation ...... 171
Example 107: Equation of Tangent to Circle Radius r Centered at the Origin, through given point .......... 172
Example 108: Intersection of two tangents to the curve y=x^2 ...................................................................... 173

Transforms ........................................................................................................................................................... 174
Example 109: Composition of reflections in parallel lines ............................................................................. 175
Example 110: Combining Reflections .............................................................................................................. 176
Example 111: Parabolic Mirror .......................................................................................................................... 178
Example 112: Billiards ....................................................................................................................................... 179
Example 113: Areas and Dilatation ................................................................................................................... 180
Example 114: Translating a circle .................................................................................................................... 181

Some Constructions ........................................................................................................................................... 182
Example 115: Equation of the Perpendicular Bisector of 2 Points ................................................................. 183
Example 116: Length of the Angle Bisector of a triangle ............................................................................... 184

Some Mechanisms .............................................................................................................................................. 185
Example 117: A Crank Piston Mechanism ........................................................................................................ 186
Example 118: A Quick Return Mechanism ...................................................................................................... 187
Example 119: Paucellier’s Linkage .................................................................................................................... 188
Example 120: Harborth Graph .......................................................................................................................... 189

Loci ...................................................................................................................................................................... 191
Example 121: Circle of Apollonius ...................................................................................................................... 192
Example 122: A Circle inside a Circle ................................................................................................................ 193
Example 123: Another Circle in a Circle ........................................................................................................... 194
Example 124: Ellipse as a locus ........................................................................................................................ 196
Example 125: Archimedes Trammel ................................................................................................................ 197
Example 126: An Alternative Ellipse Construction ......................................................................................... 198
Example 127: Another ellipse .......................................................................................................................... 199
Example 128: “Bent Straw” Ellipse Construction ............................................................................................. 200
Example 129: Similar construction for a Hyperbola ....................................................................................... 201
Example 130: Parabola as locus of points equidistant between a point and a line ...................................... 202
Example 131: Squeezing a circle between two circles ....................................................................................... 203
Example 132: Rosace a Quatre Branches ........................................................................................................ 204
Example 133: Lemniscate .................................................................................................................................. 205
Example 134: Pascal’s Limaçon ........................................................................................................................ 206
Example 135: Kulp Quartic .................................................................................................................................. 207
Example 136: The Witch of Agnesi ................................................................................................................ 208
<table>
<thead>
<tr>
<th>Example</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>137</td>
<td>Newton’s Strophoid</td>
<td>209</td>
</tr>
<tr>
<td>138</td>
<td>MacLaurin’s Trisectrix and other Such Like</td>
<td>210</td>
</tr>
<tr>
<td>139</td>
<td>Trisectrice de Delange</td>
<td>212</td>
</tr>
<tr>
<td>140</td>
<td>“Foglie del Suardi”</td>
<td>213</td>
</tr>
<tr>
<td>141</td>
<td>A Construction of Diocletian</td>
<td>214</td>
</tr>
<tr>
<td>142</td>
<td>Kappa Curve</td>
<td>215</td>
</tr>
<tr>
<td>143</td>
<td>Kepler’s Egg</td>
<td>216</td>
</tr>
<tr>
<td>144</td>
<td>Cruciform Curve</td>
<td>217</td>
</tr>
<tr>
<td>145</td>
<td>Locus of centers of common tangents to two circles</td>
<td>218</td>
</tr>
<tr>
<td>146</td>
<td>Steady Rise Cam Curve</td>
<td>219</td>
</tr>
<tr>
<td>147</td>
<td>Oscillating Flat Plate Cam</td>
<td>221</td>
</tr>
<tr>
<td>148</td>
<td>A Cam Star</td>
<td>222</td>
</tr>
<tr>
<td>149</td>
<td>Ellipse as Envelope of Circles</td>
<td>224</td>
</tr>
<tr>
<td>150</td>
<td>Hyperbola as an envelope of circles</td>
<td>225</td>
</tr>
<tr>
<td>151</td>
<td>Hyperbola as an Envelope of Lines</td>
<td>226</td>
</tr>
<tr>
<td>152</td>
<td>Caustics in a cup of coffee</td>
<td>227</td>
</tr>
<tr>
<td>153</td>
<td>A Nephroid by another route</td>
<td>228</td>
</tr>
<tr>
<td>154</td>
<td>Tschirnhausen’s Cubic</td>
<td>229</td>
</tr>
<tr>
<td>155</td>
<td>Cubic Spline</td>
<td>230</td>
</tr>
<tr>
<td>156</td>
<td>A Triangle Spline</td>
<td>231</td>
</tr>
<tr>
<td>157</td>
<td>Another Triangle Spline</td>
<td>233</td>
</tr>
</tbody>
</table>
A First Example
As a first example, we’ll do Pythagoras’ Theorem. First we draw the triangle:
As a first task, let’s make the angle ABC right. To do this, we select the segments AB and BC, then click the perpendicular button:

Notice that there is now a perpendicular constraint symbol between the two line segments.

Now we need to set the distance between A and B to be $x$. To do this, we select the segment AB and click the distance constraint button. You will get an edit box with the preset value of $a$. Change this to $x$.

Similarly, set the constraint of $y$ between B and C:
Now we are ready to display the value of the hypotenuse. Select the segment AC and the Calculate Distance button.

The => sign in front of the measurement shows that it is an output.
Triangles
The following examples explore aspects of triangles:
Example 1: Altitude of a right angled triangle

\[ \Rightarrow \frac{a \cdot b}{\sqrt{a^2 + b^2}} \]
Example 2: Altitude of the symmetrical altitude of an isosceles triangle

\[ \Rightarrow \sqrt{\frac{-b^2}{4+x^2}} \]
Example 3: Altitude of the non-symmetrical altitude of an isosceles triangle

\[ \Rightarrow b^2 \cdot \frac{b^4}{4x^2} \]

\[ \Rightarrow \frac{b^2}{2x} \]
Example 4: Altitude of a triangle defined by side lengths

Heron’s formula says that the area of a triangle is

$$\frac{\sqrt{(a + b + c)(a + b - c)(a - b + c)(-a + b + c)}}{4}$$

You’ll need to expand the expression to see the relationship to the altitude below:
Example 5: **Altitude of an Isosceles Triangle**

An isosceles triangle is defined by base length and one angle. What is its altitude?

\[ a \cdot \sin(\theta) \quad \text{or} \quad \frac{a \cdot \sin(\theta)}{2 \cdot |\cos(\theta)|} \]
Example 6: Triangle areas

Two sides and the included angle:

\[ \frac{1}{2} a \cdot b \cdot \sin(\theta) \]
Two sides and the non included angle:
Three sides:

\[ \frac{\sqrt{a+b+c} \cdot \sqrt{a+b-c} \cdot \sqrt{a-b+c} \cdot \sqrt{-a+b+c}}{4} \]
**Example 7: Intersection of the Medians of a Right angled Triangle**

Medians will be easy to specify when we have the Midpoint construction. However, they can also be specified simply by setting the distance to the end of the median to be half the specified length of the side:
Example 8: Centroid of a general triangle
We look at the centroid (intersection of the medians). We check the length of the median and the location of the incenter on that median:

We see that the centroid is located $2/3$ of the way down the median.
Alternatively, if we constrain the triangle by specifying its vertices, we obtain this formula for the location of the centroid:

\[
\begin{align*}
(x_0, y_0) & + (x_1, y_1) + (x_2, y_2) \\
\frac{x_0 + x_1 + x_2}{3}, \frac{y_0 + y_1 + y_2}{3}
\end{align*}
\]
Example 9: Angle of a Median

Given a triangle with sides length $a$, $b$ and included angle $\theta$ we derive the angle the median makes with the other side of the triangle:

\[
\arctan \left( \frac{-b \cdot \sin(\theta)}{-a + b \cdot \cos(\theta)} \right)
\]
Example 10: Orthocenter

We look at the orthocenter (intersection of the altitudes) of a triangle. Again we derive an expression for its distance from a vertex of the triangle.

\[ \Rightarrow \frac{b \cdot \left| a^2 - b^2 + c^2 \right|}{\sqrt{a+b+c \cdot (a+b-c) \cdot (a-b+c) \cdot (-a+b+c)}} \]
Example 11: Orthocenter Coordinates
This example is taken from the paper “Lernen mit einem neuen Taschenrechner”

Given that A and C are located at points (a,b) and (c,b) respectively, and that point B can move along the line y=d, we find the coordinates of the intersection of the altitudes. The locus is a quadratic, as shown both by its shape and by its equation.

\[ -x^2 - b^2 - a \cdot c + b \cdot d + X(a+c) + Y(b-d) = 0 \]
**Example 12: Circumcircle radius**

The center of the circumcircle of a triangle is the intersection of the perpendicular bisectors. Examining the expression for the distance from one vertex to the intersection point shows that it is symmetric. Hence will be the same, whichever vertex is chosen (try it).

\[
\frac{a \cdot b \cdot c}{\sqrt{(a+b+c) \cdot (a+b-c) \cdot (a-b+c) \cdot (a-b-c)}}
\]
Of course, a more direct approach is just to draw the circumcircle:

\[
\begin{align*}
\frac{abc}{\sqrt{a+b+c}\cdot\sqrt{(a+b-c)(a-b+c)(-a+b+c)}}
\end{align*}
\]
Example 13: Circumcircle radius in terms of vectors

