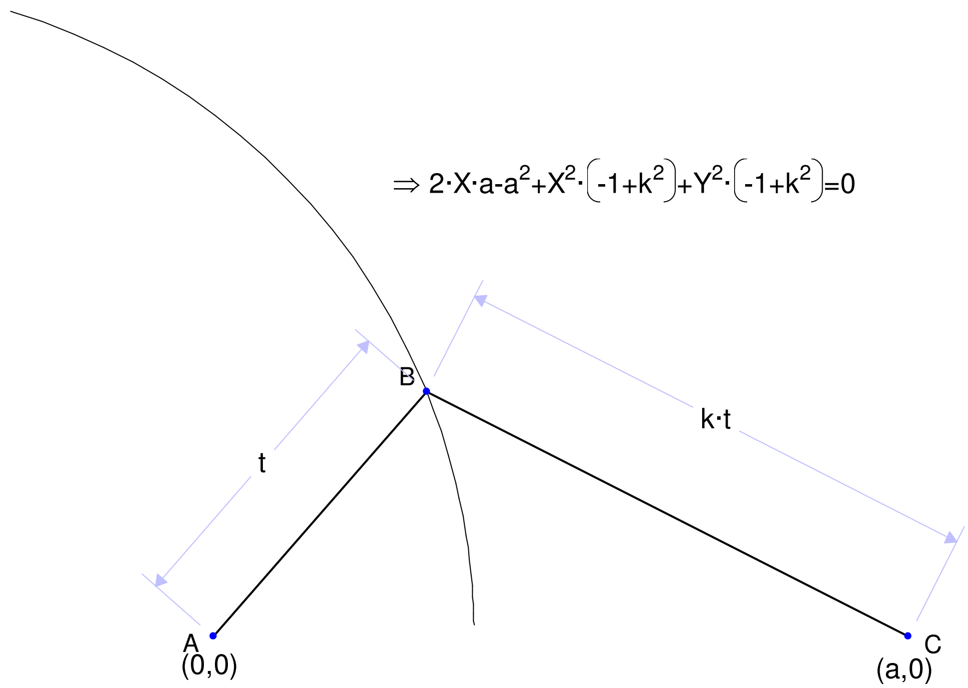
# Loci

Here are some locus examples

### Example 121: Circle of Apollonius

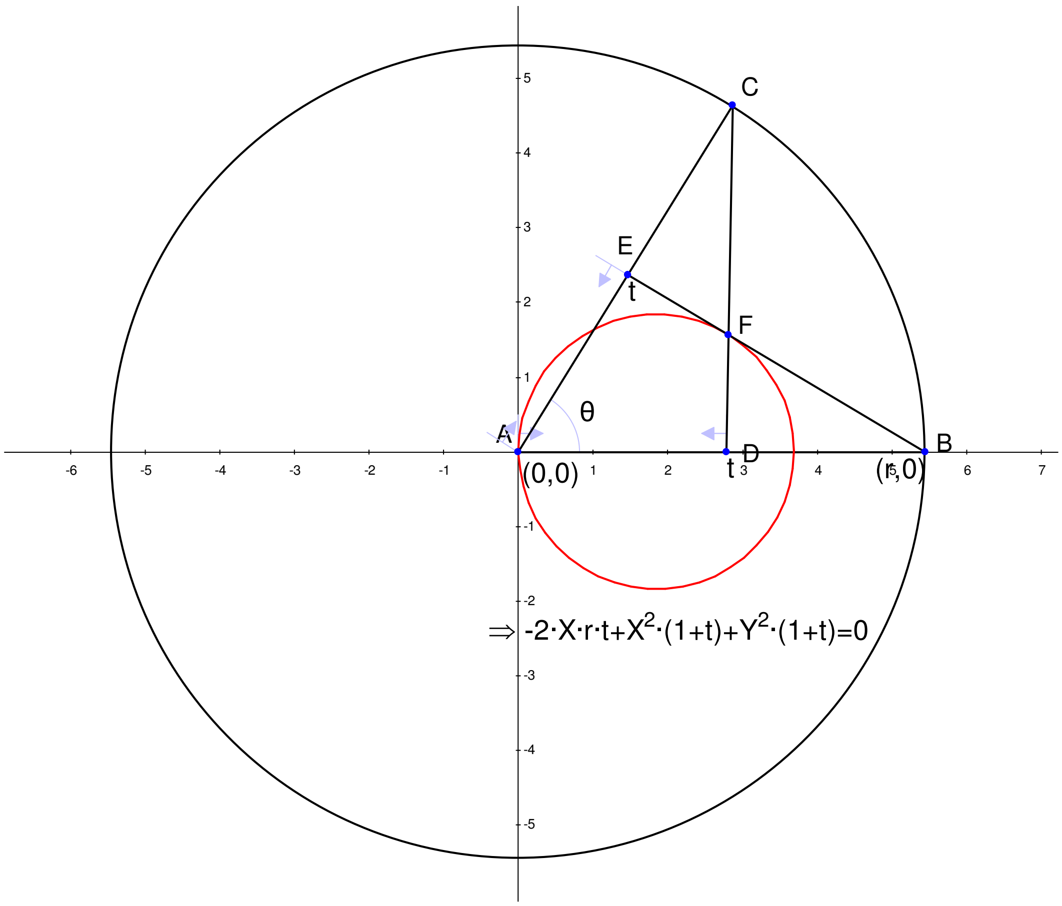
The Circle of Apollonius is the locus of points the ratio of whose distance from a pair of fixed points is constant:




What is the center and radius?

## Example 122: A Circle inside a Circle

Points D and E are proportion t along the radii AD and AC of the circle centered at the origin and radius r. The intersection of CD and DE traces a circle.

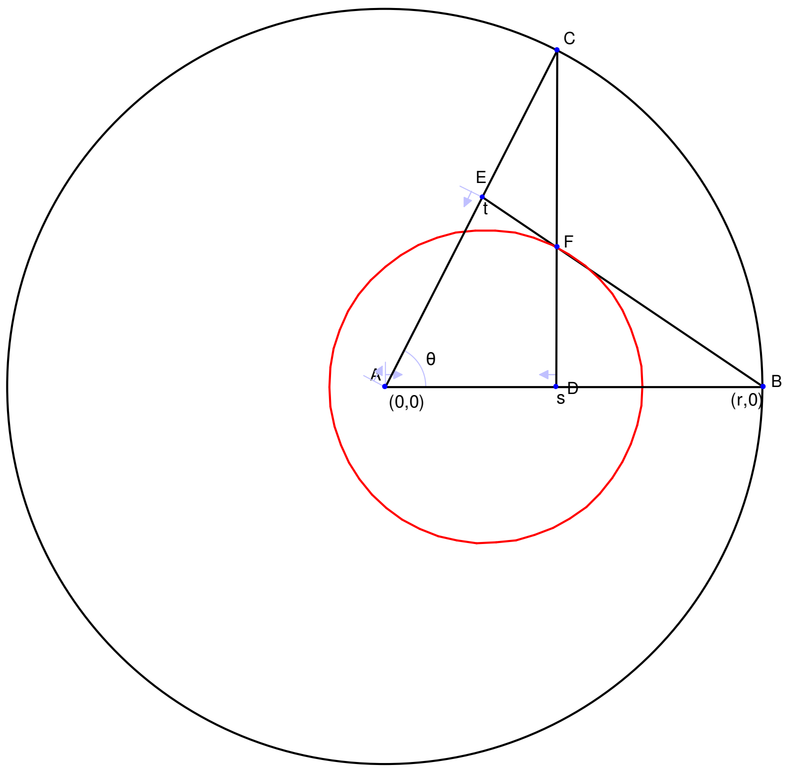



Show that it goes through the origin. What is the center of the circle? What is its radius?

### Example 123: Another Circle in a Circle

More generally if D is proportion s along AC, we have the following circle:

$$\Rightarrow r^2\cdot s^2-2\cdot r^2\cdot s^2\cdot t-r^2\cdot t^2+2\cdot r^2\cdot s\cdot t^2+X^2\cdot\left(1-2\cdot s\cdot t+s^2\cdot t^2\right)+Y^2\cdot\left(1-2\cdot s\cdot t+s^2\cdot t^2\right)+X\cdot\left(-2\cdot r\cdot s+2\cdot r\cdot s\cdot t+2\cdot r\cdot s^2\cdot t-2\cdot r\cdot s^2\cdot t^2\right)=0$$




What is the center of this circle?

Can we find the radius of this – perhaps by copying the expression into an algebra system and working on it there?

Here is one approach, in Maple. First we substitute Y=0., then solve for X to determine the x intercepts of the circle. The radius can be found by subtracting these and dividing by 2.

```
> subs(Y=0,-s^2*r^2+2*t*s^2*r^2+t^2*r^2-2*t^2*s*r^2+(-
2*t*s+1+t^2*s^2)*X^2+(-2*t*s+1+t^2*s^2)*Y^2+(-
2*t*r+2*t*s*r+2*t^2*s*r-2*t^2*s^2*r)*X = 0);
```

$$-s^2 r^2 + 2\,t\,s^2 r^2 + t^2 r^2 - 2\,t^2 s\,r^2 + (-2\,t\,s + 1 + t^2 s^2)\,X^2 + (-2\,t\,r + 2\,t\,s\,r + 2\,t^2 s\,r - 2\,t^2 s^2 r)\,X = 0$$

```
> solve(%,X);
```

$$\frac{r\,(-t+s)}{t\,s-1},\ \frac{r\,(-t-s+2\,t\,s)}{t\,s-1}$$

```
> (r*(-t+s)/(t*s-1)- r*(-t-s+2*t*s)/(t*s-1))/2;
```

$$\frac{r\,(-t+s)}{2\,(t\,s-1)} - \frac{r\,(-t-s+2\,t\,s)}{2\,(t\,s-1)}$$

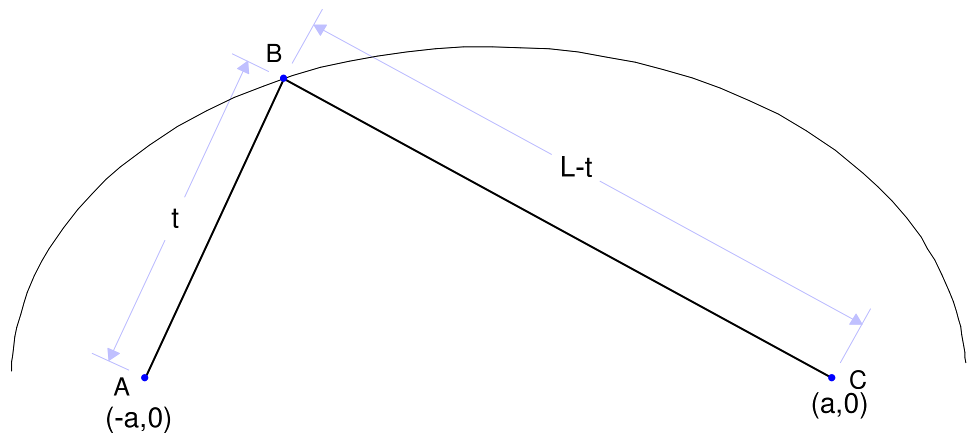
```
> simplify(%);
```

$$-\frac{r\,s\,(-1+t)}{t\,s-1}$$

## Example 124: Ellipse as a locus

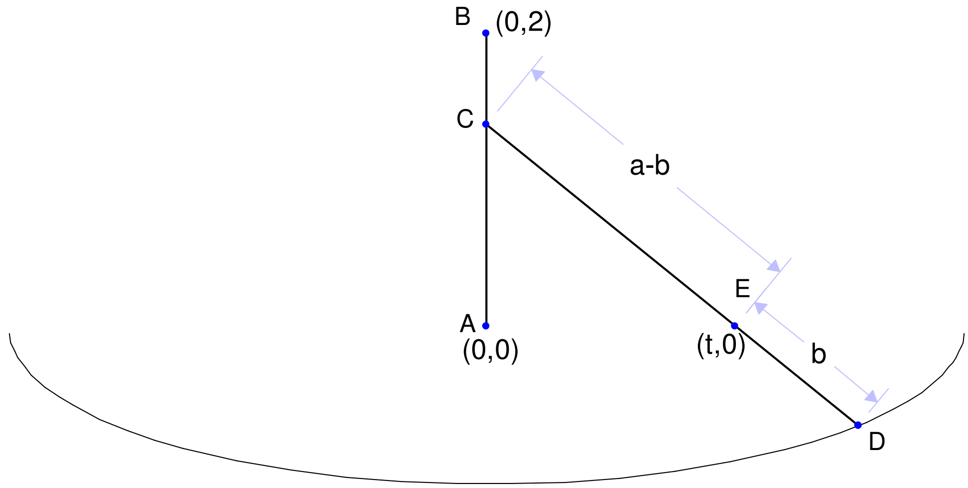
Here is the usual string based construction of an ellipse foci (-a,0) (a,0):

$$\Rightarrow -L^4+4\cdot L^2\cdot Y^2+4\cdot L^2\cdot a^2+X^2\cdot\left(4\cdot L^2-16\cdot a^2\right)=0$$



