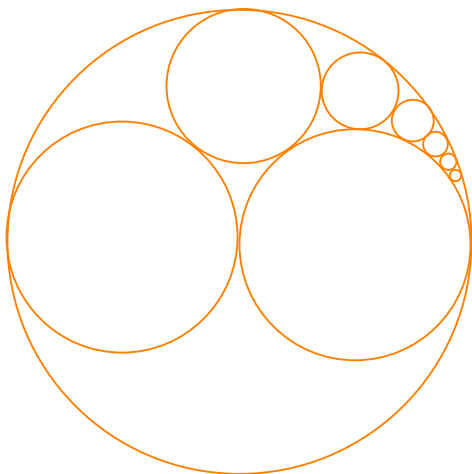


From Inductive Reasoning to Proof by Induction

A Geometry Expressions Lesson

MATHEMATICS



We use Geometry Expressions to calculate the radii of circles in a particular Pappus Chain. We will use inductive reasoning to conjecture a general form for this sequence of radii.

We will then use mathematical induction to prove the formula for a general member of the Pappus Chain.

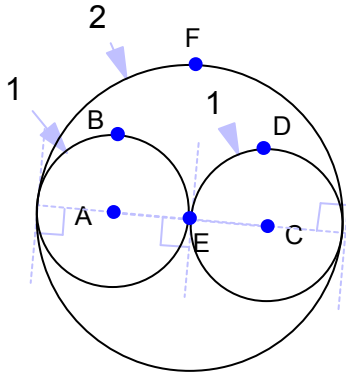
TECHNOLOGY

This lesson only uses 5 Geometry Expressions commands, and is therefore highly suitable for the beginning user. Here are the 5 commands:

- **Draw / Circle**
- **Constrain / Radius**
- **Constrain / Tangent**
- **Calculate / Radius**
- **View / Scale to Selection**

Inductive Reasoning

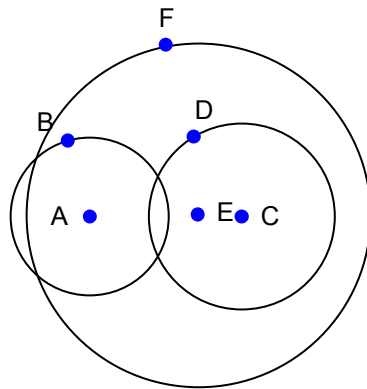
A Pappus Chain is a sequence of mutually tangent circles. We will study the radii of one such chain, while learning to use Geometry Expressions. First, we create our starting point:



Here is how to do that:

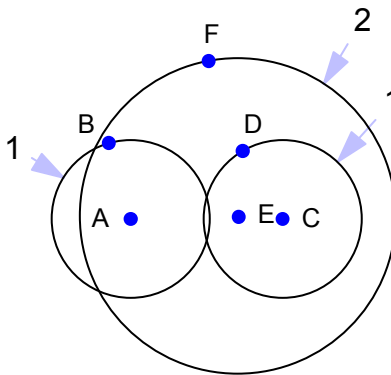
Sketch a first circle using **Draw / Circle**: click the center and drag a radius point.

Now sketch two more circles roughly configured as above (don't worry about getting it exactly right; that is what constraints are for). Your drawing should now look like:



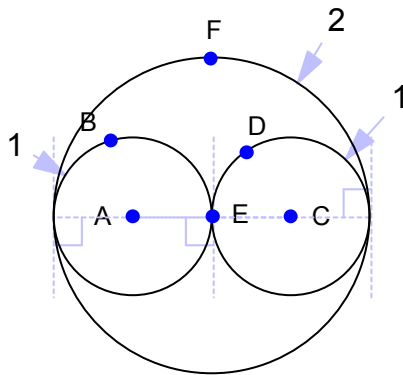
Select circle AB, then **Constrain / Radius**. Specify its radius to be 1.