If we have 2 vectors defining a circle, what is the circle’s radius:

\[
\Rightarrow \sqrt{\left(\frac{u_0 \cdot u_1}{2} + \frac{v_0^2}{2}\right) \cdot \left(\frac{-u_0 + u_1}{2} + \frac{v_0 + v_1}{2}\right)^2 + \left(\frac{u_1 \cdot v_0 + u_0 \cdot v_1}{2}\right)^2 - \left(u_0 \cdot v_0\right)^2 + \left(u_1 \cdot v_0 + u_0 \cdot v_1\right)^2}
\]
Example 14: Incircle Radius

\[ \Rightarrow \frac{\sqrt{a+b-c} \cdot \sqrt{a-b+c} \cdot \sqrt{a+b+c}}{2 \cdot \sqrt{a+b+c}} \]
Example 15: Vector joining the incircle with the vertex of a triangle defined by vectors

\[
\begin{bmatrix}
  u_0 \\
  v_0
\end{bmatrix} \Rightarrow \begin{bmatrix}
  u_1 \frac{u_0^2 + v_0^2 + u_0^2 + v_1^2}{u_0^2 + v_0^2 + u_1^2 + v_1^2} \\
  v_1 \frac{u_0^2 + v_0^2 + u_1^2 + v_1^2}{u_0^2 + v_0^2 + u_1^2 + v_1^2}
\end{bmatrix}
\]

Examine the expression – the denominator of each coefficient is the perimeter of the triangle, so the vector can be written:

\[
\left( \frac{u_0 | \mathbf{u}_1 | + u_1 | \mathbf{u}_0 |}{P} \right) \\
\left( \frac{v_0 | \mathbf{u}_1 | + v_1 | \mathbf{u}_0 |}{P} \right)
\]
Example 16: Incircle Center in Barycentric Coordinates

If we let \( a = |BC| \), \( b = |AC| \) and \( c = |AB| \) and let \( \mathbf{A}, \mathbf{B} \) and \( \mathbf{C} \) be the position vectors of the points \( A, B, C \) then the incircle center is:

\[
\frac{Aa + Bb + Cc}{a + b + c}
\]
Example 17: How does the point of contact with the incircle split a line

\[ \Rightarrow \frac{|a+b-c|}{2} \]

\[ \Rightarrow \frac{|a-b-c|}{2} \]
Example 18: Length of line joining a vertex with the point of contact between the incircle and the opposite side

\[ \frac{\sqrt{a^3 + 3ab^2 - 2b^3 - 2ab\cdot c + 2b^2\cdot c + 3ac^2 + 2bc^2 - 2c^3}}{2\sqrt{a}} \]
Example 19: A Pythagoras-Like Diagram

Draw a triangle (not necessarily right angled), and set the side lengths to be a, b and c.
Now draw squares on each side, constraining them with right angles and lengths of sides.
Now look at the distance between neighboring corners of the squares:
Example 20: A Penequilateral Triangle
Starting with a triangle whose sides are length $a,b,c$, we construct squares on each side, join the corners of the squares, then join the midpoints of these lines to create a triangle:
The triangle looks to the naked eye as if it is equilateral. Try dragging the original points and observe that the new triangle still looks equilateral:
USING SYMBOLIC GEOMETRY

Is it in fact?

\[ \Rightarrow \frac{a^2}{4} + \frac{b^2}{4} + \frac{3c^2}{4} = \sqrt{a+b+c} \sqrt{a+b-c} \sqrt{a-b+c} \sqrt{a+b+c} \]

Not quite, we observe the sides are guaranteed to be close in size, but not identical unless the original triangle is isosceles. In fact, the difference in squares of the sides of the new triangle is \(\frac{a^2}{4} - \frac{c^2}{4}\).

Notice we can repeat the process drawing squares on JK, KL and JL. The difference in squares of the ensuing sides will be \(\frac{a^2}{16} - \frac{c^2}{16}\).
By repeating this process, we can create a triangle as close as we like to an equilateral triangle, but still not exactly one.
**Example 21: Another Penequilateral Triangle**

We can do a similar construction based on equilateral triangles drawn on the sides of an original triangle:

We see that the difference in squares of the sides of the new triangle corresponding to \(a\) and \(b\) is \(\frac{a^2}{2} - \frac{c^2}{2}\). If we repeat the process, this triangle will eventually become equilateral, but not as quickly as the previous triangle.

In fact, if \(S(n)\) is the sum of the sides at the \(n\)th stage in the iteration, and \(A(n)\) is the area of the triangle on the \(n\)th stage of the iteration, the above equations for side length and area show that the following are true:

---

38
\[ S(n) = \frac{5}{8} S(n-1) + \frac{\sqrt{3}}{6} A(n-1) \]
\[ A(n) = \frac{\sqrt{3}}{32} S(n-1) + \frac{5}{8} A(n-1) \]

Which relation we can solve to give:

\[
\begin{pmatrix}
S(n) \\
A(n)
\end{pmatrix} =
\begin{pmatrix}
\frac{5}{8} & \frac{\sqrt{3}}{6} \\
\frac{\sqrt{3}}{32} & \frac{5}{8}
\end{pmatrix}^n
\begin{pmatrix}
S(0) \\
A(0)
\end{pmatrix}
\]
Example 22: Folding a right angled triangle
If you create a second right angled triangle by cutting perpendicular to the hypotenuse, then that triangle is similar to the original one. Here are its side lengths:
Example 23: Interior point of an Equilateral Triangle
If a point D is interior to an equilateral triangle, the sum of the perpendicular distances to the sides of the triangle is independent of the location of the point. The following construction shows this (try adding the three distances).
Here is a simpler way to define this one, using perpendicular distance constraints. In this case we specify $x$ to be the perpendicular distance between $D$ and $AB$, and $y$ to be the perpendicular distance from $D$ to $AC$. We then ask for the perpendicular distance from $D$ to $BC$.

\[ -\frac{\sqrt{3} \cdot a}{2} + x + y \]

Notice that the sum of these 3 distances is $\frac{\sqrt{3}}{2} a$, which is independent of $x$ and $y$. 
Example 24: Right angled triangle in semi-circle
This diagram shows that the center of the hypotenuse of a right angled triangle is half a hypotenuse away from the opposite vertex:
Example 25: Line Joining the apex to base of an isosceles triangle
Take an isosceles triangle whose equal sides have length $a$ and divide its base in lengths $x$, $y$. Join the dividing point to the opposite vertex, and examine its length. The expression is surprisingly simple:

$$\Rightarrow a^2 - x \cdot y$$

From this result, you can deduce, for example, that for any two chords through a particular point interior to a circle, the product of the lengths of the chord segments on either side of the point is the same.
Example 26: Area of Triangle defined by 2 vectors

Area of the triangle is:

\[
\frac{1}{2} | u_0 \cdot v_0 - u_0 \cdot v_1 |
\]
Example 27: Similar Triangle

Lengths on a similar triangle go by ratio:

\[
\frac{b \cdot c}{a} \quad \Rightarrow \quad \frac{b \cdot c}{a}
\]

\[
\frac{c \cdot d}{a} \quad \Rightarrow \quad \frac{c \cdot d}{a}
\]
Example 28: Areas of triangles bounded by Cevians

The ratios of areas in this diagram are simple quadratics in \( t \). Can you derive these quadratics from the sine rule?

Area of \( \triangle ABC \)

\[ z_4 \Rightarrow \frac{\sqrt{a+b+c} \cdot \sqrt{a+b+c} \cdot \sqrt{a+b+c}}{4} \]

Area of \( \triangle ADF \)

\[ z_3 \Rightarrow \frac{t \cdot \sqrt{a+b+c} \cdot \sqrt{a+b+c} \cdot \sqrt{a+b+c}}{4} \quad \frac{t^2 \cdot \sqrt{a+b+c} \cdot \sqrt{a+b+c} \cdot \sqrt{a+b+c}}{4} \]

\[ \Rightarrow \frac{z_3}{z_4} \Rightarrow t^2 \quad | t < 1 \]

Area of \( \triangle DEF/\triangle ABC \)

\[ z_0 \Rightarrow \frac{\sqrt{a+b+c} \cdot \sqrt{a+b+c} \cdot \sqrt{a+b+c}}{4} \quad \frac{3t \cdot \sqrt{a+b+c} \cdot \sqrt{a+b+c} \cdot \sqrt{a+b+c}}{4} \quad \frac{3t^2 \cdot \sqrt{a+b+c} \cdot \sqrt{a+b+c} \cdot \sqrt{a+b+c}}{4} \]

\[ \Rightarrow 1-3t+3t^2 \quad | t < 1 \]

\[ z_0 \Rightarrow \frac{\sqrt{a+b+c} \cdot \sqrt{a+b+c} \cdot \sqrt{a+b+c}}{4} \quad \frac{3t \cdot \sqrt{a+b+c} \cdot \sqrt{a+b+c} \cdot \sqrt{a+b+c}}{4} \quad \frac{3t^2 \cdot \sqrt{a+b+c} \cdot \sqrt{a+b+c} \cdot \sqrt{a+b+c}}{4} \]

\[ > 0 \]

\[ z_0 \Rightarrow \frac{\sqrt{a+b+c} \cdot \sqrt{a+b+c} \cdot \sqrt{a+b+c}}{4} \quad \frac{3t \cdot \sqrt{a+b+c} \cdot \sqrt{a+b+c} \cdot \sqrt{a+b+c}}{4} \quad \frac{3t^2 \cdot \sqrt{a+b+c} \cdot \sqrt{a+b+c} \cdot \sqrt{a+b+c}}{4} \]

\[ > 0 \]
Example 29: Decomposing a vector into components parallel and perpendicular to a second vector

\[
\begin{align*}
\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} & \Rightarrow \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \\
\begin{pmatrix} u_0 \cdot u_1 - u_0 \cdot v_0 \cdot v_1 \\ -u_0^2 - v_0^2 \\ v_0 \cdot u_0 - u_0 \cdot v_0 \cdot v_1 \\ -u_0^2 - v_0^2 \end{pmatrix} & \Rightarrow \begin{pmatrix} v_0 \cdot (u_1 \cdot v_0 + u_0 \cdot v_1) \\ -u_0^2 - v_0^2 \\ u_0 \cdot (u_1 \cdot v_0 - u_0 \cdot v_1) \\ -u_0^2 - v_0^2 \end{pmatrix}
\end{align*}
\]
Quadrilaterals
Here are some examples using quadrilaterals
Example 30: **Diagonals of a rhombus**
A rhombus has sides length $a$ and one diagonal length $b$, what is the length of the other diagonal?