### Example 125: Archimedes Trammel

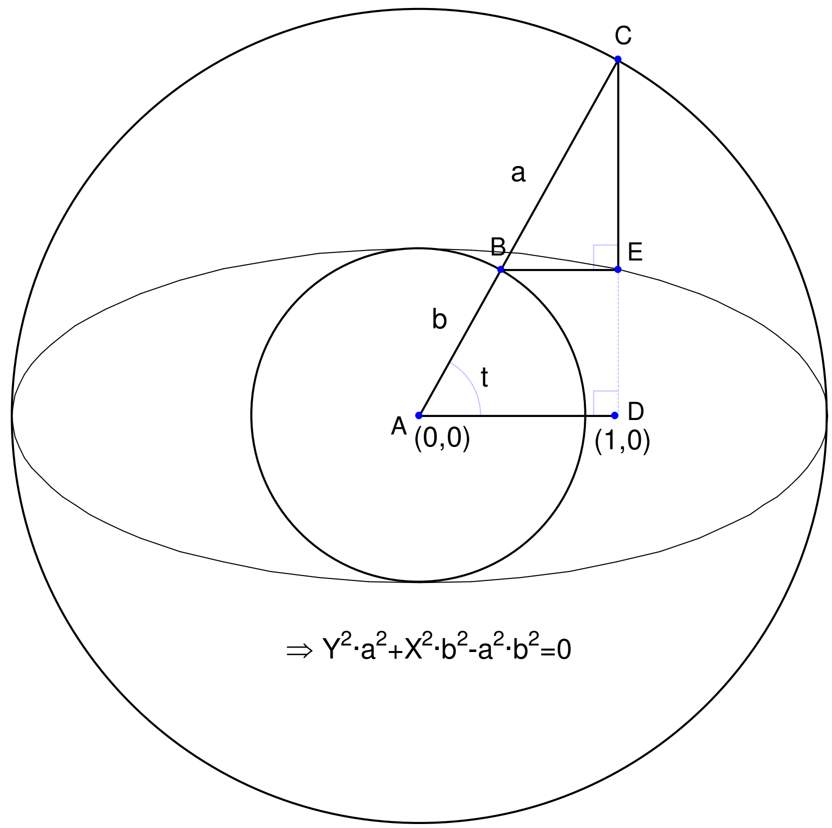
A mechanism which generates an ellipse is Archimedes Trammel. The points C and E are constrained to run along the axes, while the distance between them is set to a-b. We trace the locus of the point D distance b from E along the same line. This gives an ellipse with semi major axes a and b:




$$\Rightarrow Y^2 \cdot a^2 + X^2 \cdot b^2 - a^2 \cdot b^2 = 0$$

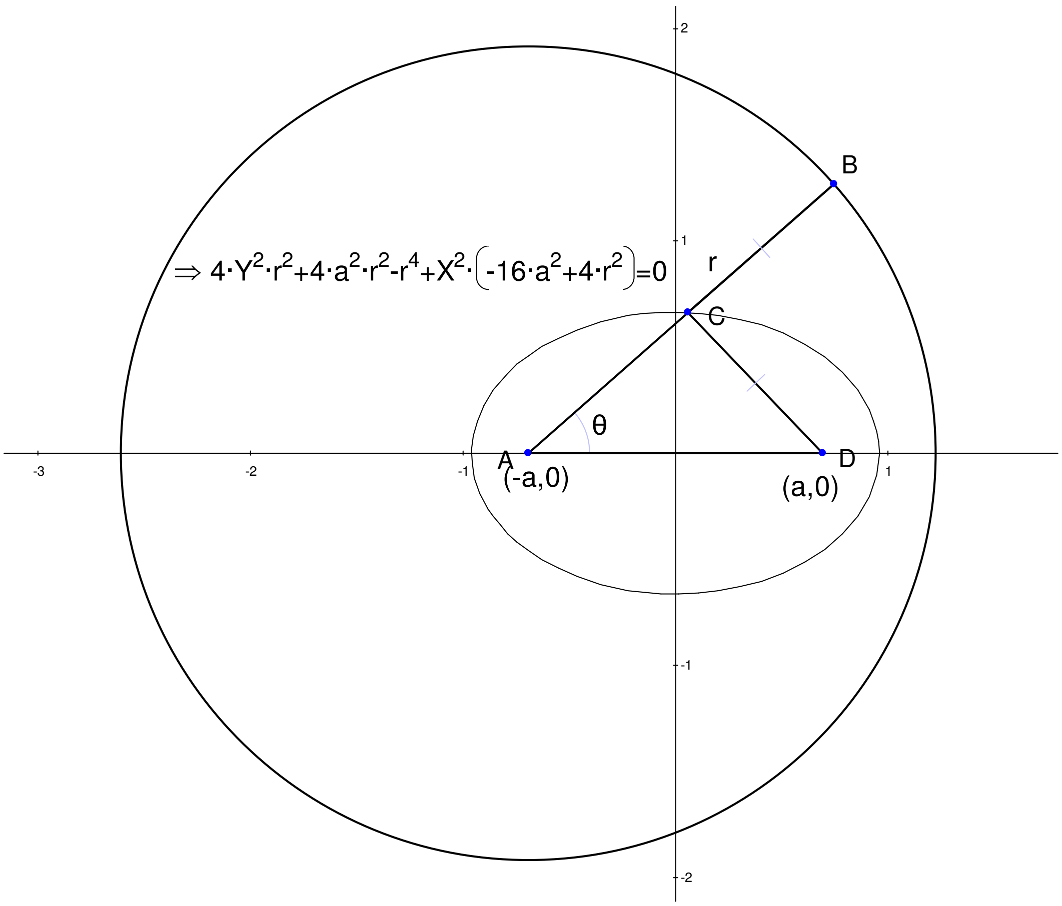
## Example 126: An Alternative Ellipse Construction

Here is a construction (ascribed to Newton) which builds the ellipse from concentric circles radius equivalent to the semi major axes



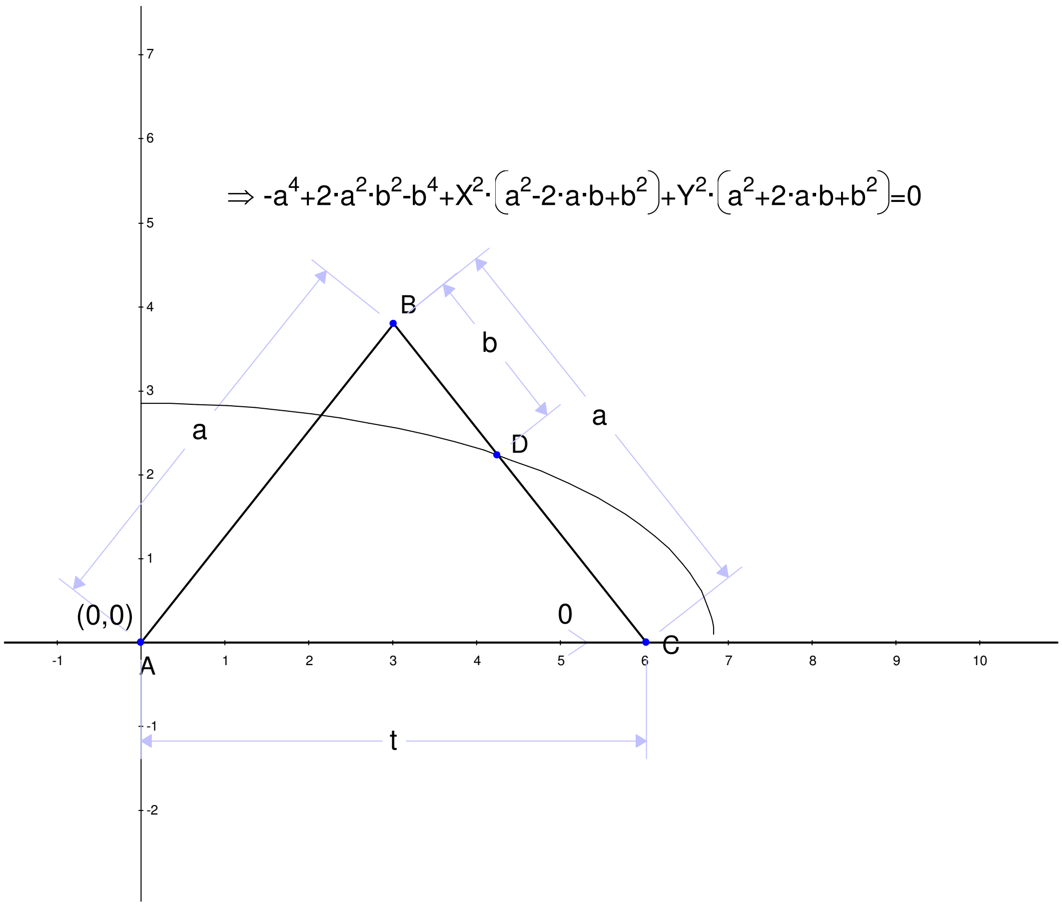
## Example 127: Another ellipse

This time take a circle and a point, and the location of all points equidistant from the circle and the point:



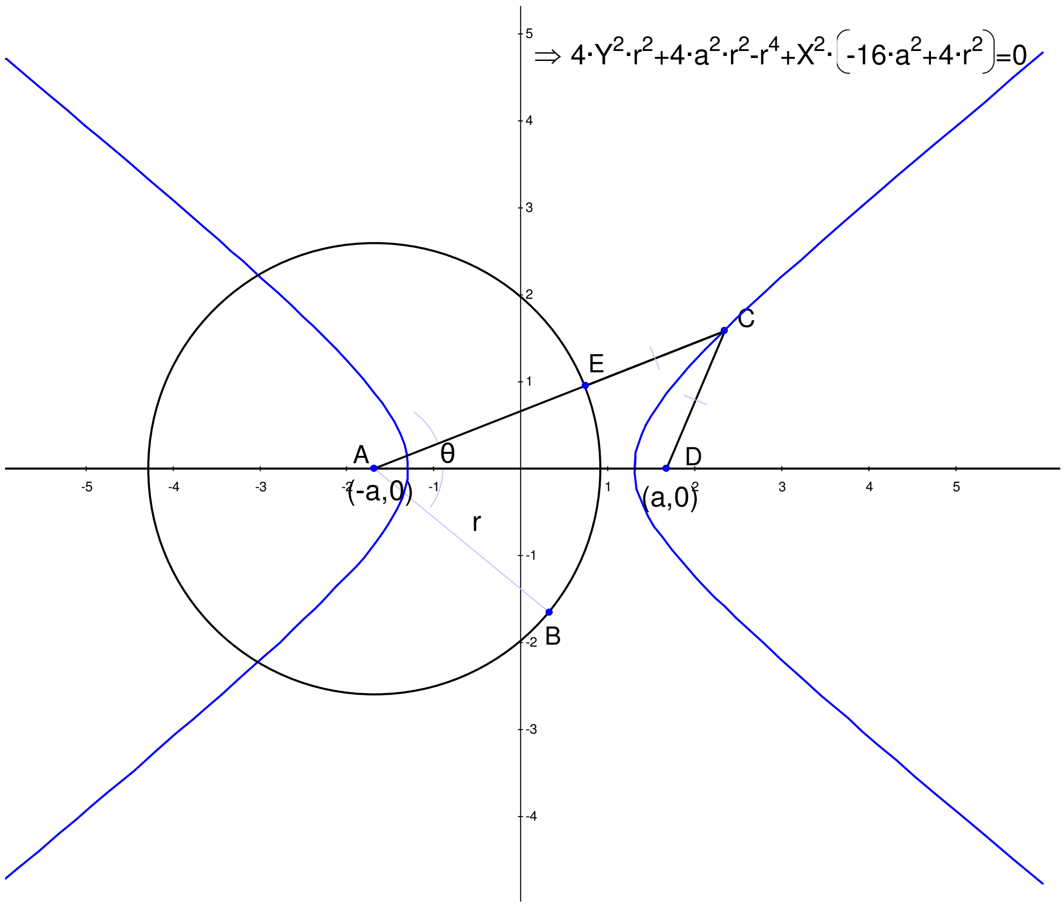
## Example 128: “Bent Straw” Ellipse Construction

Here is another ellipse construction. Geometrically observe that the semi major axes are x-a and x+a. Can you verify this from the algebraic expression?



