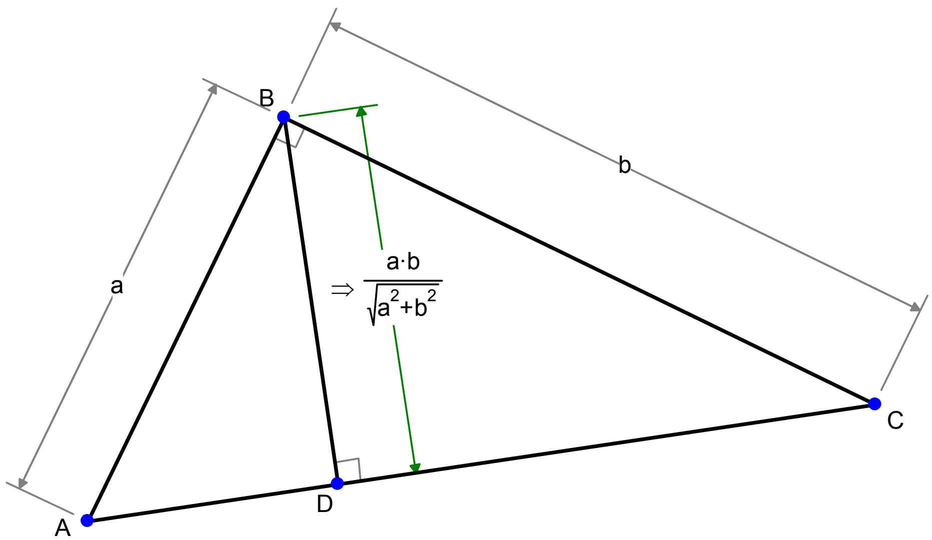
Napoleon and Pythagoras with Geometry Expressions

Introduction

Geometry Expressions automatically generates algebraic expressions from geometric figures. For example in the diagram below, the user has specified that the triangle is right and has short sides length a and b. The system has calculated an expression for the length of the altitude:

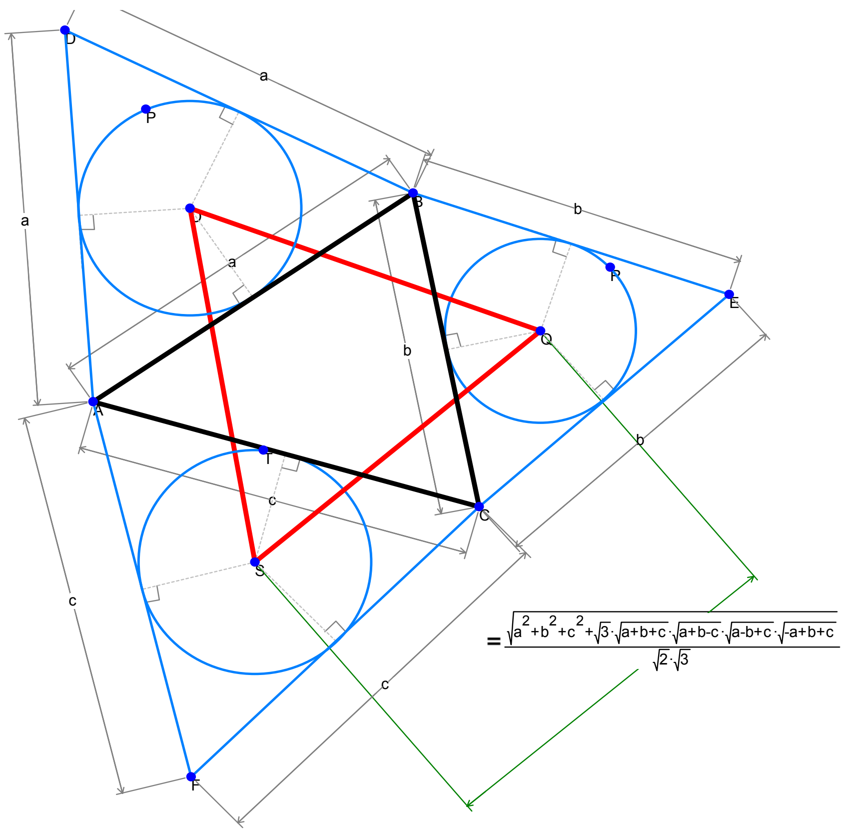


We present a collection of worked examples using Geometry Expressions.

These examples explore diagrams similar to that of Pythagoras' Theorem, and Napoleon's Theorem: in which similar figures are constructed on each side of an original triangle (or in one case quadralateral).