$$\Rightarrow \sqrt{4 \cdot a^2 - b^2}$$
Example 31: Diagonals of a parallelogram
Given a parallelogram whose sides measure a and b, and one diagonal is c, what is the length of the other diagonal?

\[ \Rightarrow 2 \cdot a^2 + 2 \cdot b^2 - c^2 \]

A simple enough result, but can you derive it? I used the cosine rule, but can you do it by Pythagoras alone?
Example 32: Diagonals of a Kite
Continuing in this theme, if we have a kite whose non-axis diagonal is length c, and whose sides are length a and b, what is the length of the other diagonal?

\[ a^2 + b^2 - c^2 = \sqrt{\frac{2\cdot a + c \cdot \sqrt{2\cdot a - c^2}}{2}} + \sqrt{\frac{2\cdot c + b \cdot \sqrt{2\cdot b - c^2}}{2}} \]

Can you see the similarity to the parallelogram?
What about the non-convex kite with the same side lengths and diagonal?
How about if we are specified the axis, what is the non-axis diagonal?

\[ 2 \cdot a^2 + 2 \cdot b^2 - \frac{a^4}{d^2} + \frac{2 \cdot a^2 \cdot b^2}{d^2} - \frac{b^4}{d^2} - d^2. \]

This is very similar to the equation of the altitude of a triangle. Why?
Example 33: **Cyclic Quadrilateral**

A quadrilateral is inscribed in a circle of radius $a$. 3 sides of the quadrilateral have length $b$. What is the length of the fourth side?

![Diagram of a cyclic quadrilateral]

What is the relationship between $a$ and $b$ when the fourth side has length $b$?
Example 34: A Right Trapezoid

\[ h^2 + (-a+b)^2 \]
Example 35: Areas of Quadrilaterals

Trapezium:

\[ \text{Area} = \frac{h \cdot (a+b)}{2} \]
Kite:

\[
\Rightarrow \frac{ab}{2}
\]
Parallelogram:

\[ a \cdot b \cdot \sin(\theta) \]
Example 36: Areas of triangles in a trapezoid

One pair of triangles formed by the diagonals of a trapezoid are equal in area.

\[ \Rightarrow \frac{a \cdot b \cdot h}{2(a+b)} \]
The other pair are not:

\[ a \cdot h = \frac{a^2 \cdot h}{2 \cdot (a+b)} \]

\[ b \cdot h = \frac{b^2 \cdot h}{2 \cdot (a+b)} \]
Example 37: Diameter of the circumcircle
Here is a diagram which allows us to find the diameter of the circumcircle of the triangle whose sides have lengths $a$, $b$ and $c$. Why?

\[ \frac{2 \cdot a \cdot b \cdot c}{\sqrt{a+b+c} \cdot \sqrt{(a+b-c)(a-b+c)(a-b-c)}} \]
**Example 38: Quadrilateral with perpendicular diagonals and one right angle**

A Quadrilateral has 3 sides length a, b, c and a right angle. Its diagonals are perpendicular. What is the length of the remaining side?
Example 39: Finding the diameter of an arc given the perpendicular offset from the chord.

Here is the diagram. Verify that it is correct:
**Example 40: Napoleon’s Theorem**

Napoleon’s Theorem states that if you take a general triangle and draw an equilateral triangle on each side, then the triangle formed by joining the incenters of these new triangles is equilateral. You can see that the length is symmetrical in a,b,c and hence identical for the three sides of the triangle.

\[
\Rightarrow \frac{a^2}{6} + \frac{b^2}{6} + \frac{c^2}{6} + \sqrt{\frac{a+b+c}{6}} \cdot \sqrt{\frac{a+b-c}{6}} \cdot \sqrt{\frac{a-b+c}{6}} \cdot \sqrt{\frac{a-b-c}{6}}
\]
Example 41: An Isosceles Triangle Theorem

ABC is isosceles. AE=DC. We show that EF=FC.
Example 42: A Quadrilateral with Perpendicular Diagonals

Given two sides, the lengths of the diagonals and the fact that they are perpendicular, what are the lengths of the other two sides of a quadrilateral?
**Example 43: Intersection of Common Tangent with Axis of Symmetry of Two Circles**

Line AB has length a. A is perpendicular distance x from line CD, and B is perpendicular distance y from CD. Find the distance of the intersection point between AB and CD from A and B:
What if we change the diagram slightly so that E is external to AB:
Example 44:  **Slope of the Angle Bisector**
What is the slope of the angle bisector of a line with slope 0 and a line with slope $m$?

\[
\Rightarrow \frac{-1 + \sqrt{1 + m^2}}{m}
\]
Example 45: Location of intersection of common tangents
Circles AB and CD have radii \( r \) and \( s \) respectively. If the centers of the circles are \( a \) apart, and \( E \) is the intersection of the interior common tangent with the line joining the two centers, what are the lengths \( AE \) and \( CE \)?
How about the exterior common tangent?
Example 46: Altitude of Cyclic Trapezium defined by common tangents of 2 circles

Given circles radii $r$ and $s$ and distance $a$ apart, what is the altitude of the trapezium formed by joining the intersections of the 4 common tangents with one of the circles?

Notice that this is symmetrical in $r$ and $s$, and hence the trapezium in circle $AB$ has the same altitude.
Example 47: Areas Cyclic Trapezia defined by common tangents of 2 circles

Look at the ratio of the areas of the trapezia in the previous example:

\[
\frac{z_0}{z_1} \Rightarrow \frac{r}{s}
\]

\[
\frac{2\cdot r \cdot s \cdot \sqrt{a^2 - r^2 - 2\cdot r \cdot s + s^2}}{a^2}
\]
**Example 48:** Triangle formed by the intersection of the interior common tangents of three circles

Notice that if $A$ is the area of the triangle formed by the centers of the circles, then area $STU$ is:

$$\frac{2rstA}{(r+s)(s+t)(r+t)}$$

\[\Rightarrow \frac{r \cdot s \cdot t \cdot \frac{a+b+c}{2} \cdot \frac{a+b-c}{2} \cdot \frac{a-b+c}{2} \cdot \frac{-a+b+c}{2}}{2 \cdot (r+s)(s+t)(r+t)}\]
Example 49:  **Distance between sides of a rhombus**

Given a rhombus with side length a and diagonal b, what is the perpendicular distance between opposite sides?

\[ b \cdot \frac{2 \cdot a + b \cdot \sqrt{2 \cdot a - b}}{2 \cdot a} \]
Example 50: Angles of Specific Triangles
Here are some triangles, with their angles displayed.
\[ \Rightarrow \pi / 4 \]

\[ \Rightarrow \arctan \left( \frac{b}{a} \right) \]

\[ \Rightarrow \arctan \left( \frac{a}{b} \right) \]
Example 51: Sides of Specific Triangles
Here are some specific angle-defined triangles:

![Diagram of triangle with sides labeled as follows:
- $\triangle ABC$ with $\angle BAC = \pi/4$ and side lengths $a$, $\sqrt{2}a$, $\sqrt{2}a$.
- $\triangle ABC$ with $\angle BAC = \pi/6$ and side lengths $a$, $\sqrt{3}a$, $2a$.]

79
\[ \Rightarrow a^2 \cdot \sqrt{2 - \sqrt{3}} \]

\[ \Rightarrow a \cdot \sqrt{2 + \sqrt{3}} \]
Example 52: Angles in the general triangle

3 sides defined

\[ \Rightarrow \arccos \left( \frac{-a^2 + b^2 + c^2}{2bc} \right) \]
USING SYMBOLIC GEOMETRY

TRIANGLE WITH 2 SIDES AND INCLUDED ANGLE

\[ \Rightarrow \pi + \arctan \left( \frac{-b \cdot \sin(A)}{c + b \cdot \cos(A)} \right) \]
TRIANGLE WITH 2 SIDES AND NON-INCLUDED ANGLE

\[ \Rightarrow \pi \cdot \arcsin \left( \frac{b \cdot \sin(C)}{c} \right) \]
Here's the exterior angle:
Example 53: Some implied right angles
The median of an isosceles triangle is also its perpendicular bisector, and altitude:
A triangle whose median is the same length as half its base is right angled:
The diagonals of a kite are perpendicular
The line joining the intersection points of two circles is perpendicular to the line joining their centers:
Here is a particular triangle (from the book “The Curious Incident of the Dog in the Night”)
Example 54: Triangle defined by 2 angles and a side

\[
\Rightarrow a \cdot \sin(t) \quad \Rightarrow \frac{a \cdot \sin(s)}{\sin(s+t)}
\]

\[
\Rightarrow \pi - s - t
\]

\[
\Rightarrow a \cdot \sin(s) \quad \frac{\sin(s+t)}{\sin(s+t)}
\]
Example 55: Triangle defined by two sides and the included angle

\[ \Rightarrow \arctan \left( \frac{-b \cdot \sin(A)}{c + b \cdot \cos(A)} \right) \]

\[ \Rightarrow \sqrt{b^2 + c^2 - 2 \cdot b \cdot c \cdot \cos(A)} \]
Example 56: Triangle defined by 2 sides and the non-included angle

\[ \Rightarrow \arctan \left( \frac{\sqrt{a^2 - c^2 \cdot \sin(A)^2} - c \cdot \cos(A)}{c + \sqrt{a^2 - c^2 \cdot \sin(A)^2} \cdot \cos(A) + c \cdot \cos(A)} \right) \]
Example 57: Incenter