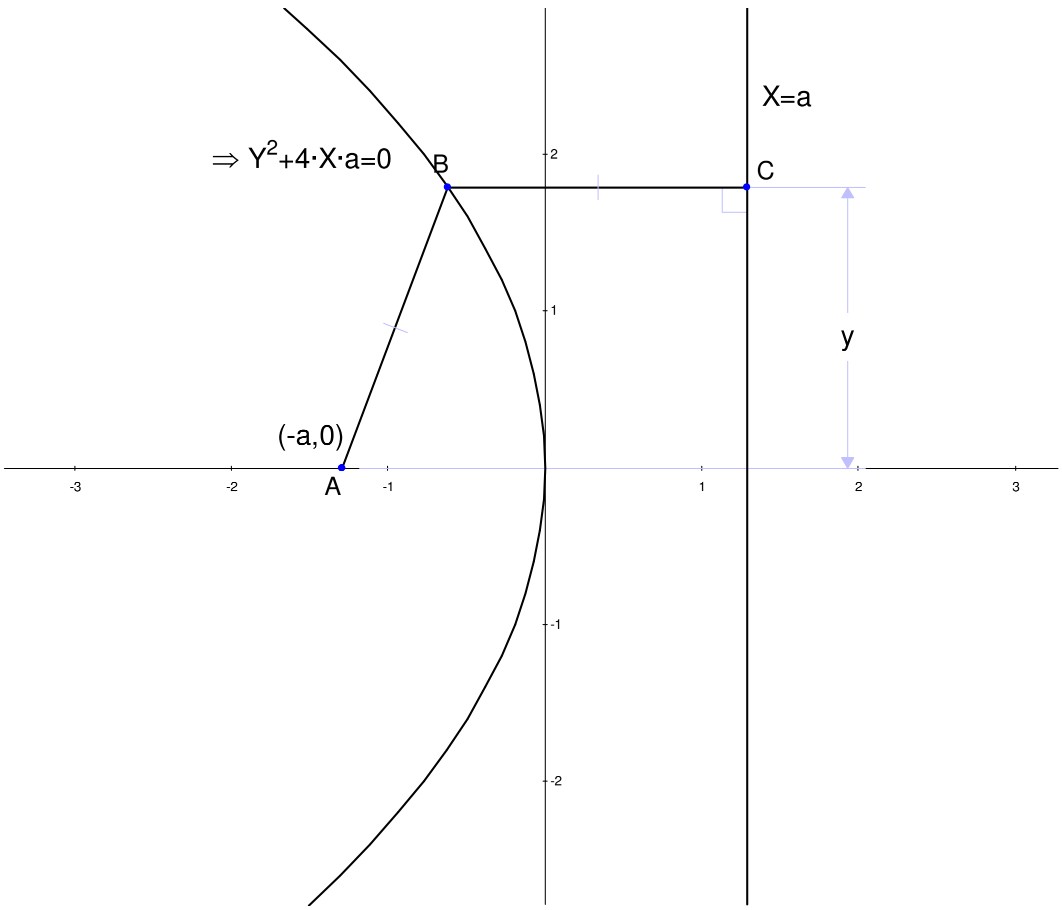
### Example 129: Similar construction for a Hyperbola

If we do a similar construction, with the generating point outside the circle, we get a hyperbola:



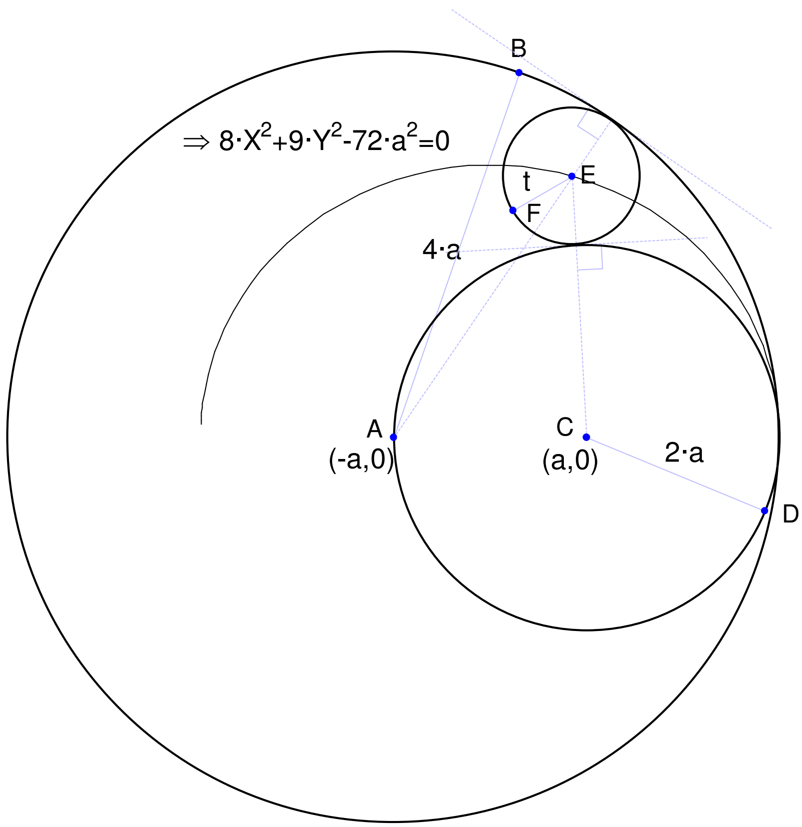
## Example 130: Parabola as locus of points equidistant between a point and a line

Here is the equation of the parabola which is the locus of points equidistant from the point (-a,0) and the line X=a:



### Example 131: Squeezing a circle between two circles

Take a circle radius 2a centered at (a,0) and a circle radius 4a centered at (-a,0). Now look at the locus of the center of the circle tangent to both.

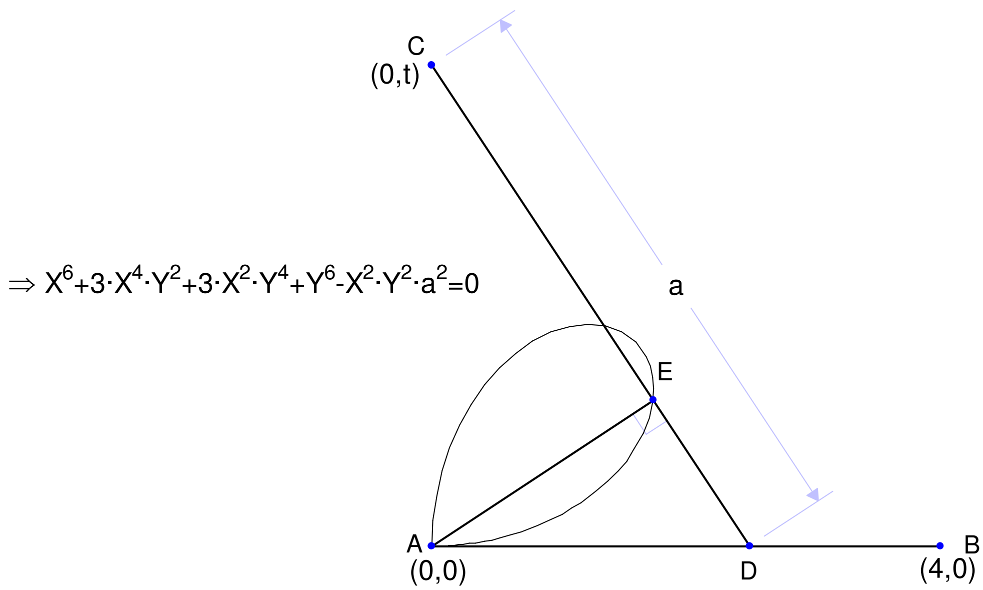



It's an ellipse. From the drawing we can see that the semi major axis in the x direction is 3a. What is the semi major axis in the y direction?

## Example 132: Rosace a Quatre Branches

This example comes from the September 2003 edition of the Casio France newsletter.

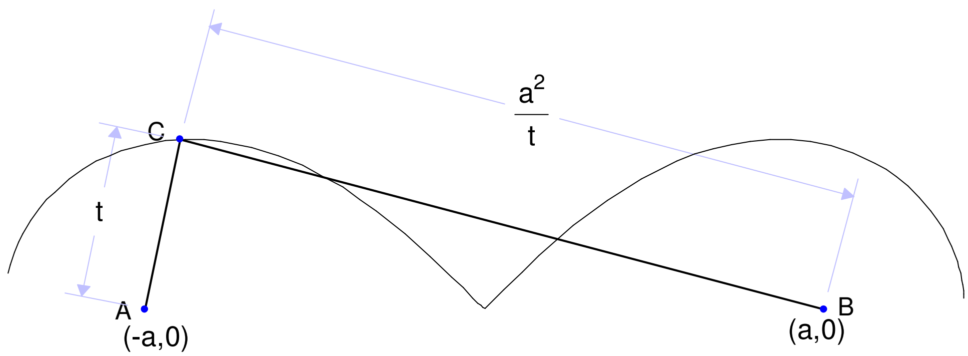
A line segment of length a has its ends on the x and y axes. We create the locus of the orthogonal projection of the origin onto this segment. Apparently this curve was studied in 1723-1728 by Guido Grandi.



### Example 133: Lemniscate

Given foci at (-a,0) and (a,0), the lemniscate is the locus of points the product of whose distance from the foci is a^2:

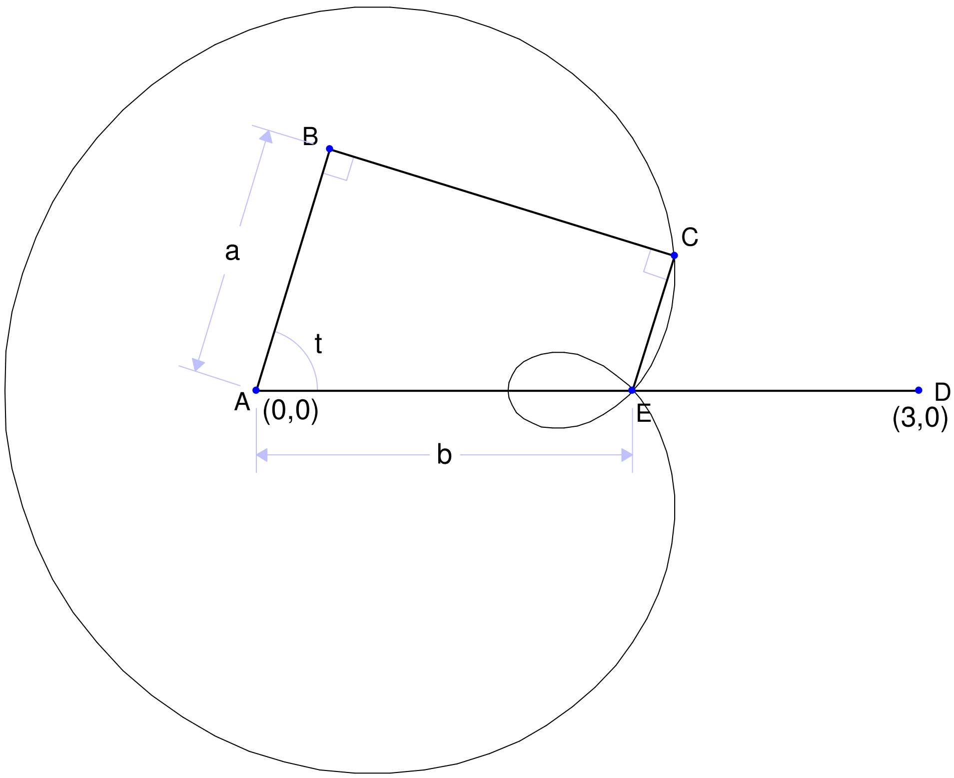
$$\Rightarrow -X^4-2\cdot X^2\cdot Y^2-Y^4+2\cdot X^2\cdot a^2-2\cdot Y^2\cdot a^2=0$$



## Example 134: Pascal's Limaçon

Named after Etienne Pascal (1588-1651), father of Blaise.