Similarly constrain Circle CD to have radius 1, and circle EF to have radius 2. Your drawing should look like this:

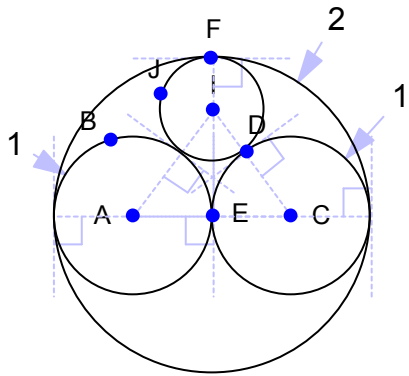


Select circle AB, hold the Shift key down and select circle CD. Now **Constrain / Tangent**

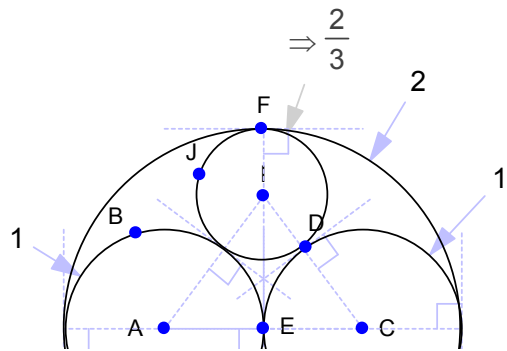
Similarly constrain circle AB to be tangent to circle EF and circle CD to be tangent to circle EF. Your drawing should now look something like this:



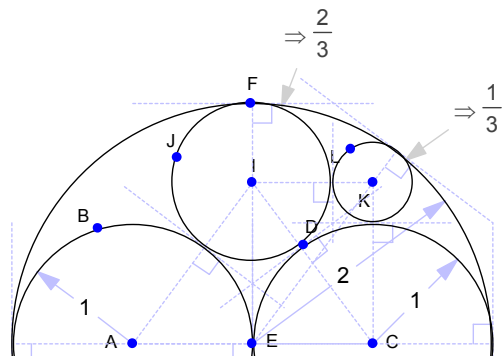
We now want to add a fourth circle inside circle EF but outside circles AB and CD, and constrain it to be tangent to all three circles. Your drawing should look like this:



Select the new circle and use **Calculate / Radius** to get Geometry Expressions to display the new circle's radius. (Notice the arrow sign in front of the radius, which indicates that the quantity was calculated by Geometry Expressions, not inputted by you).



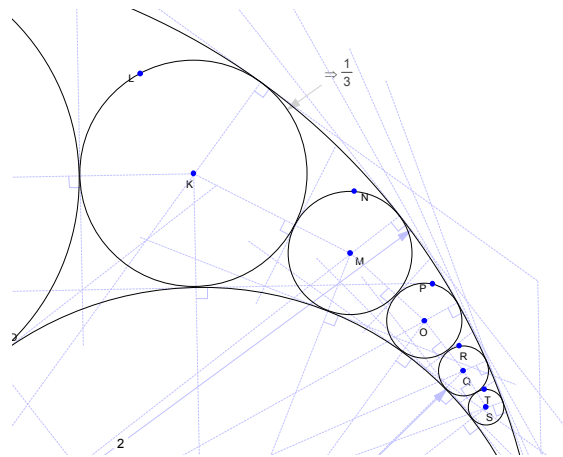
Now, repeat the procedure to create a Pappus Chain; each new circle should be tangent to circle CD and circle EF, and to the previous circle in the chain. Here is the next step:



What do you think the next value will be? Take a guess. Fill the guess into the table below, then create another circle and see if your guess is correct.

n	1	2	3	4	5	6	7	8
guess								
actual	$\frac{2}{3}$	$\frac{1}{3}$						

Now draw circles 4, 5, and 6 in our chain, each time filling in a guess before you measure the actual value from the model. (If you click **View / Scale Geometry to Selection** then drag a rectangle, you can focus in on the required small area to draw your circles).



Now come up with a guess for the 7th circle, but don't measure it yet.

Hint

Would the sequence be easier to deal with if all the radii had the same numerator?
How could you make them all have the same numerator?

Did the hint help you? Calculate the radius of the 7th circle and see if your guess was right. Hopefully, by now you can fill in the radius of the 8th circle.

Can you come up with a formula involving n, which gives the actual radius for each n in the table?