The incenter is the intersection of the angle bisectors. We have a triangle with angles $2a$ and $2b$ and base $d$:

$$d \cdot \frac{\cos(a+b) - \cos(-a+b)}{2 \cdot \sin(a+b)}$$

$$d \cdot \frac{\cos(a+b)}{2 \cdot \sin(a+b)} - d \cdot \frac{\cos(-a+b)}{2 \cdot \sin(a+b)}$$

We see that the perpendicular distance to the other sides is the same. This shows that B is the center of a circle tangent to AD and CD.
Example 58: Quadrilateral Formed by Joining the Midpoints of the sides of a Quadrilateral

Quick inspection of the side lengths shows that the new figure is a parallelogram
Example 59: Some measurements on the Pythagoras Diagram

Draw a right triangle and subtend a square on each side. The two red lines in the diagram are equal in length.
Example 60: An unexpected triangle from a Pythagoras-like diagram
Regardless of the original triangle the resulting triangle from this diagram is a right angled-isosceles:

Examination of the length AJ shows that it is symmetric in a and b, and hence identical to AK.
Example 61:  A Theorem on Quadrilaterals

This theorem states that if you draw a square on each side of a quadrilateral, then connect the center of opposite sides, the resulting lines have the same length, and are perpendicular. Here is the result in Geometry Expressions

If we create the length of the other side we can by careful examination see that the lengths are identical. Alternatively, we can do some simplification. Our constraints are necessarily asymmetric – Geometry Expressions will not let you over-constrain the diagram, and one diagonal is sufficient to define the quadrilateral. However, we might expect the formula to be simpler if expressed in terms of both diagonals.
Close inspection of the formula for the length shows that it incorporates the square of the other diagonal of the figure, as well as Heron’s formula for the areas of the triangles ABC and ACD. The following Mathematica worksheet contains the formulas from Geometry Expressions for \( L \) the length of the desired line, and \( f \) the length of the other diagonal. A little manipulation gives a simple formula for \( L^2 - f^2/2 \). This can be simplified further by noting that the remaining terms are \( e^2/2 \) and twice the area of the quadrilateral:

\[
L^2 = \frac{e^2}{2} + \frac{f^2}{2} + 2A
\]

From which we can derive that:
Areas are simpler when expressed in terms of angles. Here is a revision of the diagram with angles inserted. This gives us more of a clue of how to prove the result:

\[
\Rightarrow \sqrt{b^2 + c^2 - 2bc \cos(\theta)}
\]

\[
\Rightarrow \sqrt{c^2 + d^2 - 2cd \cos(\phi)}
\]

\[
\Rightarrow \sqrt{b^2 + c^2 - 2bc \cos(\theta) + c^2 + d^2 - 2cd \cos(\phi)}
\]

\[
\Rightarrow \frac{b^2}{2} + \frac{c^2}{2} + \frac{d^2}{2} + bc \sin(\theta) + cd \sin(\phi) - bd \sin(\theta + \phi) - bc \cos(\theta) - cd \cos(\phi)
\]
Example 62: Rectangle Circumscribing an Equilateral Triangle

Directly from the diagram we have the following theorem:

The area of the larger right triangle is the sum of the areas of the smaller two.

This appears in page 19-21 of Mathematical Gems, by Ross Honsberger (and various other places).
Polygons
Some examples with pentagons, hexagons…
Example 63: Regular Pentagons and more
What length should the side of a regular pentagon be?

We can address this problem by making a pentagon with all but one side length \( a \), then solving such that the final side is also length \( a \):

\[
\sqrt{16a^2 - \frac{a^8}{r^6} + \frac{8a^6}{r^4} - \frac{20a^4}{r^2}} = a
\]

\[
> \text{solve}(\sqrt{16a^2 - a^8/r^6 + 8a^6/r^4 - 20a^4/r^2} = a, a);
\]

\[
\sqrt{5} \frac{\sqrt{5} + 1}{2} \sqrt{\frac{a}{r^2}}, \sqrt{5} \frac{\sqrt{5} - 1}{2} \sqrt{\frac{a}{r^2}}, \sqrt{3} \sqrt{\frac{a}{r^2}}, 0
\]
Ignoring the 0 solution, there are 3 solutions. Let\'s pick the middle one and paste it in as $a$:

We see this indeed gives us the regular pentagon. What do the other solutions give us?
The first solution gives the regular five pointed star. How about the third solution?
At first sight we seem to have lost 2 of the sides. But they are in fact there, sitting on top of BC.
Example 64: A Regular Pentagon Construction
Here is a construction for a regular pentagon inside a circle of radius $r$. D is the midpoint of AC, DE is congruent to BD and BF is congruent to BE.

\[
\Rightarrow r \cdot \frac{\sqrt{5} \cdot \sqrt{5}}{2}
\]

We see this yields a side of the requisite length.
Example 65: Area of a Hexagon bounded by Triangle side trisectors

The area can be seen to be 1/10 the area of the original triangle
Angles and Circles
Example 66: Rounding a Corner
Assume we have a corner of angle $\theta$, and we round it off with a curve of radius $r$, how far away from the corner does the round start? And what is the chord length?
Example 67: Cyclic Quadrilaterals
Opposite angles of Cyclic quadrilaterals add up to 180, so the exterior angle equals the opposite interior angle:
Example 68: Angles subtended by a chord
The angle at the center subtended by a chord is twice the angle at the circumference subtended by that chord:
Why?
Example 69: Angle subtended by a point outside the circle

We generalize the above result for the angle subtended by a point outside the circle:
Example 70: Angle at intersection of two circles:
Here is a familiar method of making the 120 degree angle needed to draw a hexagon:
Example 71: Angle subtended by two tangents

\[ \Rightarrow \frac{\pi}{2} - \frac{a}{2} \]
Circles
Example 72: Distance between the incenter and circumcenter

First, an isosceles triangle

Then, more generally:
\[ \Rightarrow \sqrt{a \cdot b \cdot c^4 + c^3 \cdot (-a-b)+c^2 \cdot (-a^2+3 \cdot a \cdot b-b^2)+c \cdot (a^2-a^2 \cdot b-a \cdot b^2+b^3)} \]

\[ \sqrt{a+b+c \cdot \sqrt{(a+b-c) \cdot (a-b+c) \cdot (-a+b+c)}} \]
Example 73: Radius of Circle through 2 vertices of a triangle tangent to one side

\[
\Rightarrow \frac{a^2 \cdot b}{\sqrt{a+b+c} \cdot \sqrt{(a+b-c)(a-b+c)(-a+b+c)}}
\]
Example 74:  **Length of the common tangent to two tangential circles**

A succinct formula:

\[ \Rightarrow 2\sqrt{a \cdot b} \]

Also the internal common tangent bisects this:
\[ \Rightarrow 2\sqrt{a \cdot b} \]
\[ \Rightarrow \sqrt{a \cdot b} \]
\[ \Rightarrow \sqrt{a \cdot b} \]
Example 75: Tangents to the Radical Axis of a Pair of Circles

The radical axis of a pair of circles is the line joining the points of intersection. The lengths of tangents from a given point on this axis to the two circles are the same.
Example 76: Inverting a segment in a circle

To invert a point $C$ in a circle $AB$ radius $r$, you find a point $E$ on the line $AC$ such that

$$AC \cdot AE = r^2$$

If $|AC| = a$ and $|AD| = b$ and $|CD| = c$, then we find the length of the inverted segment $C_1D_1$
Example 77: Inverting a circle
We can invert a circle, by inverting its points of tangency:
Example 78: The nine point circle

The nine point circle is the circle through the midpoint of each side of a triangle. Taking a triangle with one vertex at the origin, another at \((x,0)\) and another at \((x_1,y_1)\), we look at the center and radius of the nine point circle:

\[
(x_1, y_1) \Rightarrow x + \frac{x_1}{4}, \frac{x_1}{2}, y_1 + y_1^2 \frac{x_1}{4}
\]

\[
\Rightarrow \sqrt{x^2 - 2 \cdot x \cdot x_1 + x_1^2 + y_1^2 - x_1^2 y_1^2} \cdot |x| \frac{4 \cdot y_1}{4 \cdot x \cdot y_1}
\]
Example 79: Excircles
The three excircles of a triangle are tangent to the three sides but exterior to the circle:
We examine the triangle joining the centers of the excircles:
Example 80: Circle tangential to two sides of an equilateral triangle and the circle centered at their intersection through the other vertices
This circle has radius one third the side of the triangle…
Example 81: Circle tangent to 3 sides of a circular segment

Here are the radii of the circle tangent to two sides and the circular arc of a circular segment, for one or two popular angles:
\[
(\Rightarrow a \cdot (3 + 2\sqrt{3})
\]

\[ \Rightarrow \frac{\sqrt{2 \cdot (2 \cdot a \cdot a \cdot \sqrt{8 + 4 \cdot 2})}}{2 \cdot (3 + 2 \cdot \sqrt{2})} \]
Example 82: Various Circles in an Equilateral Triangle