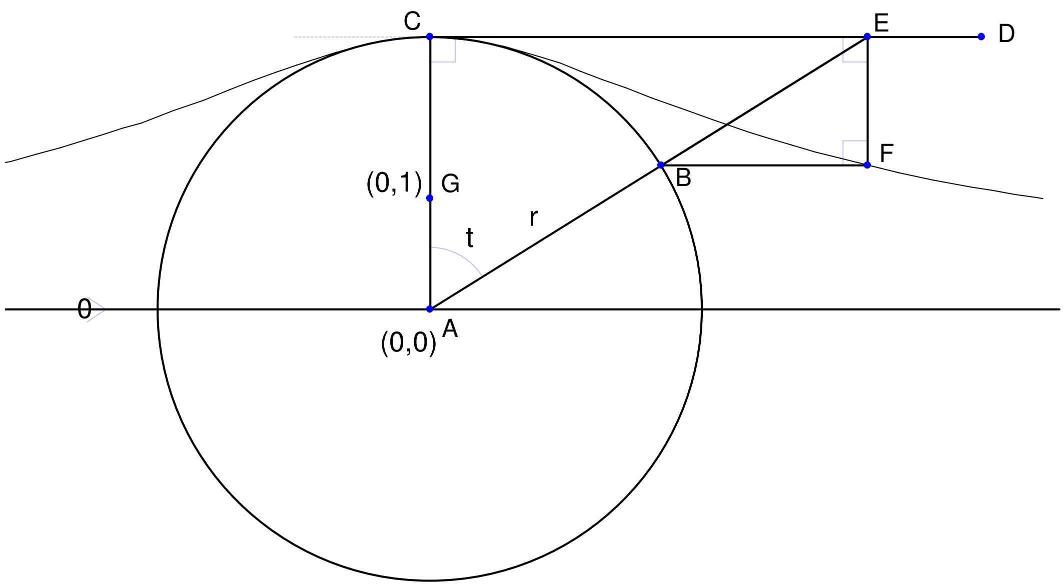
$$\Rightarrow X^4+2\cdot X^2\cdot Y^2+Y^4-Y^2\cdot a^2-2\cdot X^3\cdot b-2\cdot X\cdot Y^2\cdot b+2\cdot X\cdot a^2\cdot b-a^2\cdot b^2+X^2\cdot\left(-a^2+b^2\right)=0$$



## Example 135: Kulp Quartic

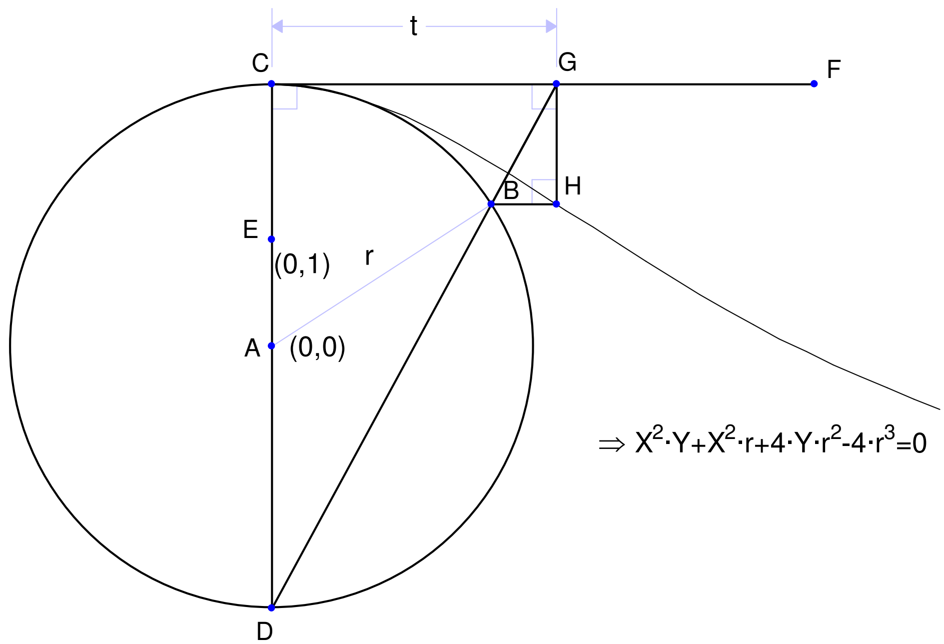
Studied by, you guessed it – Kulp, in 1868:

$$\Rightarrow X^2 \cdot Y^2 + Y^2 \cdot r^2 - r^4 = 0$$

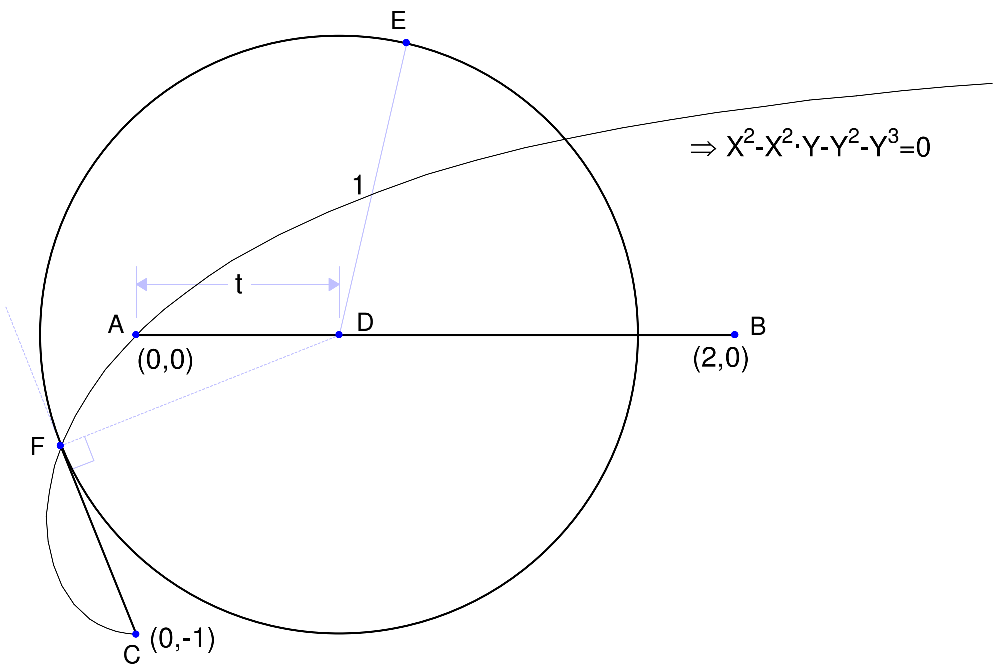


## Example 136: The Witch of Agnesi

Named after Maria Gaetana Agnesi (1748)

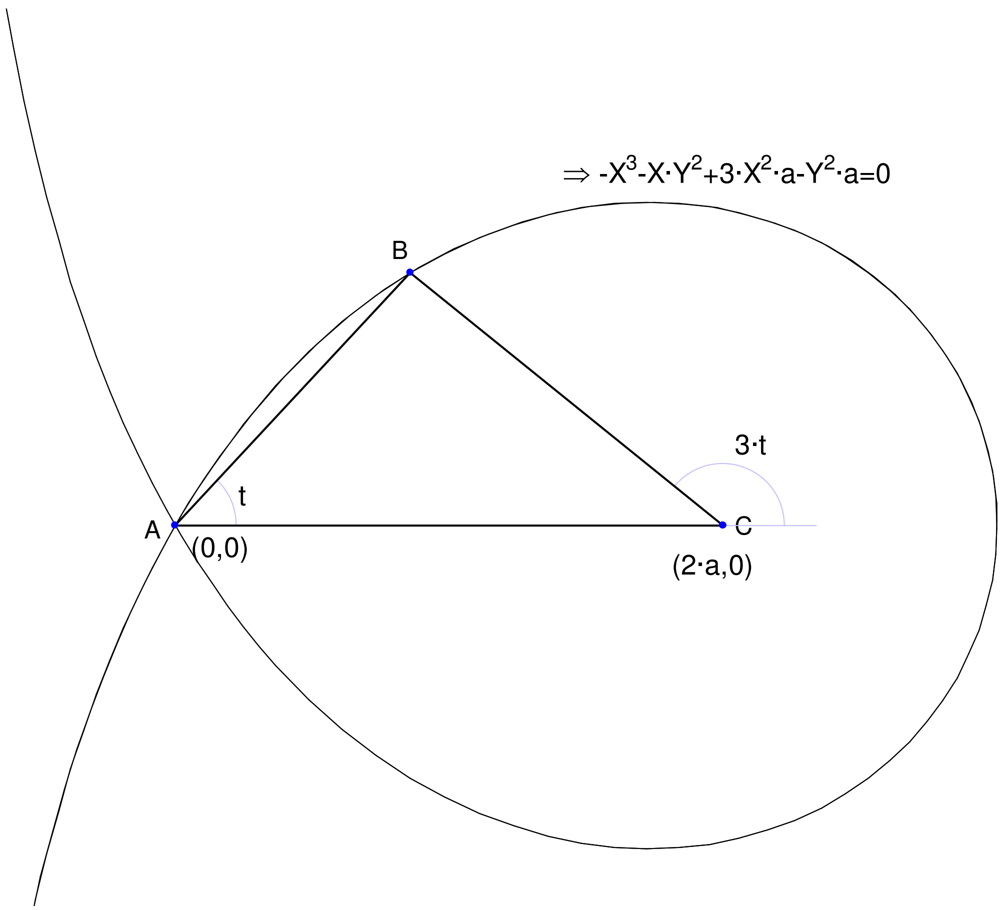


### Example 137: Newton's Strophoid

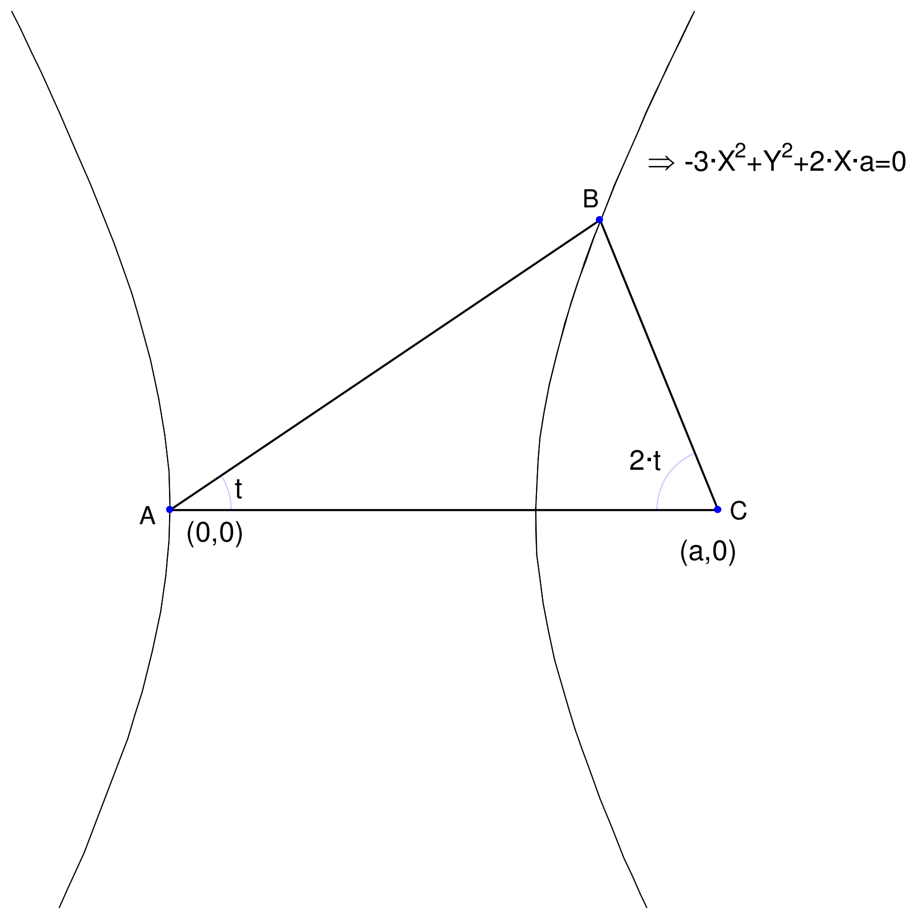


### Example 138: MacLaurin's Trisectrix and other Such Like

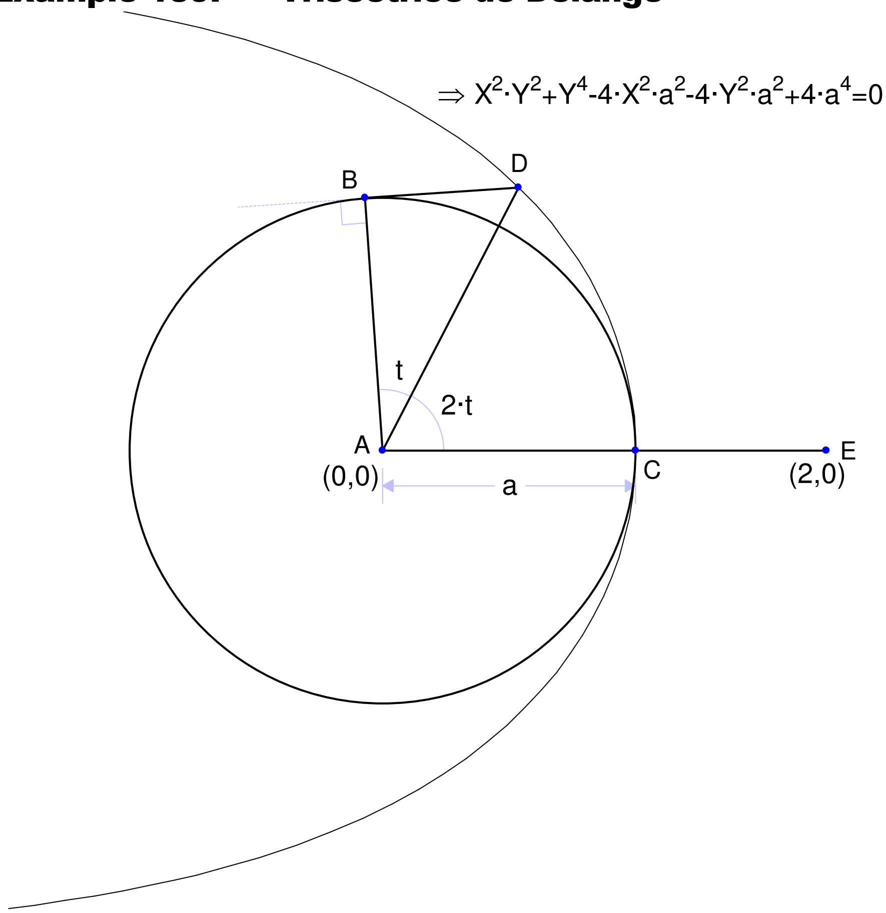
A cubic derived from the intersection of two lines rotating at different speeds