Write it here: _____

Proof by Induction

In the previous section you should have shown that for the first several values of n , the radius of the n^{th} circle is:

$$\frac{2}{n^2 + 2}$$

Now we may strongly suspect that this is true for any value of $n > 1$; however we have not proved it. A powerful method of proof for this type of proposition is mathematical induction. To apply mathematical induction to some proposition $P(n)$, we need to show:

1. $P(1)$ is true.
2. if $P(n-1)$ is true, then $P(n)$ is true.

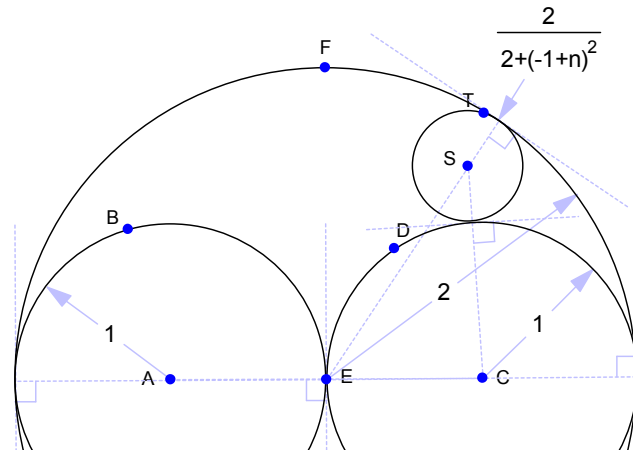
Then, by induction, $P(N)$ is true for any $N > 0$.

$P(n)$ for us is the proposition that the n^{th} circle in our chain has radius $\frac{2}{n^2 + 2}$.

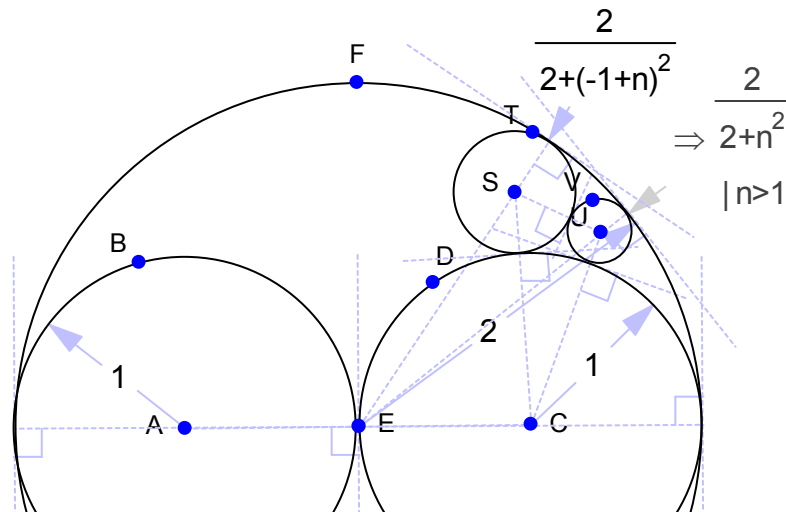
$P(1)$ is the proposition that the first circle in the chain has radius $\frac{2}{1^2 + 2} = \frac{2}{3}$, which we have already shown.

We need to prove that if the $(n-1)^{\text{st}}$ circle has radius $\frac{2}{(n-1)^2 + 2}$, then the n^{th} circle has radius $\frac{2}{n^2 + 2}$.

To do this, remove all circles except the original 3, and now create a generic member of the sequence. It should be tangential only to CD and EF. Then constrain the radius of this new circle to be $\frac{2}{(n-1)^2 + 2}$. This is the $(n-1)^{\text{st}}$ circle. Your drawing should look like this:



Now add the next circle in the sequence, and get Geometry Expressions to calculate its radius.



This shows that $P(n-1)$ implies $P(n)$ and hence by induction the proposition is true for any positive integer n .

(Note: if you get a more complicated expression involving an absolute value, you should select the expression and **Properties** from the right click menu. Change **Use Assumptions** to **true**)

Extensions

1. Try starting with circles radii 1, 2 and 3.

n	1	2	3	4	5	6
guess						
actual						

General formula: _____

2. Generalize by starting with circles radii r , s , and $r+s$.

n	1	2	3	4
guess				
actual				

General formula: _____