We look at the radii of various circles in an equilateral triangle:
What would the next length in the sequence be?
Example 83: Circle tangential to base of equilateral triangle and constructing circles
Now we create a circle centered on C through A and B, and make circle DE tangential to it rather than to line AB. We see the circle has radius $3a/8$. 
Example 84: Radius of the circle through two vertices of a triangle and tangent to one side

\[
\Rightarrow \frac{b^2 \cdot c}{\sqrt{a+b+c} \cdot \sqrt{(a+b-c)(a-b+c)(a-b-c)}}
\]
Example 85: Circle Tangent to 3 Circles with the same radii

And the external one:
\[ 3 \cdot r + 2 \cdot \sqrt{3} \cdot r + r \cdot 28 + 16 \cdot \sqrt{3} \]
Example 86: Circles Tangent to 3 Circles of Different Radii:

\[
\begin{align*}
1 & \Rightarrow \frac{1}{r^2} + \frac{1}{s^2} + \frac{1}{t^2} + 2 \cdot \frac{\frac{1}{rs} + \frac{1}{rt} + \frac{1}{st}}{r^2s^2 + r^2t^2 + s^2t^2} \\
1 & \Rightarrow \frac{-1}{r^2} + \frac{1}{s^2} + \frac{1}{t^2} + 2 \cdot \frac{\frac{1}{rs} + \frac{1}{rt} + \frac{1}{st}}{r^2s^2 - r^2t^2 - s^2t^2}
\end{align*}
\]
Example 87: Circle Tangential to 3 circles same radius

3 circles of the same radius, two are tangential and the third is distance b and c from those

\[ \Rightarrow \sqrt{2a^2bc} \sqrt{2a^2bc} \sqrt{2a^2bc} \sqrt{2a^2bc} \sqrt{8abc + 4a^2 \sqrt{2a^2bc} \sqrt{2a^2bc} \sqrt{2a^2bc} \sqrt{2a^2bc}} \]

\[ 4 \left( 16a^2 - 8a^2 b^2 + c^2 + c^2 \cdot \sqrt{8a^2 - 2b} \right) \]
Example 88: Two circles inside a circle twice the radius, then a third

\[ \frac{2 \cdot r}{3} \]
And if we keep on going:
The general case looks like this:

We can copy this expression into Maple to generate the above sequence:

\[
\frac{1}{1/(1/2*1/r+1/x+2*sqrt(-1/2*1/(r^2)+1/2/x/r))};
\]

\[
\frac{1}{2 \cdot r} + \frac{1}{x} + \sqrt{-2 \cdot \frac{1}{r^2} + \frac{2}{x \cdot r}}
\]

\[
\text{subs}(r=1, \%);
\]

\[
\frac{1}{2} + \frac{1}{x} + \sqrt{-2 + \frac{2}{x}}
\]
\[ f := x \mapsto \frac{1}{\frac{1}{2} + \frac{1}{x} + \sqrt{-2 + \frac{2}{x}}} \]

\[ f(1) ; \]
\[ \frac{2}{3} \]

\[ f\left(\frac{2}{3}\right) ; \]
\[ \frac{1}{3} \]

\[ f\left(\frac{1}{3}\right) ; \]
\[ \frac{2}{11} \]

\[ f\left(\frac{2}{11}\right) ; \]
\[ \frac{1}{9} \]

\[ f\left(\frac{1}{9}\right) ; \]
\[ \frac{2}{27} \]

\[ f\left(\frac{2}{27}\right) ; \]
\[ \frac{1}{19} \]

A little analysis of the series can lead us to postulate the formula \(2/(n^2+2)\) for the \(n\)’th circle:

Let’s feed the \(n\)-th term into Maple:

\[ f\left(\frac{2}{\left((n-1)^2+2\right)}\right) ; \]
\[ \frac{1}{\frac{3}{2} + \frac{1}{2} (n-1)^2 + \sqrt{(n-1)^2}} \]

In order to get the expression to simplify, we make the assumption that \(n>1\):
> assume(n>1);
> simplify(f(2/((n-1)^2+2)));

\[ \frac{2}{2 + n^2} \]

We see that this is the next term in the series. By induction, we have shown that the \( n \)'th circle has radius \( \frac{2}{2 + n^2} \).
Example 89: A theorem old in Pappus’ time
A theorem which was old in Pappus’ days (3rd century AD) relates the radii to height of the circles in figures like the above:
Applying the general model, we get a formula:

\[ \Rightarrow -\frac{2 \cdot r \cdot \sqrt{x} \cdot \sqrt{2 \cdot r - 2 \cdot x - 2 \cdot x^2} \cdot \sqrt{2 \cdot r - 2 \cdot x}}{r + x} \]

Again, we can copy this into Maple for analysis:

\[ > -(-2 \cdot \text{sqrt}(x) \cdot x^3 - 4 \cdot x^{3/2} \cdot r - 2 \cdot x^{5/2} \cdot r) \cdot \text{sqrt}(2 \cdot r - 2 \cdot x)/(r+x)^2/r; \]

\[ > \text{simplify}(%); \]

\[ > \text{subs}(x=2/(n^2+2),r=1,%); \]
\[2 \sqrt{2} \sqrt{\frac{1}{2 + n^{-2}}} \sqrt{2 - \frac{4}{2 + n^{-2}}}\]

> \texttt{simplify(\%);}

\[4 \frac{n^{-}}{2 + n^{-2}}\]

We see that the height above the centerline for these circles is the radius multiplied by 2n.
**Example 90: Yet Another Family of Circles**

We generalize the situation from a couple of examples ago. We look at the family generated by two circles of radius $a$ and $b$ inside a circle of radius $a+b$:

\[
\Rightarrow \frac{a \cdot b \cdot (a+b)}{a^2 + a \cdot b + 4 \cdot b^2} \Rightarrow \frac{a \cdot b \cdot (a+b)}{a^2 + a \cdot b + 9 \cdot b^2} \Rightarrow \frac{a \cdot b \cdot (a+b)}{a^2 + a \cdot b + 16 \cdot b^2}
\]

The pattern is pretty obvious this time: the radius of the $n$th circle is:

\[
\frac{ab(a + b)}{n^2 a^2 + ab + b^2}
\]
To prove this, we derive the formula for the general circle radius $x$, and analyze in Maple:

\[
\Rightarrow \frac{1}{\frac{1}{b} + \frac{1}{x} + \frac{1}{a+b} + 2 \sqrt{\frac{1}{b\cdot x} - \frac{1}{b\cdot(a+b)} - \frac{1}{x\cdot(a+b)}}}
\]

Now we try feeding in one of the circle radii into this formula in maple (first making the assumption that the radii are positive (along with $n>1$ for later use):

\[
\text{assume}(a>0,b>0,n>1);
\]

\[
\text{f:=x}\rightarrow\frac{1}{\frac{1}{b} + \frac{1}{x} - \frac{1}{a+b} + 2 \sqrt{\frac{1}{b\cdot x} - \frac{1}{b\cdot(a+b)} - \frac{1}{x\cdot(a+b)}}};
\]

\[
\text{f((a+b)*b*a/(9*a^2+b*a+b^2))};
\]
\[
1 \left( \frac{1}{b^2} + \frac{9a^2 + b^2 - a + b^2}{(a + b)(b - a)} - \frac{1}{a + b}\right) \\
+ 2 \sqrt[3]{ \frac{9a^2 + b^2 - a + b^2}{(a + b)(b^2 - a^2)} - \frac{1}{a + b}\left( - \frac{9a^2 + b^2 - a + b^2}{(a + b)^2(b - a)} \right) }
\]

\[\text{simplify(\%);}\]
\[
\frac{(a + b)(b - a)}{16a^2 + b^2 - a + b^2}
\]

Let's try the general case, feeding in the formula for the n-1st radius:

\[\text{simplify(f((a+b)*b*a/((n-1)^2*a^2+b*a+b^2))};\]
\[
\frac{b - a - (a + b)}{b - a - a^2 - n^2 + b^2}
\]

By induction, we have proved the general result.
\[
\Rightarrow \frac{1}{\left( \frac{1}{b} + \frac{1}{x} + \frac{1}{a+b} \right)^2} \cdot \frac{1}{\frac{1}{b \cdot x} - \frac{1}{b \cdot (a+b)} - \frac{1}{x \cdot (a+b)}}
\]
Example 91: Archimedes Twins
The given circles are mutually tangential with radius a, b and a+b. Archimedes twins are the circles tangential to the common tangent of the inner circles. We see from the symmetry of the radius expression that they are congruent.
Example 92: Circles tangential to two Touching circles and their common tangent
Here is another family of circles, based on original circles of radius $r$
Notice that the two red ones have the same radii:
Example 93: The triangle joining the points of tangency of 3 circles
Given tangential circles radii $a, b, c$, we find the side length of the triangle joining the points of tangency:
Example 94: The triangle tangential to 3 tangential circles
We look at the triangle formed by the common tangents to these 3 circles – in the case where the circles all have radius $a$, this is an equilateral triangle, with length as shown below:

$$\Rightarrow a\cdot\left(2+2\sqrt{3}\right)$$
Example 95: Center and Radius of a Circle Given Equation

Find the center and radius of the circle whose equation is

\[ x^2 + y^2 + ax + by + c = 0 \]
Example 96: A limit point
In the example, E is the intersection of BD with the x axis. We examine the x coordinate of E as r tends to 0

This can be done by means of a picture:

Or analytically in Maple:
\( \text{limit(-r^2/(sqrt(-r+2)*sqrt(r+2)-2), r=0)}; \)
Example 97: Buehler's Circle
Given a circle inside and tangent to another circle, we create an isosceles triangle whose base connects the intersection between the axis of symmetry and the two circles, and whose apex lies on the outer circle. We get the height of this triangle:
Using this as input, we construct the circle tangent to both circles, whose center is on the line through G perpendicular to AC. We examine its radius and the distance from its center to the line. We infer that it is tangential to this line.

\[ \Rightarrow \frac{-2 \cdot r \cdot (r-s)}{r+s} \]

\[ \Rightarrow \frac{2 \cdot r \cdot (-r+s)}{r+s} \]

\[ |s^2 > r^2| \]

\[ \sqrt{r+s} \cdot \sqrt{-r+s} \]
Example 98: Circle to two circles on orthogonal radii of a third
In the diagram, Circles DE and FG are centered on perpendicular radius lines AB and AC. Circle HI is tangential to all three. We observe that AFHD is a rectangle.
Equations of Lines and Circles
Example 99: Intersection of Two Lines
What is the intersection of the lines $Y = c + mX$ and $Y = d + nX$?
Example 100: Equation of Line Through Two Points

\[ x_1 \cdot y_0 + x_0 \cdot y_1 + y \cdot (-x_0 + x_1) + X \cdot (y_0 - y_1) = 0 \]
Example 101: Intersection of a Line with a Circle

\[
\begin{align*}
B \frac{(A+B+y+C_0)^2}{A^2+B^2} + A \frac{(A+B+y+C_0)^2}{A^2+B^2} &= \frac{A^2}{A^2+B^2} \frac{(A+B+y+C_0)^2}{A^2+B^2} + B \frac{(A+B+y+C_0)^2}{A^2+B^2} \\
&= \frac{C_0+x+y+e=0}{A^2+B^2}
\end{align*}
\]
Example 102: Equation of the Line Joining the Intersection of Two Circles

Given two circles, what is the equation of the line joining their intersections?

\[ Y = \frac{-X \cdot a_1 + X \cdot a_2 - c_1 + c_2}{b_1 - b_2} \]

\[ X^2 + Y^2 + X \cdot a_1 + Y \cdot b_1 + c_1 = 0 \]

\[ X^2 + Y^2 + X \cdot a_2 + Y \cdot b_2 + c_2 = 0 \]
Example 103: Projecting a Point onto a Line

Project the point \((x,y)\) onto the line \(c + aX + bY = 0\):

\[
\Rightarrow \frac{c + a \cdot x + b \cdot y}{\sqrt{a^2 + b^2}}
\]

\[
\Rightarrow \left( \frac{a \cdot c + b \cdot (-b \cdot x + a \cdot y)}{-a^2 b^2}, \frac{b \cdot c + a \cdot b \cdot x - a^2 y}{-a^2 b^2} \right)
\]
Example 104: Radius of the incircle of a Triangle formed by 3 Lines Whose Equations are Known

This equation is simpler if we assume that the line equations are normalized, and hence that the square roots in the denominator are equal to 1.
Example 105: Equation of the Altitude of a Triangle Defined by lines with Given Equations

\[ Y = \frac{-X \cdot a_0 \cdot c_0}{b_0} \]

\[ Y = \frac{-X \cdot a_1}{b_1} \]

\[ Y = \frac{-X \cdot a_2 \cdot c_2}{b_2} \]

\[ Y = \frac{-X \cdot a_2 \cdot b_0 \cdot b_1 + X \cdot a_0 \cdot b_1 \cdot b_2 + a_1 \cdot a_2 \cdot c_0 + b_1 \cdot b_2 \cdot c_0 \cdot a_0 \cdot a_1 \cdot c_2 \cdot b_0 \cdot b_1 \cdot c_2}{a_1 \left(-a_2 \cdot b_0 + a_0 \cdot b_2\right)} \]
Example 106: Equation of the Line through a Given Point at 45 degrees to a Line of Given Equation

\[ Y = \frac{x \cdot a + x \cdot b \cdot x \cdot y + a \cdot y \cdot y}{a \cdot b} \]

\[ c + x \cdot a + y \cdot b = 0 \]
Example 107: Equation of Tangent to Circle Radius r
Centered at the Origin, through given point

\[
Y = \frac{-X \cdot r \cdot x + r \cdot x^2 + r \cdot y^2 - X \cdot y \cdot \sqrt{r^2 + x^2 + y^2}}{r \cdot y \cdot x \cdot \sqrt{r^2 + x^2 + y^2}}
\]
Example 108: Intersection of two tangents to the curve \( y=x^2 \)

We create the point \((x,x^2)\) and draw its locus as \(x\) goes from -3 to 3. Now we create two tangents point proportional along the curve, and examine their intersection.

The x coordinate of the intersection is the average and the y coordinate the product of the x coordinates of the tangent points.
Transforms
Here are some examples using transforms:
**Example 109: Composition of reflections in parallel lines**

If we have a line with equation $\cos(t)X + \sin(t)Y + C = 0$, and a second line parallel and distance $a$ away, and if we reflect a point in the first line, then reflect its image in the second, the result is equivalent to a translation of twice $a$:

$$\Rightarrow (x_0 + 2a\cdot\cos(t), y_0 + 2a\cdot\sin(t))$$
Example 110: Combining Reflections

Reflection in two lines through the origin is equivalent to rotation about the origin of twice the angle between the lines:

\[
(y \cdot \sin(2 \cdot \theta) + x \cdot \cos(2 \cdot \theta), -x \cdot \sin(2 \cdot \theta) + y \cdot \cos(2 \cdot \theta))
\]
Reflection in two parallel lines is equivalent to translation of twice the distance between the lines:

\[ (x, y) \Rightarrow (2u + x, 2v + y) \]
Example 111:  **Parabolic Mirror**  
A parabolic mirror focuses parallel rays into a single point. Where is that point? We create the parabola \( y = a \cdot x^2 \) and reflect a ray parallel to the y axis in the tangent to the curve. We examine the y intercept of the image:
Example 112: Billiards

In American billiards, you need to bounce your cue ball off 2 cushions before hitting the other ball. What is the length of this path? We can create the correct path by reflecting the target ball in the two cushions, drawing a line between cue ball and twice-reflected target, then reflecting this line back.

The path length is the same as the distance between cue ball and twice reflected target.

Note the y coordinate of point C. Of course, this is the critical thing to compute when playing the game.
Example 113: Areas and Dilatation
Create a triangle with sides length $a$, $b$ and angle $\theta$. Look at its area. Now Dilate by a factor of $k$. Look at the area of the transformed triangle.

\[ \Rightarrow \frac{a \cdot b \cdot \sin(\theta)}{2} \]

Notice it is $k^2$ times the original area.
Example 114: Translating a circle

A circle radius \( r \) is translated by the vector \((u,v)\). What is the length of the line joining the intersection points between the circle and its image?

\[
\Rightarrow \sqrt{4r^2-u^2-v^2}
\]
Some Constructions
We have the constructions Midpoint, Perpendicular Bisector, and Angle Bisector. Here are some examples using these:
Example 115: Equation of the Perpendicular Bisector of 2 Points

\[
Y = \frac{-2 \cdot X \cdot x_0 + x_0^2 + 2 \cdot X \cdot x_1 - x_1^2 + y_0^2 - y_1^2}{2 \cdot (y_0 - y_1)}
\]
Example 116:  Length of the Angle Bisector of a triangle
Given a triangle side lengths a, b, c, what is the length of a perpendicular bisector?

\[
\frac{a \cdot b}{b + c}
\]

\[
\frac{a \cdot c}{b + c}
\]

\[
\Rightarrow \sqrt{b \cdot 2 \cdot b \cdot c^2 + c^3 + c \cdot \left( -a^2 + b^2 \right)}
\]

\[
\Rightarrow \frac{b + c}{c}
\]
Some Mechanisms

Some examples involving mechanisms:
Example 117: A Crank Piston Mechanism

For crank length \( c \) and connecting rod length \( L \), we compute piston displacement:

\[
\Rightarrow |b^2 - a^2 \cdot \sin(t)^2 + a \cdot \cos(t)|
\]
**Example 118: A Quick Return Mechanism**

The crank DG operates a quick return mechanism whose end-effector is at point F. The formula shows the horizontal displacement of F in terms of t and the various parameters of the geometry: a, b, u, v:

\[
\begin{align*}
\Rightarrow b^2 & \left[ \frac{-u+\frac{a\cdot u}{\sqrt{1+u^2-2\cdot u\cdot \cos(t)}} - \frac{a\cdot \cos(t)}{\sqrt{1+u^2-2\cdot u\cdot \cos(t)}}}{\sqrt{1+u^2-2\cdot u\cdot \cos(t)}} \right]^2 - \frac{a\cdot \sin(t)}{\sqrt{1+u^2-2\cdot u\cdot \cos(t)}}
\end{align*}
\]
Example 119: Paucellier's Linkage
In Paucellier’s linkage, we look at the height of the end-effector:

We see this is invariant in t.

The mechanism was the first to convert purely rotary motion (of the crank AB) to exact linear motion of the end effector D.
Example 120: Harborth Graph

The Harborth graph looks like this:

It is the minimal known 4-regular matchstick graph. That is a planar graph, each of whose vertices are adjacent to 4 edges, and each edge being the same length (if you look closely at the above picture the lengths are not all the same).