A similar construction can give a range of other curves.  For example, a hyperbola:

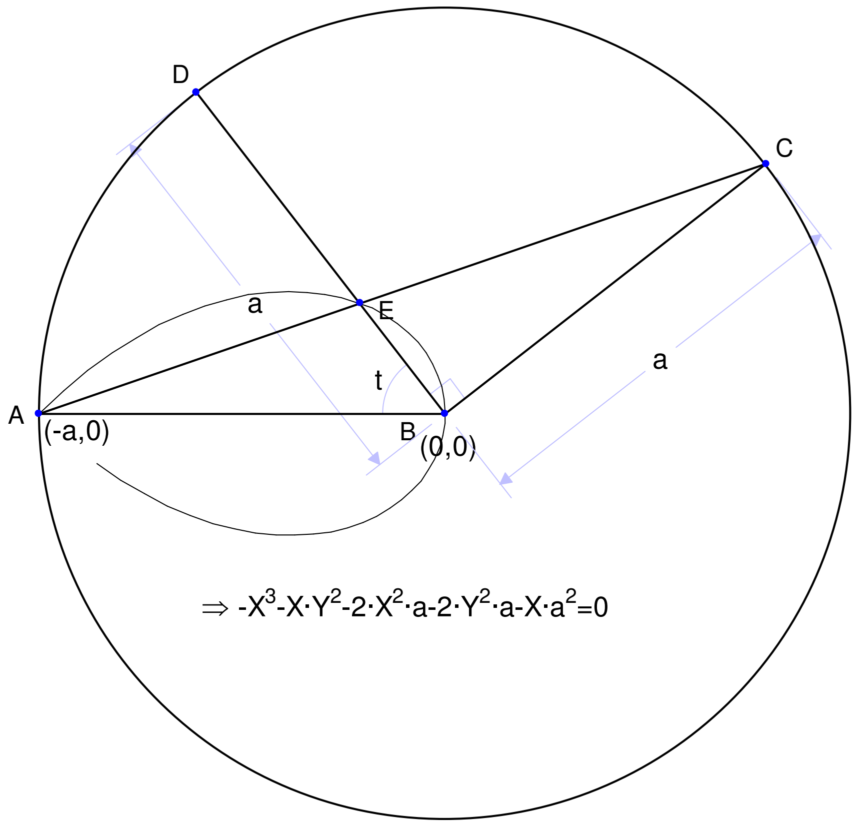


## Example 139: Trisectrice de Delange



### Example 140: “Foglie del Suardi”

Here is a cubic which can be drawn by a mechanism consisting of intersecting a particular radius with a particular chord of a circle.

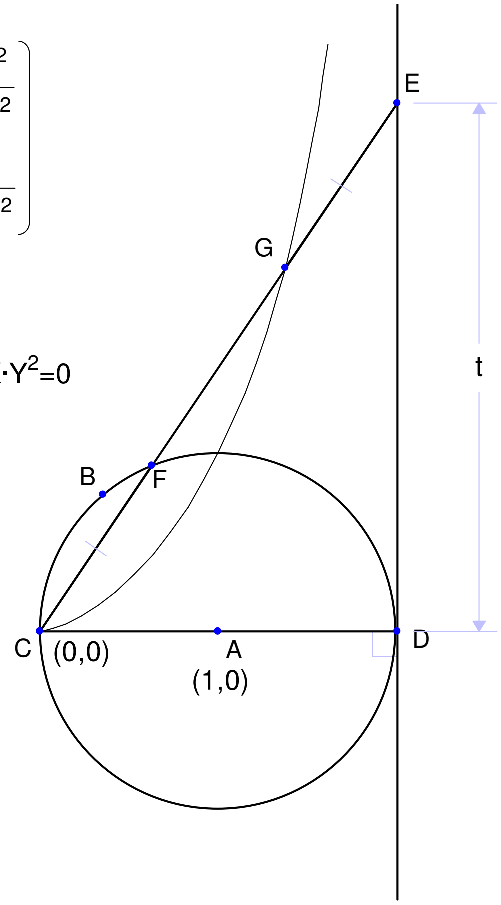



$$\Rightarrow -X^3 - X\cdot Y^2 - 2\cdot X^2\cdot a - 2\cdot Y^2\cdot a - X\cdot a^2 = 0$$

## Example 141: A Construction of Diocletian

$$\Rightarrow \begin{pmatrix} X=\dfrac{2\cdot t^2}{4+t^2} \\ Y=\dfrac{t^3}{4+t^2} \end{pmatrix}$$

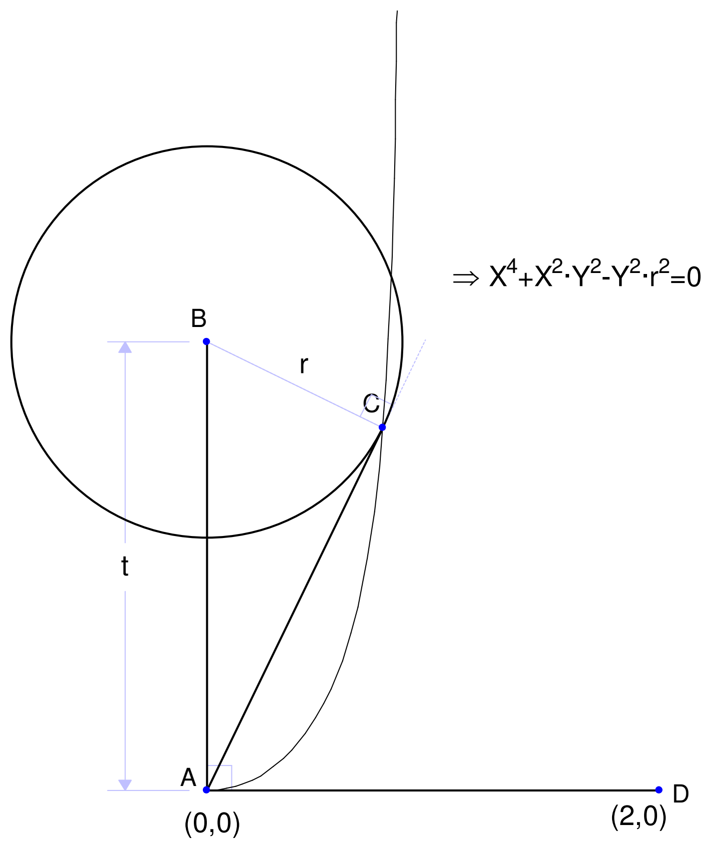
$$\Rightarrow -X^3+2\cdot Y^2-X\cdot Y^2=0$$




Segment CF is defined to be congruent to GE. Diocletian used this construction to define a cubic curve.

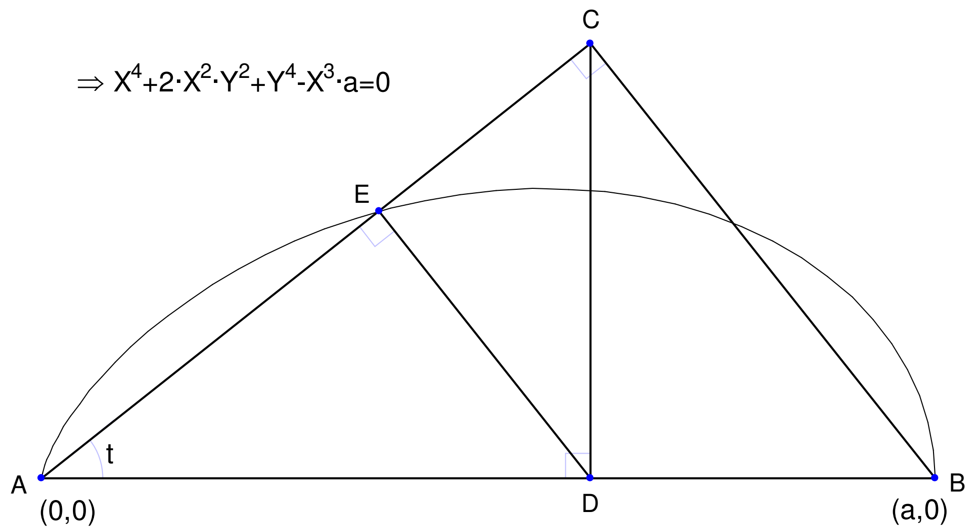
## Example 142: Kappa Curve

Studied by Gutschoven in 1662, the locus of the intersection between a circle and its tangent through the origin as the circle slides up the y-axis:

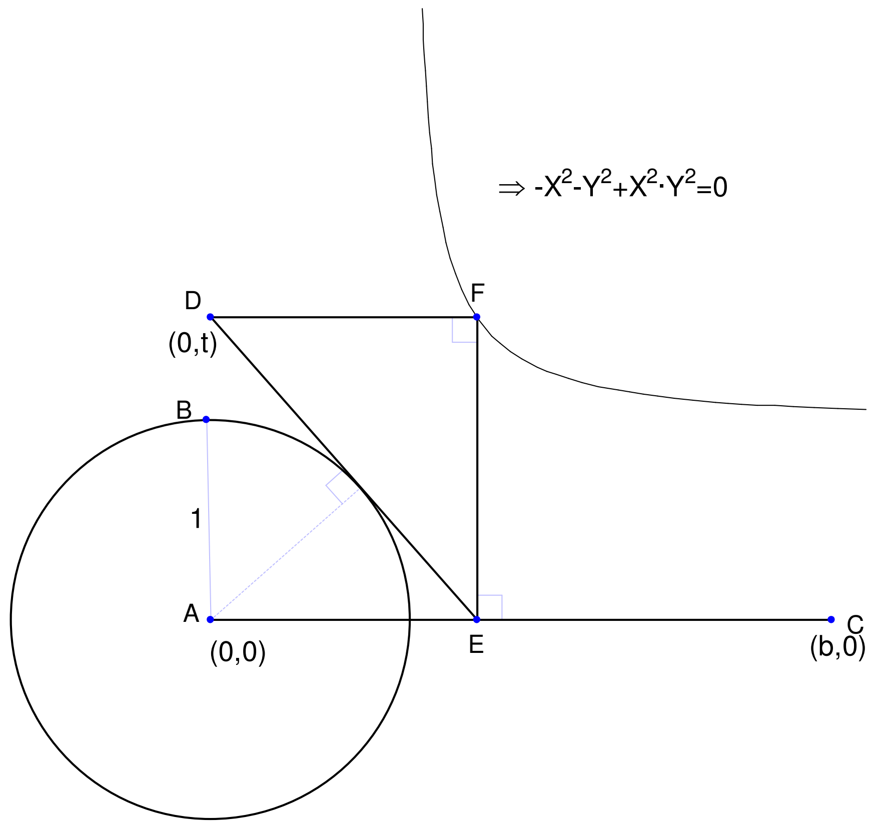


### Example 143: Kepler's Egg

An egg shape defined by projecting B onto AC, then back onto AB then back onto AC:



## Example 144: Cruciform Curve

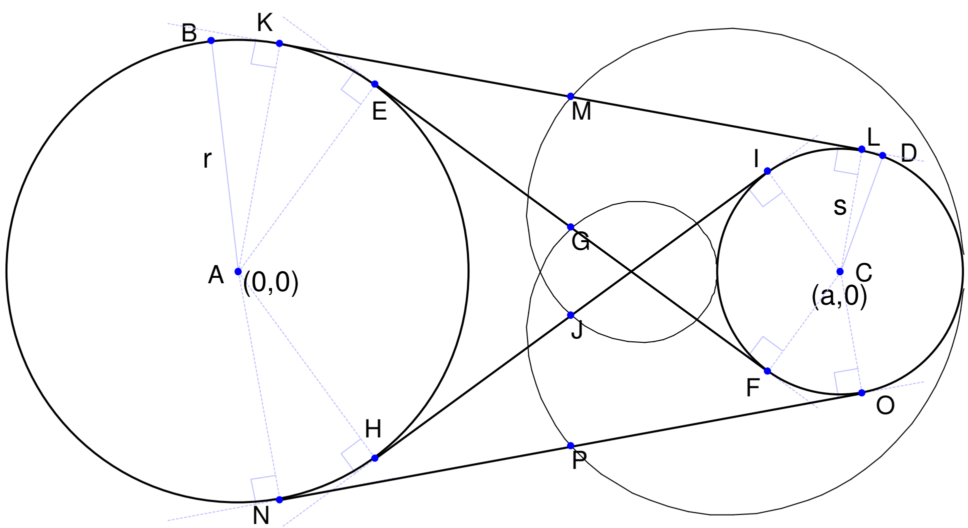



this curve can be rewritten in the form: $\frac{1}{x^2}+\frac{1}{y^2}=1$

## Example 145: Locus of centers of common tangents to two circles

We take the locus as the radius r of the left circle varies. The midpoints of all four common tangents lie on the same fourth order curve

$$\Rightarrow 4\cdot X^4+8\cdot X^2\cdot Y^2+4\cdot Y^4-12\cdot X^3\cdot a-12\cdot X\cdot Y^2\cdot a+a^4-a^2\cdot s^2+Y^2\cdot\left(4\cdot a^2-4\cdot s^2\right)+X^2\cdot\left(13\cdot a^2-4\cdot s^2\right)+X\cdot\left(-6\cdot a^3+4\cdot a\cdot s^2\right)=0$$