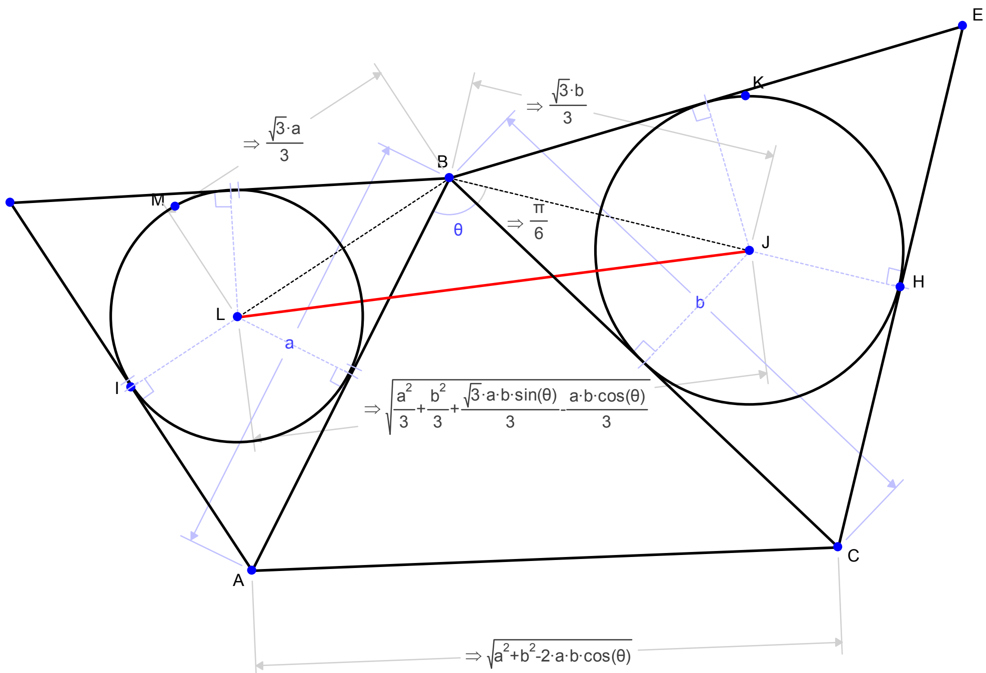
Example 1: Napoleon's Theorem

Napoleon's Theorem states that if you take a general triangle and draw an equilateral triangle on each side, then the triangle formed by joining the incenters of these new triangles is equilateral.



Can you prove the theorem from this diagram by inspection of the expression for the length of QS?

Now we need to prove the expression in the diagram. One way of doing this is to remove the constraint on the length of AC, and replace it by a constraint on the angle ABC.

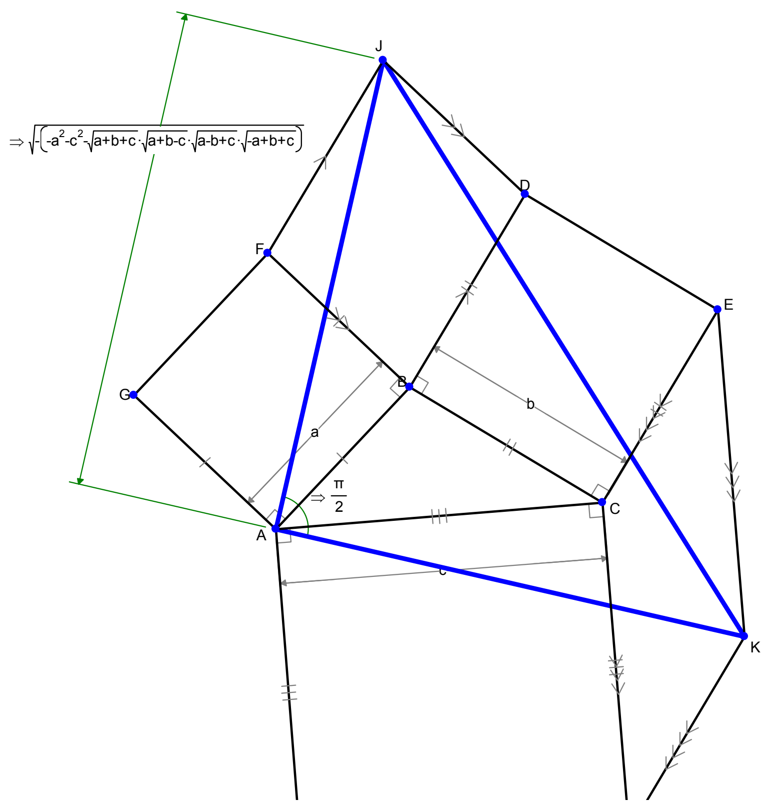


Can you use the cosine rule to show that the expression for LJ displayed above is correct?

Given that can you prove the expression in terms of lengths a,b,c?