Here is the accurate drawing from Geometry Expressions
Here is how this quite complicated diagram is put together. It is based on a mechanism that looks like this (which is then replicated by reflecting in AI and then in GF):

The trapezoid is 3 units on the long base and 2 units on the short base.

The angle EDA is tweaked until GF is approximately perpendicular to AI. This makes the reflected mechanisms join up nicely.

The appropriate value is 78.5743 degrees.
Loci

Here are some locus examples
Example 121: Circle of Apollonius

The Circle of Apollonius is the locus of points the ratio of whose distance from a pair of fixed points is constant:

\[ \frac{2 \cdot X \cdot a^2 + X^2 \cdot (1+k^2)}{2} + Y^2 \cdot (1+k^2) = 0 \]

What is the center and radius?
Example 122: A Circle inside a Circle

Points D and E are proportion $t$ along the radii AD and AC of the circle centered at the origin and radius $r$. The intersection of CD and DE traces a circle.

Show that it goes through the origin. What is the center of the circle? What is its radius?
Example 123: Another Circle in a Circle

More generally if D is proportion s along AC, we have the following circle:

$$r^2 - 2rs^2 - t^2 + 2r^2s^2 + X^2 \left(1 - 2st + s^2t^2\right) + Y^2 \left(1 - 2st + s^2t^2\right) + X^2 \left(-2rs + 2rst + 2rs^2 - 2rs^2 - t^2\right)$$

What is the center of this circle?

Can we find the radius of this – perhaps by copying the expression into an algebra system and working on it there?
Here is one approach, in Maple. First we substitute $Y=0$, then solve for $X$ to determine the x intercepts of the circle. The radius can be found by subtracting these and dividing by 2.

```maple
> subs(Y=0, -s^2*r^2+2*t*s^2*r^2+t^2*r^2-2*t^2*s*r^2+(-2*t*s+1+t^2*s^2)*X^2+(-2*t*s+1+t^2*s^2)*Y^2+(-2*t*r+2*t*s*r+2*t^2*s*r-2*t^2*s^2*r)*X = 0);
-\(s^2 r^2 + 2 t s^2 r^2 + t^2 r^2 - 2 t^2 s r^2 + (-2 t s + 1 + t^2 s^2) X^2 + (-2 t r + 2 t s r + 2 t^2 s r - 2 t^2 s^2 r) X = 0\)

> solve(%,X);
\(r (-t + s)/t s - 1, r (-t - s + 2 t s)/t s - 1\)

> (r*(-t+s)/(t*s-1)- r*(-t-s+2*t*s)/(t*s-1))/2;
\(r (-t + s)/2 (t s - 1), r (-t - s + 2 t s)/2 (t s - 1)\)

> simplify(%);
\(-r s (-1 + t)/t s - 1\)
Example 124: Ellipse as a locus
Here is the usual string based construction of an ellipse foci (-a,0) (a,0):

\[ \Rightarrow -L^4 + 4L^2 \cdot Y^2 + 4L^2 \cdot a^2 + X^2 \cdot (4L^2 - 16 \cdot a^2) = 0 \]
Example 125: Archimedes Trammel
A mechanism which generates an ellipse is Archimedes Trammel. The points C and E are constrained to run along the axes, while the distance between them is set to \(a-b\). We trace the locus of the point D distance \(b\) from E along the same line. This gives an ellipse with semi major axes \(a\) and \(b\):

\[
Y^2 - a^2 + X^2 - b^2 - a^2 \cdot b^2 = 0
\]
Example 126: An Alternative Ellipse Construction

Here is a construction (ascribed to Newton) which builds the ellipse from concentric circles radius equivalent to the semi major axes.

\[ Y^2 \cdot a^2 + X^2 \cdot b^2 - a^2 \cdot b^2 = 0 \]
Example 127: Another ellipse
This time take a circle and a point, and the location of all points equidistant from the circle and the point:

\[ 4 \cdot Y^2 \cdot r^2 + 4 \cdot a^2 \cdot r^2 - X^2 \cdot (16 \cdot a^2 + 4 \cdot r^2) = 0 \]
Example 128: “Bent Straw” Ellipse Construction
Here is another ellipse construction. Geometrically observe that the semi major axes are x-a and x+a. Can you verify this from the algebraic expression?

\[ \Rightarrow -a^4 + 2a^2b^2 - b^4 + X^2 \left( a^2 - 2a - b + b^2 \right) + Y^2 \left( a^2 + 2a - b + b^2 \right) = 0 \]
Example 129: Similar construction for a Hyperbola
If we do a similar construction, with the generating point outside the circle, we get a hyperbola:

\[ 4 \cdot Y^2 \cdot r^2 + 4 \cdot a^2 \cdot r^2 \cdot r^4 + X^2 \cdot (-16 \cdot a^2 + 4 \cdot r^2) = 0. \]
Example 130: Parabola as locus of points equidistant between a point and a line
Here is the equation of the parabola which is the locus of points equidistant from the point (-a,0) and the line X=a:

\[ y^2 + 4ax = 0 \]
Example 131:  **Squeezing a circle between two circles**

Take a circle radius 2a centered at (a,0) and a circle radius 4a centered at (-a,0). Now look at the locus of the center of the circle tangent to both.

\[ 8X^2 + 9Y^2 - 72a^2 = 0 \]

It’s an ellipse. From the drawing we can see that the semi major axis in the x direction is 3a. What is the semi major axis in the y direction?
Example 132: Rosace a Quatre Branches
This example comes from the September 2003 edition of the Casio France newsletter.

A line segment of length $a$ has its ends on the $x$ and $y$ axes. We create the locus of the orthogonal projection of the origin onto this segment. Apparently this curve was studied in 1723-1728 by Guido Grandi.

\[
X^6 + 3X^4Y^2 + 3X^2Y^4 + Y^6 - X^2Y^2a^2 = 0
\]
**Example 133: Lemniscate**

Given foci at (-a,0) and (a,0), the lemniscate is the locus of points the product of whose distance from the foci is $a^2$:

$$\Rightarrow -X^4 \cdot 2 \cdot X^2 \cdot Y^2 - Y^4 + 2 \cdot X^2 \cdot a^2 - 2 \cdot Y^2 \cdot a^2 = 0$$
Example 134: Pascal's Limaçon
Named after Etienne Pascal (1588-1651), father of Blaise.

\[ X^4 + 2X^2Y^2 - Y^4 + Y^2 - a^2 - 2X^3b - 2X^2Y^2b + 2Xa^2b - b^2 + X^2(a^2+b^2) = 0 \]
Example 135: Kulp Quartic
Studied by, you guessed it – Kulp, in 1868:

\[ X^2 \cdot Y^2 + Y^2 \cdot r^2 = 0 \]
Example 136: The Witch of Agnesi
Named after Maria Gaetana Agnesi (1748)

\[ X^2 \cdot Y + X^2 \cdot r + 4 \cdot Y \cdot r^2 - 4 \cdot r^3 = 0 \]
Example 137: Newton's Strophoid

\[ \Rightarrow X^2 - X^2\cdot Y^2 - Y^3 = 0 \]
Example 138:  MacLaurin's Trisectrix and other Such Like
A cubic derived from the intersection of two lines rotating at different speeds

\[ X^3 - X \cdot Y^2 + 3 \cdot X^2 \cdot a - Y^2 \cdot a = 0 \]
A similar construction can give a range of other curves. For example, a hyperbola:

\[ -3X^2 + Y^2 + 2Xa = 0 \]
Example 139: Trisectrice de Delange

\[ X^2 Y^2 + Y^4 - 4X^2 a^2 - 4Y^2 a^2 + 4a^4 = 0 \]
Example 140: “Foglie del Suardi”
Here is a cubic which can be drawn by a mechanism consisting of intersecting a particular radius with a particular chord of a circle.

\[ \Rightarrow -X^3 \cdot X \cdot Y^2 \cdot 2 \cdot X^2 \cdot a - 2 \cdot Y^2 \cdot a \cdot X \cdot a^2 = 0 \]
Example 141: A Construction of Diocletian

\[
\begin{align*}
X &= \frac{2 \cdot t^2}{4 + t^2} \\
Y &= \frac{t^3}{4 + t^2}
\end{align*}
\]

\[\Rightarrow -X^3 + 2 \cdot Y^2 \cdot X \cdot Y^2 = 0\]

Segment CF is defined to be congruent to GE. Diocletian used this construction to define a cubic curve.
Example 142: Kappa Curve
Studied by Gutschoven in 1662, the locus of the intersection between a circle and its tangent through the origin as the circle slides up the y-axis:

\[ X^4 + X^2 \cdot Y^2 - Y^2 \cdot r^2 = 0 \]
Example 143: **Kepler’s Egg**

An egg shape defined by projecting B onto AC, then back onto AB then back onto AC:

$$X^4 + 2X^2Y^2 + Y^4X^3 = 0$$
Example 144: Cruciform Curve

\[ X^2 - Y^2 + X^2Y^2 = 0 \]

this curve can be rewritten in the form: \[ \frac{1}{x^2} + \frac{1}{y^2} = 1 \]
Example 145: Locus of centers of common tangents to two circles
We take the locus as the radius $r$ of the left circle varies. The midpoints of all four common tangents lie on the same fourth order curve

$$4X^4 + 8X^2Y^2 + 12X^3a + a^2s^2Y^2 + (4a^2 - 4s^2)X^2 + (6a^3 + 4sa^2) = 0$$

We can use Maple to solve for the intersections with the x axis:

```maple
> solve(%,X);
```

218
a - s, a + s, \frac{1}{2} a, \frac{1}{2} a
Example 146: Steady Rise Cam Curve
Assuming a Flat Plate reciprocating follower, here is the cam curve for a linear rise of \( k^*t + c \). This is the Envelope of the line BE.

\[
\begin{align*}
X &= -k \cdot \sin(t) + (c + k \cdot t) \cdot \cos(t) \\
Y &= c \cdot \sin(t) + k \cdot t \cdot \sin(t) + k \cdot \cos(t)
\end{align*}
\]
**Example 147: Oscillating Flat Plate Cam**

Here is a cam curve for an oscillating flat plate cam follower, where the follower rise is linear in the cam angle: \( \text{rise} = u + t \cdot v \)

Given points: 
- \((0,0)\) 
- \((b,0)\)

Let the cam angle be \(t\) and the follower rise be \(u\) and \(v\).

The equations for \(X\) and \(Y\) are:

\[
\begin{align*}
X &= a \left( \frac{(1+2 \cdot \cos(t) \cdot v)}{(1+2 \cdot \cos(t) \cdot \sin(t) \cdot \cos(t))} \frac{(1+2 \cdot \sin(t) \cdot \cos(t)) \cdot \cos(t) + 2 \cdot (1+2 \cdot \cos(t) \cdot \sin(t) \cdot \cos(t)) \cdot \sin(t) \cdot \cos(t) \cdot \sin(t) \cdot \cos(t) \cdot v}{2 \cdot (1+2 \cdot \cos(t) \cdot \sin(t) \cdot \cos(t))} \right) \\
Y &= a \left( \frac{(1+2 \cdot \cos(t) \cdot v)}{(1+2 \cdot \cos(t) \cdot \sin(t) \cdot \cos(t))} \frac{(1+2 \cdot \sin(t) \cdot \cos(t)) \cdot \sin(t) + 2 \cdot (1+2 \cdot \cos(t) \cdot \sin(t) \cdot \cos(t)) \cdot \sin(t) \cdot \cos(t) \cdot \sin(t) \cdot \cos(t) \cdot v}{2 \cdot (1+2 \cdot \cos(t) \cdot \sin(t) \cdot \cos(t))} \right)
\end{align*}
\]

Where 
- \(|b| > 0\)
- \(|a| > 0\)
Example 148: A Cam Star

Based on the previous model, let's take the simple case where the follower angle is twice the cam angle:

\[
\begin{align*}
X &= \frac{3 \cdot a \cdot \cos(t)}{2} + \frac{a \cdot \cos(3 \cdot t)}{2} \\
Y &= \frac{3 \cdot a \cdot \sin(t)}{2} - \frac{a \cdot \sin(3 \cdot t)}{2}
\end{align*}
\]
Can we get an implicit definition of the curve? Yes.

\[ \Rightarrow X^5 + 3X^4\gamma^2 + 3X^2\gamma^4 + \gamma^6 - 12X^4a^2 + 84X^2\gamma^2a^2 - 12\gamma^4a^2 + 48X^2a^4 + 48\gamma^2a^4 - 64a^6 = 0 \]
Example 149: Ellipse as Envelope of Circles
Take the envelope of the circles whose centers lie on the x-axis and which have extrema which lie on the unit circle. We find it is an ellipse:

\[ -2X^2 + 2Y^2 = 0 \]
Example 150: Hyperbola as an envelope of circles
Take the envelope of a family of circles centered on a line and whose radius is an eccentricity times the distance from a focus.

\[ e^2 + y^2 - e^4 + x^2 \cdot (-1 + e^2) = 0 \]
Example 151: Hyperbola as an Envelope of Lines

We take the envelope of the perpendicular bisectors of the line CD as C traverses the circle AB.

The result is a hyperbola with foci A and B.

What happens if D lies inside the circle?
Example 152: Caustics in a cup of coffee

The Nephroid curve generated by reflecting a set of parallel rays in a circle, and then taking the envelope of the reflected rays:

\[ 64 X^6 + 192 X^4 Y^2 + 192 X^2 Y^4 + 64 \gamma^8 X^4 r^2 - 96 X^2 Y^2 r^2 + 48 \gamma^4 r^2 - 15 X^2 r^4 + 12 \gamma^2 r^4 r^6 = 0 \]
Example 153: A Nephroid by another route

The envelope of the circles whose centers lie on a circle and which are tangential to the
diameter form the same type of curve:

\[ 4 \cdot X^6 + 12 \cdot X^4 \cdot Y^2 + 12 \cdot X^2 \cdot Y^4 + 4 \cdot Y^6 - 12 \cdot X^4 \cdot a^2 - 24 \cdot X^2 \cdot Y^2 \cdot a^2 - 12 \cdot Y^4 \cdot a^2 + 12 \cdot X^2 \cdot a^4 + 15 \cdot Y^2 \cdot a^4 - 4 \cdot a^6 = 0 \]
Example 154: Tschirnhausen's Cubic

Studied by Ehrenfried Tschirnhausen in 1690, this is the caustic of a set of parallel rays perpendicular to the axis of a parabola:

\[ 108X^2 - 81Y + 72Y^2 - 16Y^3 = 0 \]
Example 155: Cubic Spline

This diagram shows an algorithm for constructing the cubic spline from its control points:

Point E is proportion $t$ along the line AB. Point F is proportion $t$ along BC. Point G is proportion $t$ along CD. Point H is proportion $t$ along EF. Point I is proportion $t$ along FG. Point J is proportion $t$ along HI. The spline curve is the locus as $t$ runs from 0 to 1.
**Example 156: A Triangle Spline**

We can create another spline curve from 3 control points ABC in the following way: Point D is located proportion $t$ along AB. Point E is located proportion $t$ along BC. We take the locus of the intersection of AE and CD:

\[
X = \frac{x_0 - 2t^2x_0 + t^2x_1 - t^3x_1 + t^2x_2}{1-t^2} \\
Y = \frac{y_0 - 2t^2y_0 + t^2y_1 - t^3y_1 + t^2y_2}{1-t^2}
\]

Copy the $x$ coordinate into Maple and differentiate to get:

```maple
> u := diff((x[`0`] - 2*x[`0`]*t + x[`0`]*t^2 + x[`1`]*t - x[`1`]*t^2 + x[`2`]*t^2 + x[`2`]*t^2) / (-t+1+t^2), t);
```

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231
**USING SYMBOLIC GEOMETRY**

\[
u := \frac{-2x_0 + 2x_0t + x_j - 2x_jt + 2x_2t}{-t + 1 + t^2} - \frac{(x_0 - 2x_0t + x_0t^2 + x_1t - x_jt^2 + x_2t^2)(-1 + 2t)}{(-t + 1 + t^2)^2}
\]

Substituting \(t=0\) and \(t=1\):

\[
> \text{subs}(t=0,u);
\]

\[-x_0 + x_j
\]

\[
> \text{subs}(t=1,u);
\]

\[-x_j + x_2
\]

Comparable result for \(y\) shows that the curve is tangent to the control triangle at the end points
Example 157: Another Triangle Spline

We can also create a spline from a control triangle by taking the locus of a point $G$ proportion $t$ along $DE$.

Observing the parametric form of the curves we see that one is a parametric quadratic, while the other is a rational quadratic. Implicit forms are both conics (and almost, but not quite, identical).

\[
\begin{align*}
X &= 2b\cdot t + t^2(a - 2b) \\
Y &= 2c\cdot t(1-t)
\end{align*}
\]

\[4Yabc + 4X^2c^2 - 4Xac^2 + Y^2(a^2 - 4ab + 4b^2) + X\cdot Y(4ac - 8bc) = 0\]

\[
\begin{align*}
X &= \frac{t(b + a\cdot t - b\cdot t)}{1+t^2} \\
Y &= \frac{c\cdot t^2}{1+t^2}
\end{align*}
\]

\[Yabc + X^2c^2 - Xac^2 + Y^2(a^2 - a\cdot b + b^2) + X\cdot Y(ac - 2b\cdot c) = 0\]
What types of conics are they? Extending the curves a little can give a clue:

The blue curve looks like a parabola, the red certainly does not.

Copying the blue curve equation into Maple and examining the quadratic form shows that it is indeed a parabola:

```maple
> 4*c*b*a*Y+4*c^2*X^2-4*c^2*a*X+(a^2-4*b*a+4*b^2)*Y^2+(4*c*a-8*c*b)*Y*X = 0;
```

```maple
> <<4*c^2 |(4*c^2*a-8*c*b)/2>,<(4*c^2*a-8*c*b)/2 |(a^2-4*b*a+4*b^2)>>;
```
\[
\begin{array}{ccc}
\hat{e} & 4c^2 & 2ca - 4cb \\
\hat{e} & 2ca - 4cb & a^2 - 4ba + 4b^2
\end{array}
\]

\[> \text{Determinant(\%)}; \]
\[0\]

How about the red curve:

\[> c^2b^2aY+c^2X^2-c^2aX+(a^2-ba+b^2)Y^2+(ca-2cb)YX = 0; \]
\[cbay+c^2X^2-c^2aX+(a^2-ba+b^2)Y^2+(ca-2cb)YX = 0\]

\[> \langle c^2 | (ca-2cb)/2 \rangle, \langle (ca-2cb)/2 \rangle | (a^2-ba+b^2) \rangle; \]
\[
\begin{array}{ccc}
\hat{e} & c^2 & \frac{1}{2} ca - cb \\
\hat{e} & \frac{1}{2} ca - cb & a^2 - ba + b^2
\end{array}
\]

\[> \text{Determinant(\%)}; \]
\[\frac{3}{4} c^2 a^2\]

We see that the determinant is positive. This means we will always have a portion of an ellipse.