We can use Maple to solve for the intersections with the x axis:

```
> subs(Y=0,4*X^4+8*Y^2*X^2+4*Y^4-12*a*X^3-
12*a*Y^2*X+a^4-s^2*a^2+(4*a^2-4*s^2)*Y^2+(13*a^2-
4*s^2)*X^2+(-6*a^3+4*s^2*a)*X );
```

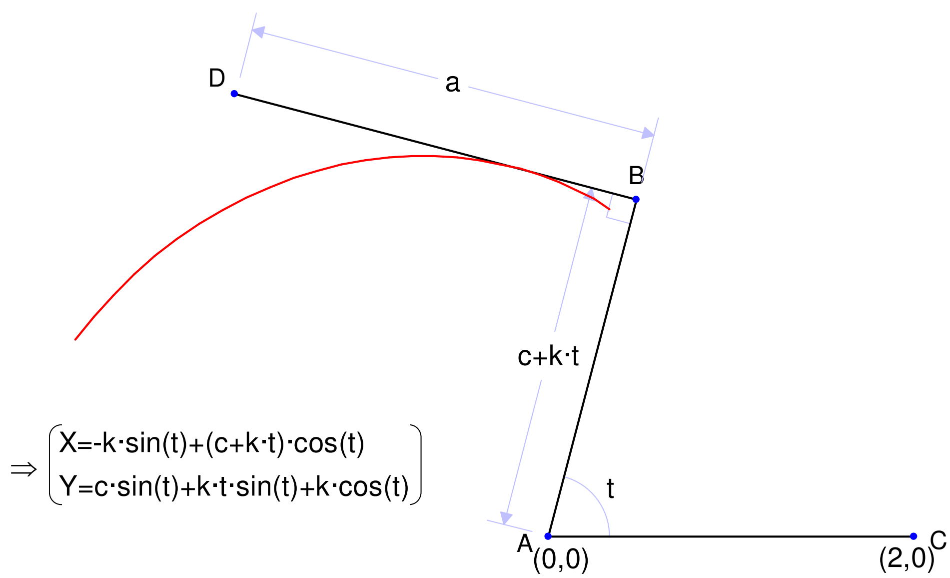
$$4X^4-12aX^3+a^4-s^2a^2+(13a^2-4s^2)X^2+(-6a^3+4s^2a)X$$

```
> solve(%,X);
```

$$a - s, a + s, \frac{1}{2} a, \frac{1}{2} a$$

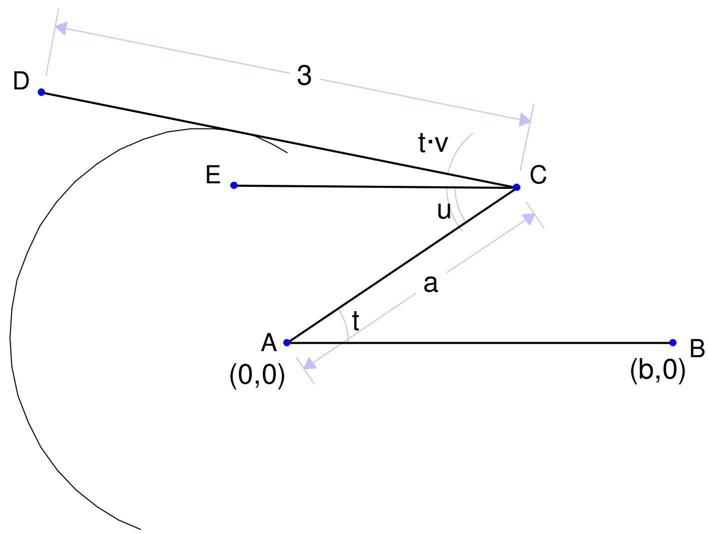
## Example 146: Steady Rise Cam Curve

Assuming a Flat Plate reciprocating follower, here is the cam curve for a linear rise of k*t+c. This is the Envelope of the line BE.



## Example 147: Oscillating Flat Plate Cam

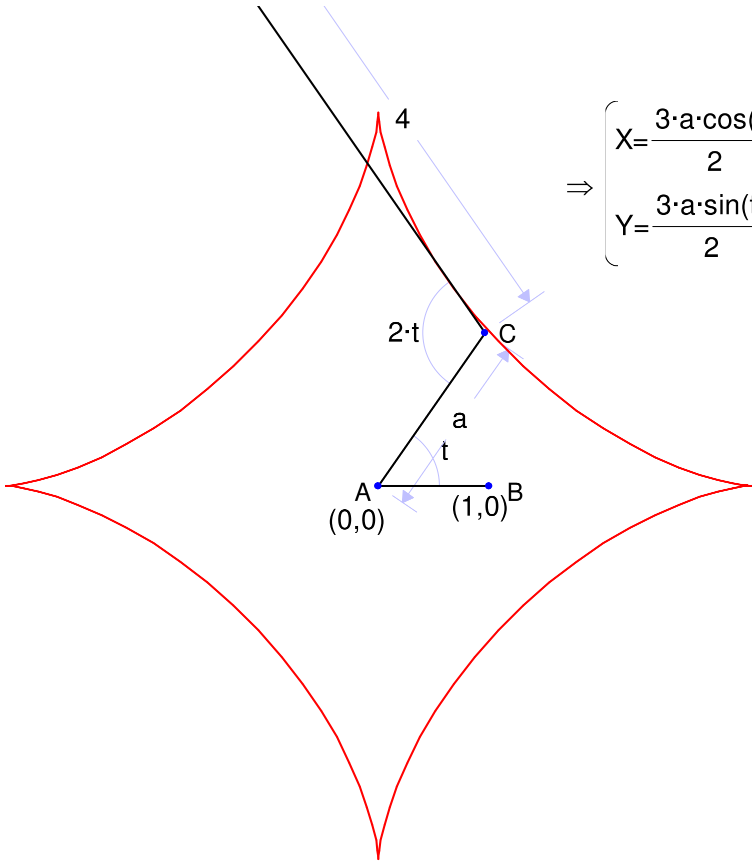
Here is a cam curve for an oscillating flat plate cam follower, where the follower rise is linear in the cam angle: rise = u+t*v




$$\Rightarrow \left\{ \begin{array}{l} X=\dfrac{a\cdot\left(\left(-1+2\cdot\cos(t\cdot v)^2\right)\cdot\left(\left(-1+2\cdot\cos(u)^2\right)\cdot\cos(t)+2\cdot\sin(t)\cdot\sin(u)\cdot\cos(u)\right)-\cos(t)+2\cdot v\cdot\cos(t)-2\cdot\left(-\left(-1+2\cdot\cos(u)^2\right)\cdot\sin(t)+2\cdot\sin(u)\cdot\cos(t)\cdot\cos(u)\right)\cdot\sin(t\cdot v)\cdot\cos(t\cdot v)\right)}{2\cdot(-1+v)} \\ \mid b>0 \\ Y=\dfrac{a\cdot\left(-\left(\left(-1+2\cdot\cos(t\cdot v)^2\right)\cdot\left(-\left(-1+2\cdot\cos(u)^2\right)\cdot\sin(t)+2\cdot\sin(u)\cdot\cos(t)\cdot\cos(u)\right)+2\cdot\left(\left(-1+2\cdot\cos(u)^2\right)\cdot\cos(t)+2\cdot\sin(t)\cdot\sin(u)\cdot\cos(u)\right)\cdot\sin(t\cdot v)\cdot\cos(t\cdot v)\right)-\sin(t)+2\cdot v\cdot\sin(t)\right)}{2\cdot(-1+v)} \\ \mid b>0 \end{array} \right.$$

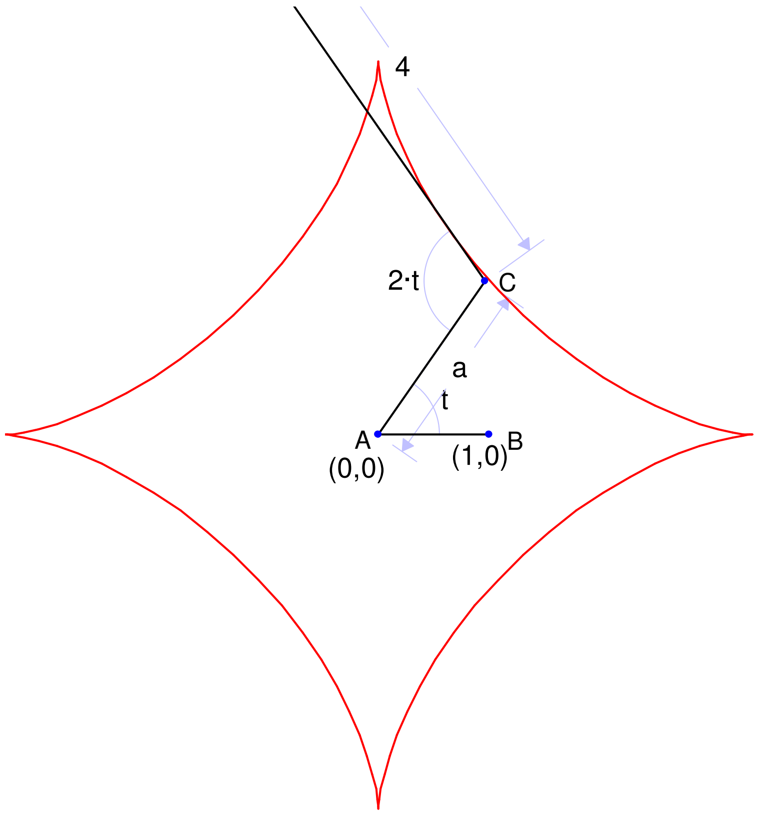
## Example 148: A Cam Star

Based on the previous model, let's take the simple case where the follower angle is twice the cam angle:




$$\Rightarrow \begin{pmatrix} X=\frac{3\cdot a\cdot \cos(t)}{2}+\frac{a\cdot \cos(3\cdot t)}{2} \\ Y=\frac{3\cdot a\cdot \sin(t)}{2}-\frac{a\cdot \sin(3\cdot t)}{2} \end{pmatrix}$$

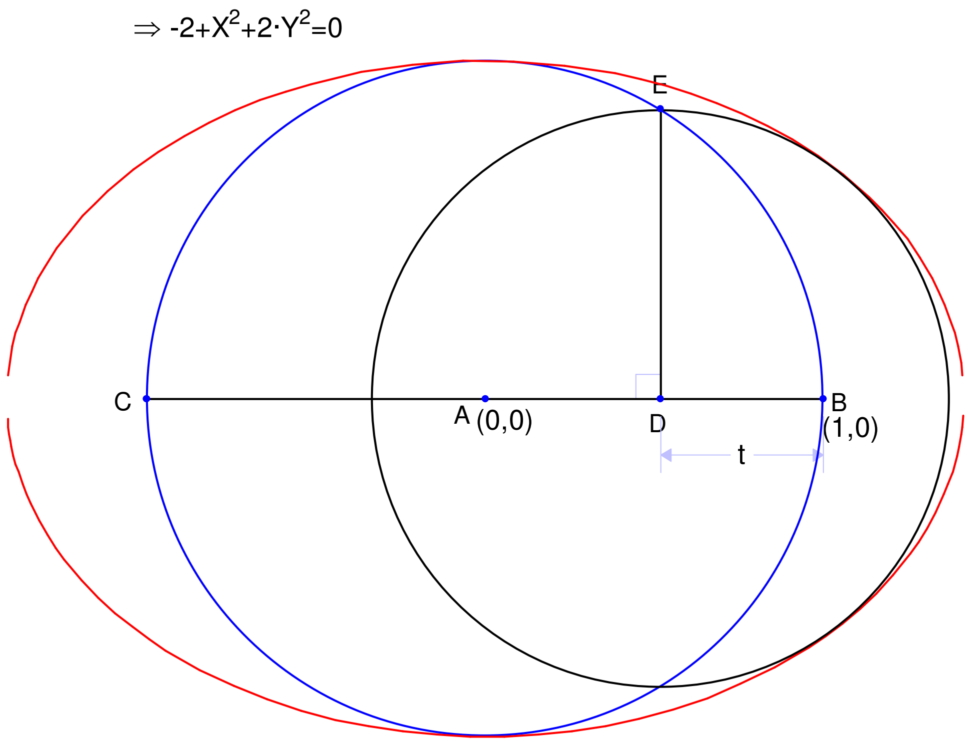
Can we get an implicit definition of the curve?  Yes.




$$\Rightarrow X^6+3\cdot X^4\cdot Y^2+3\cdot X^2\cdot Y^4+Y^6-12\cdot X^4\cdot a^2+84\cdot X^2\cdot Y^2\cdot a^2-12\cdot Y^4\cdot a^2+48\cdot X^2\cdot a^4+48\cdot Y^2\cdot a^4-64\cdot a^6=0$$

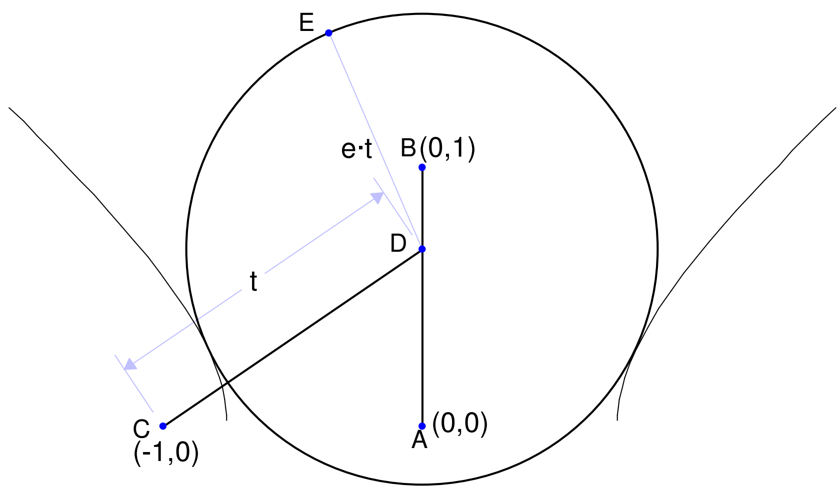
### Example 149: Ellipse as Envelope of Circles

Take the envelope of the circles whose centers lie on the x-axis and which have extrema which lie on the unit circle. We find it is an ellipse:



## Example 150: Hyperbola as an envelope of circles

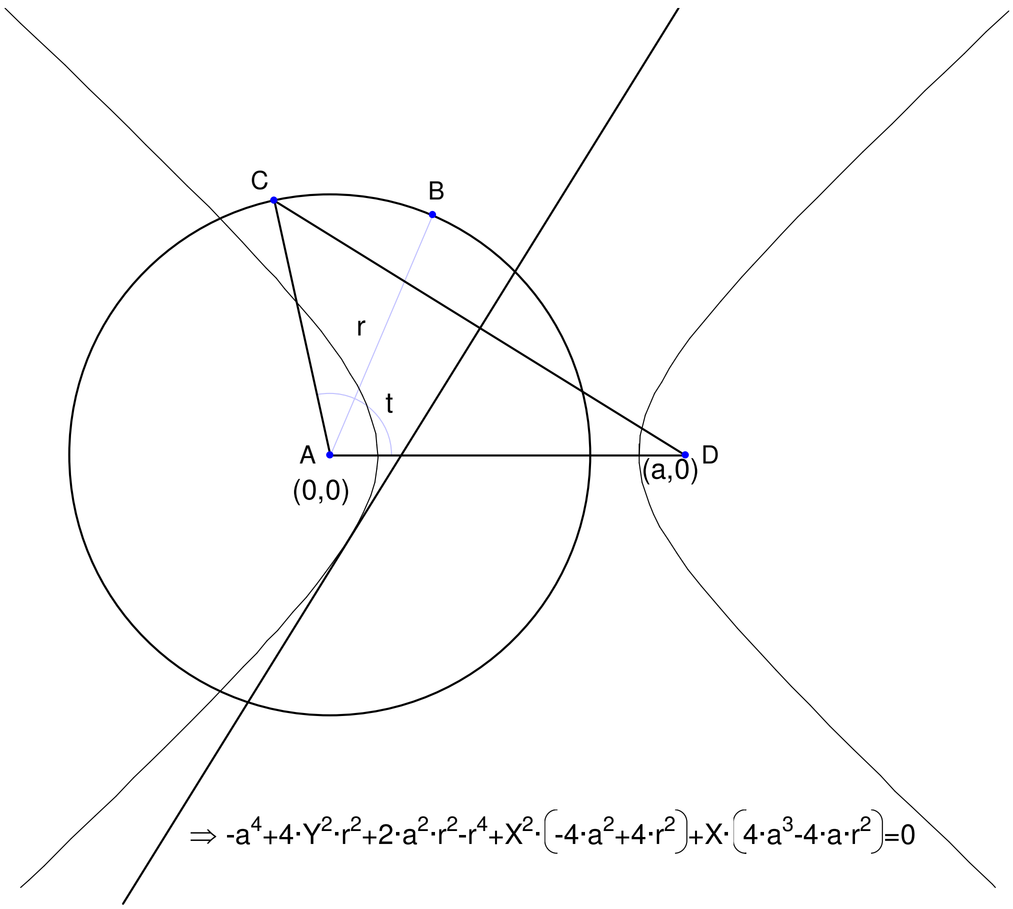
Take the envelope of a family of circles centered on a line and whose radius is an eccentricity times the distance from a focus.




$$\Rightarrow e^2+Y^2 \cdot e^2-e^4+X^2 \cdot \left(-1+e^2\right)=0$$

## Example 151: Hyperbola as an Envelope of Lines

We take the envelope of the perpendicular bisectors of the line CD as C traverses the circle AB.

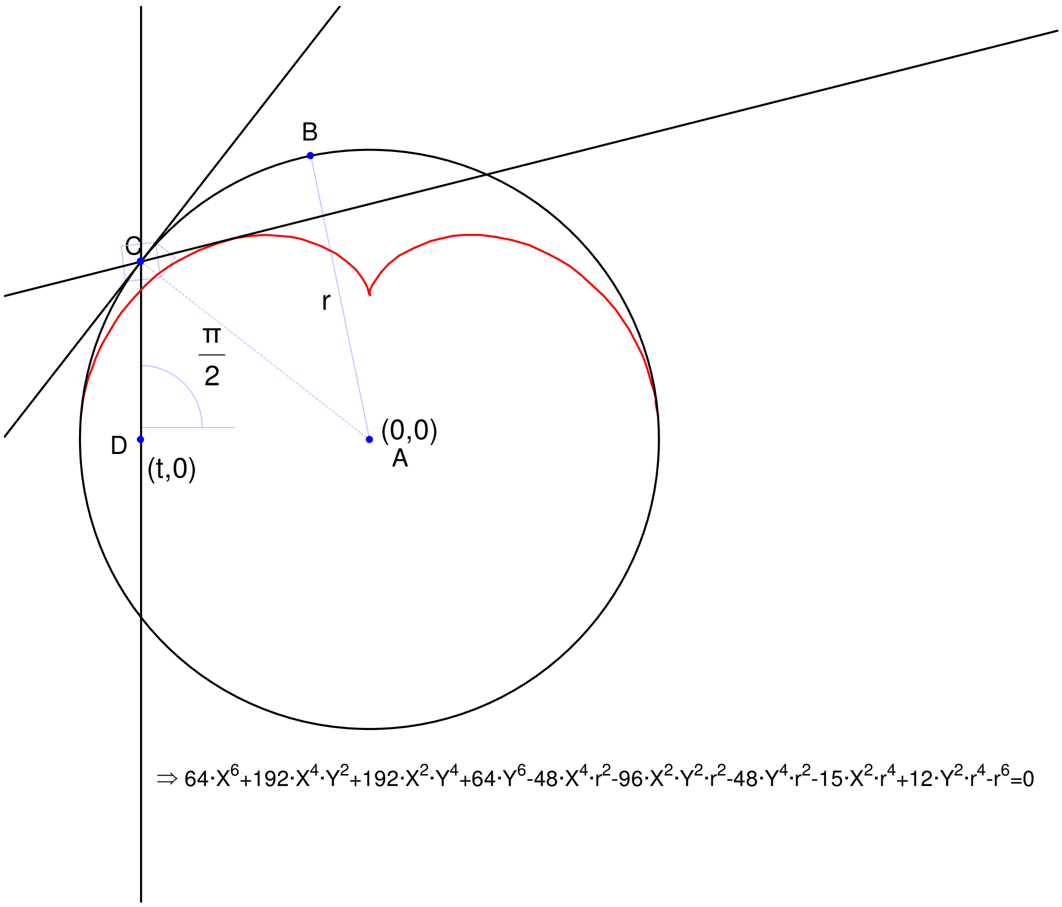



The result is a hyperbola with foci A and B.

What happens if D lies inside the circle?

## Example 152: Caustics in a cup of coffee

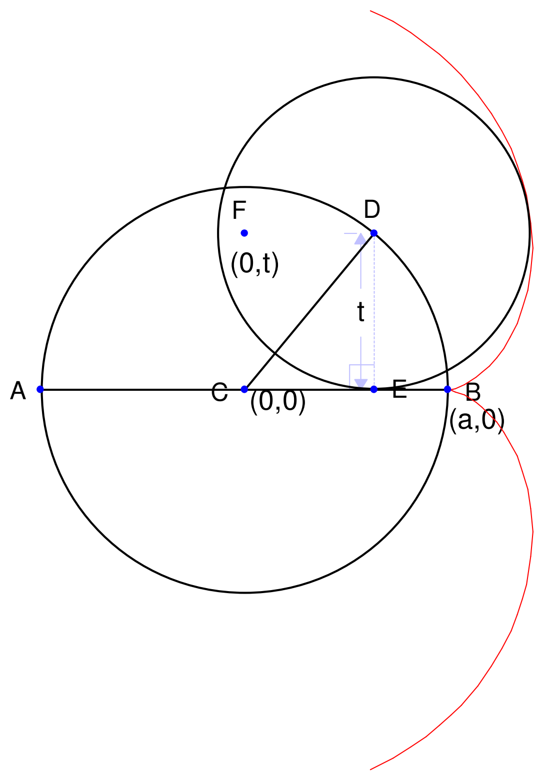
The Nephroid curve generated by reflecting a set of parallel rays in a circle, and then taking the envelope of the reflected rays:




$$\Rightarrow 64\cdot X^6+192\cdot X^4\cdot Y^2+192\cdot X^2\cdot Y^4+64\cdot Y^6-48\cdot X^4\cdot r^2-96\cdot X^2\cdot Y^2\cdot r^2-48\cdot Y^4\cdot r^2-15\cdot X^2\cdot r^4+12\cdot Y^2\cdot r^4-r^6=0$$

## Example 153: A Nephroid by another route

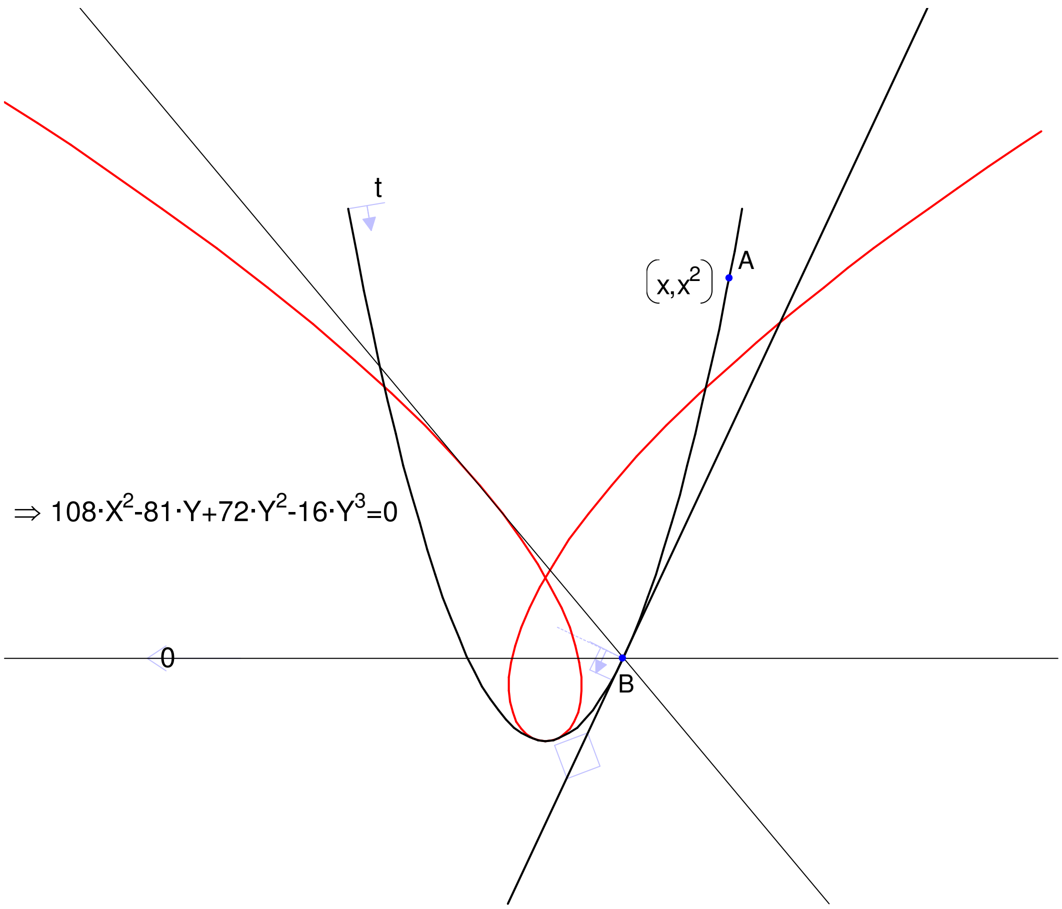
The envelope of the circles whose centers lie on a circle and which are tangential to the diameter form the same type of curve:




$$\Rightarrow 4\cdot X^6+12\cdot X^4\cdot Y^2+12\cdot X^2\cdot Y^4+4\cdot Y^6-12\cdot X^4\cdot a^2-24\cdot X^2\cdot Y^2\cdot a^2-12\cdot Y^4\cdot a^2+12\cdot X^2\cdot a^4-15\cdot Y^2\cdot a^4-4\cdot a^6=0$$

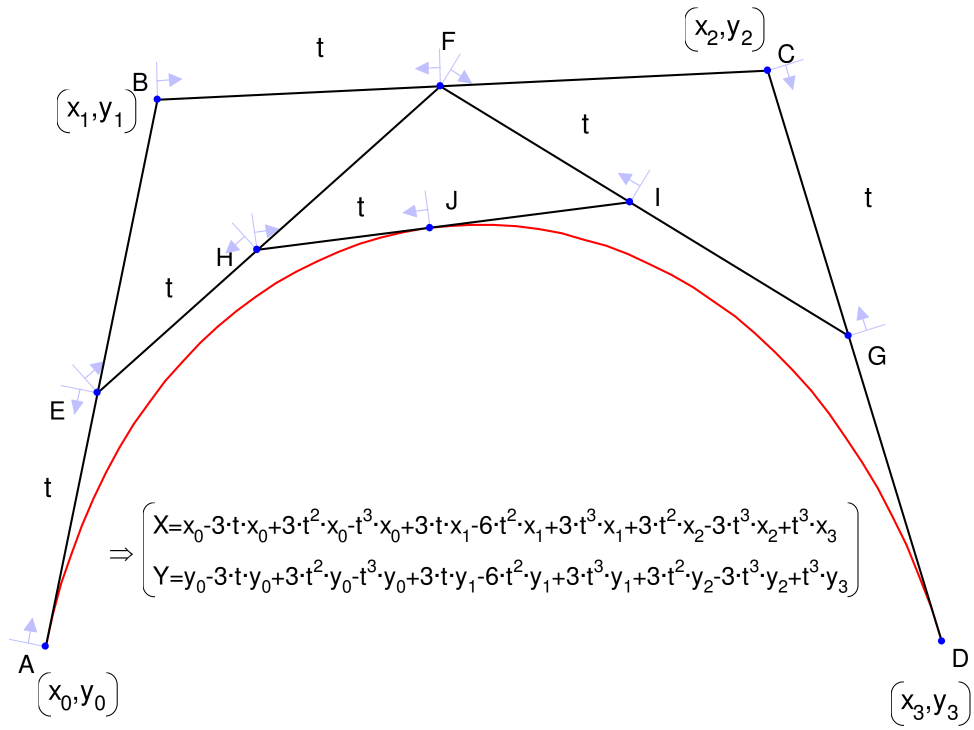
## Example 154: Tschirnhausen's Cubic

Studied by Ehrenfried Tschirnhausen in 1690, this is the caustic of a set of parallel rays perpendicular to the axis of a parabola:



## Example 155: Cubic Spline

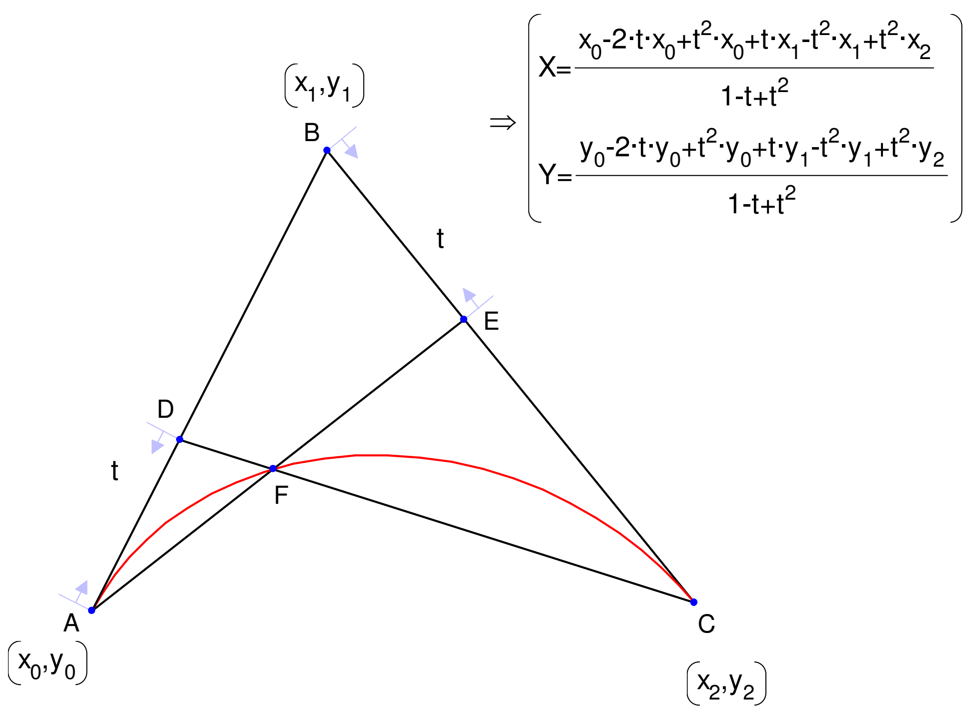
This diagram shows an algorithm for constructing the cubic spline from its control points:




Point E is proportion t along the line AB. Point F is proportion t along BC. Point G is proportion t along CD. Point H is proportion t along EF. Point I is proportion t along FG. Point J is proportion t along HI. The spline curve is the locus as t runs from 0 to 1.

## Example 156: A Triangle Spline

We can create another spline curve from 3 control points ABC in the following way: Point D is located proportion t along AB. Point E is located proportion t along BC. We take the locus of the intersection of AE and CD:




Copy the x coordinate into Maple and differentiate to get:

```
> u:= diff((x[`0`]-2*x[`0`]*t+x[`0`]*t^2+x[`1`]*t-
x[`1`]*t^2+x[`2`]*t^2)/(-t+1+t^2),t);
```

$$u := \frac{-2\,x_0 + 2\,x_0\,t + x_1 - 2\,x_1\,t + 2\,x_2\,t}{-t + 1 + t^2} - \frac{(x_0 - 2\,x_0\,t + x_0\,t^2 + x_1\,t - x_1\,t^2 + x_2\,t^2)\,(-1 + 2\,t)}{(-t + 1 + t^2)^2}$$

Substituting t=0 and t=1::

```
> subs(t=0,u);
```

$$-x_0 + x_1$$

```
> subs(t=1,u);
```

$$-x_1 + x_2$$

Comparable result for y shows that the curve is tangent to the control triangle at the end points

## Example 157: Another Triangle Spline

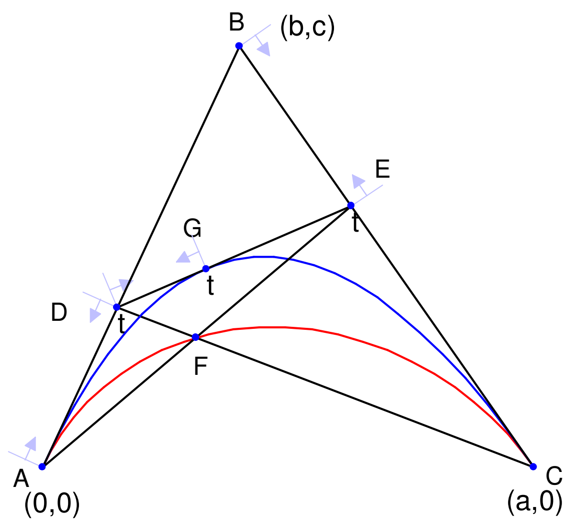
We can also create a spline from a control triangle by taking the locus of a point G proportion t along DE.

Observing the parametric form of the curves we see that one is a parametric quadratic, while the other is a rational quadratic. Implicit forms are both conics (and almost, but not quite, identical).

Locus of G

$$\Rightarrow \begin{pmatrix} X=2\cdot b\cdot t+t^2\cdot(a-2\cdot b) \\ Y=2\cdot c\cdot t\cdot(1-t) \end{pmatrix}$$

$$\Rightarrow 4\cdot Y\cdot a\cdot b\cdot c+4\cdot X^2\cdot c^2-4\cdot X\cdot a\cdot c^2+Y^2\cdot\left(a^2-4\cdot a\cdot b+4\cdot b^2\right)+X\cdot Y\cdot(4\cdot a\cdot c-8\cdot b\cdot c)=0$$

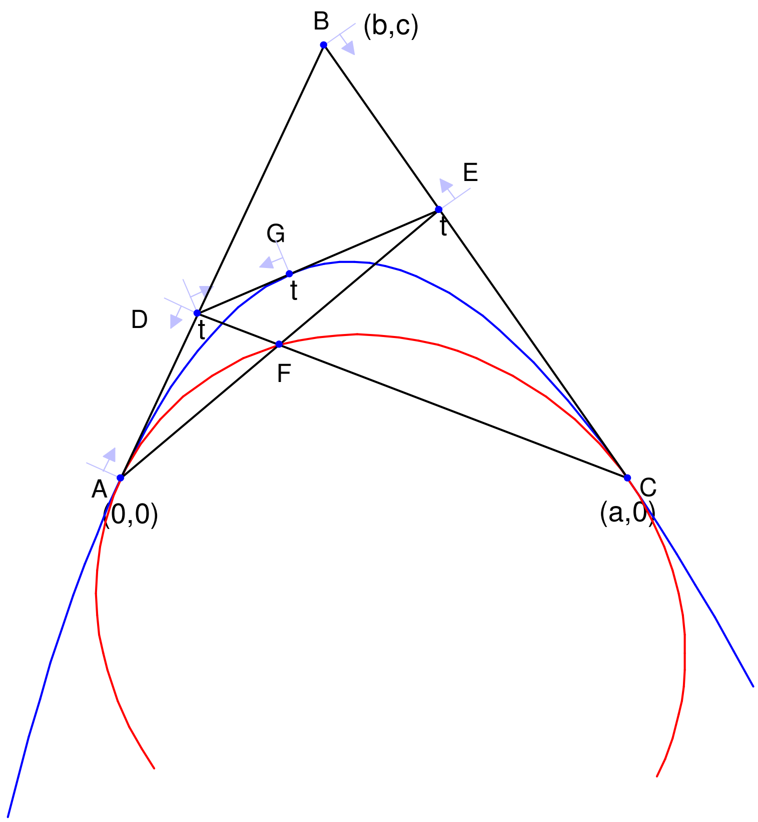



Locus of F

$$\Rightarrow \begin{pmatrix} X=\dfrac{t\cdot(b+a\cdot t-b\cdot t)}{1-t+t^2} \\ Y=\dfrac{c\cdot\left(t-t^2\right)}{1-t+t^2} \end{pmatrix}$$

$$\Rightarrow Y\cdot a\cdot b\cdot c+X^2\cdot c^2-X\cdot a\cdot c^2+Y^2\cdot\left(a^2-a\cdot b+b^2\right)+X\cdot Y\cdot(a\cdot c-2\cdot b\cdot c)=0$$

What types of conics are they?  Extending the curves a little can give a clue:




The blue curve looks like a parabola, the red certainly does not.

Copying the blue curve equation into Maple and examining the quadratic form shows that it is indeed a parabola:

```
> 4*c*b*a*Y+4*c^2*X^2-4*c^2*a*X+(a^2-
4*b*a+4*b^2)*Y^2+(4*c*a-8*c*b)*Y*X = 0;
```

$$4\,c\,b\,a\,Y + 4\,c^2\,X^2 - 4\,c^2\,a\,X + (a^2 - 4\,b\,a + 4\,b^2)\,Y^2 + (4\,c\,a - 8\,c\,b)\,Y\,X = 0$$

```
> <<4*c^2 |(4*c*a-8*c*b)/2>, <(4*c*a-8*c*b)/2 |(a^2-
4*b*a+4*b^2)>>;
```

$$\begin{bmatrix} 4c^2 & 2ca-4cb \\ 2ca-4cb & a^2-4ba+4b^2 \end{bmatrix}$$

```
> Determinant(%);
```

$$0$$

How about the red curve:?

```
> c*b*a*Y+c^2*X^2-c^2*a*X+(a^2-b*a+b^2)*Y^2+(c*a-
2*c*b)*Y*X = 0;
```

$$c\,b\,a\,Y + c^2 X^2 - c^2 a\,X + (a^2 - b\,a + b^2)\,Y^2 + (c\,a - 2\,c\,b)\,Y X = 0$$

```
> <<c^2 |(c*a-2*c*b)/2>, <(c*a-2*c*b)/2 |(a^2-
b*a+b^2)>>;
```

$$\begin{bmatrix} c^2 & \frac{1}{2}ca-cb \\ \frac{1}{2}ca-cb & a^2-ba+b^2 \end{bmatrix}$$

```
> Determinant(%);
```

$$\frac{3}{4}c^2 a^2$$

We see that the determinant is positive. This means we will always have a portion of an ellipse.