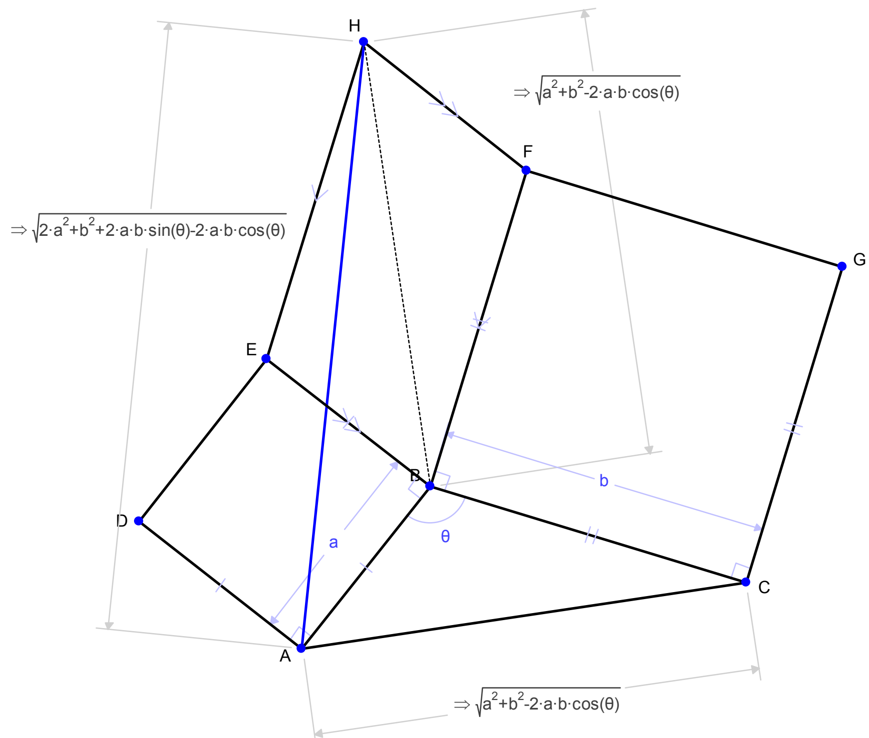
Example 2: An unexpected triangle from a Pythagoras-like diagram

Regardless of the original triangle the resulting triangle from this diagram is a right angled-isosceles:



Examination of the length AJ shows that it is symmetric in a and b, and hence identical to AK.

This result can also be proved by replacing the distance c with an angle:

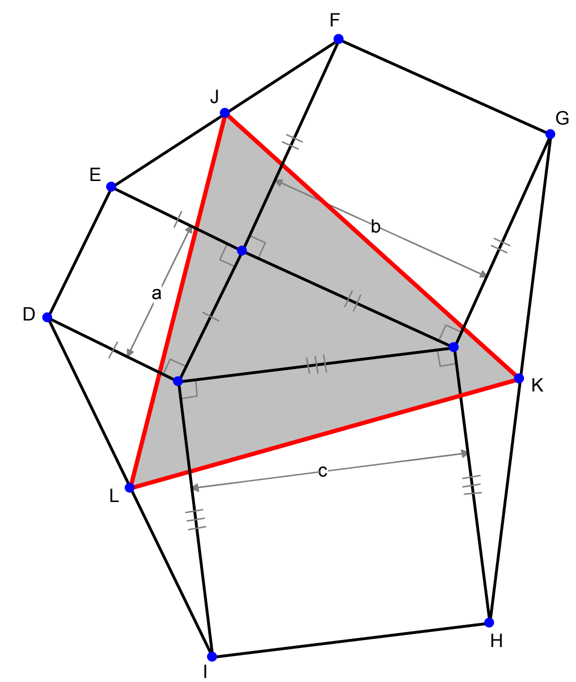


Given the expression above for AH, can you derive the formula in terms of a,b,c?

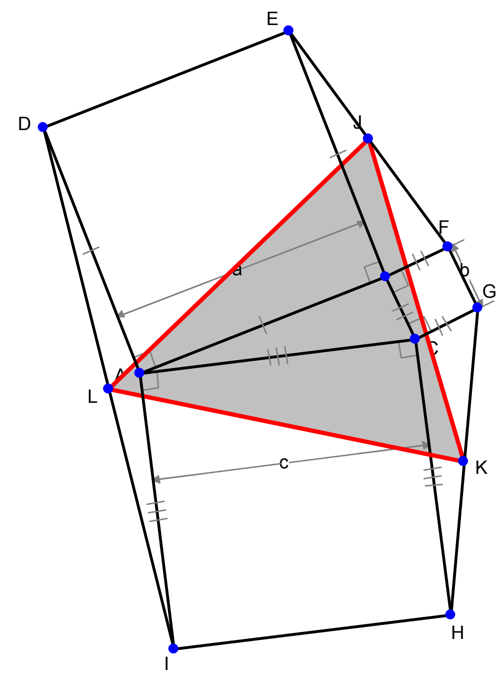
Can you derive the expression for AH? (Hint: which angle of the original triangle is the same as EBH?

Example 3:   A Penequilateral Triangle

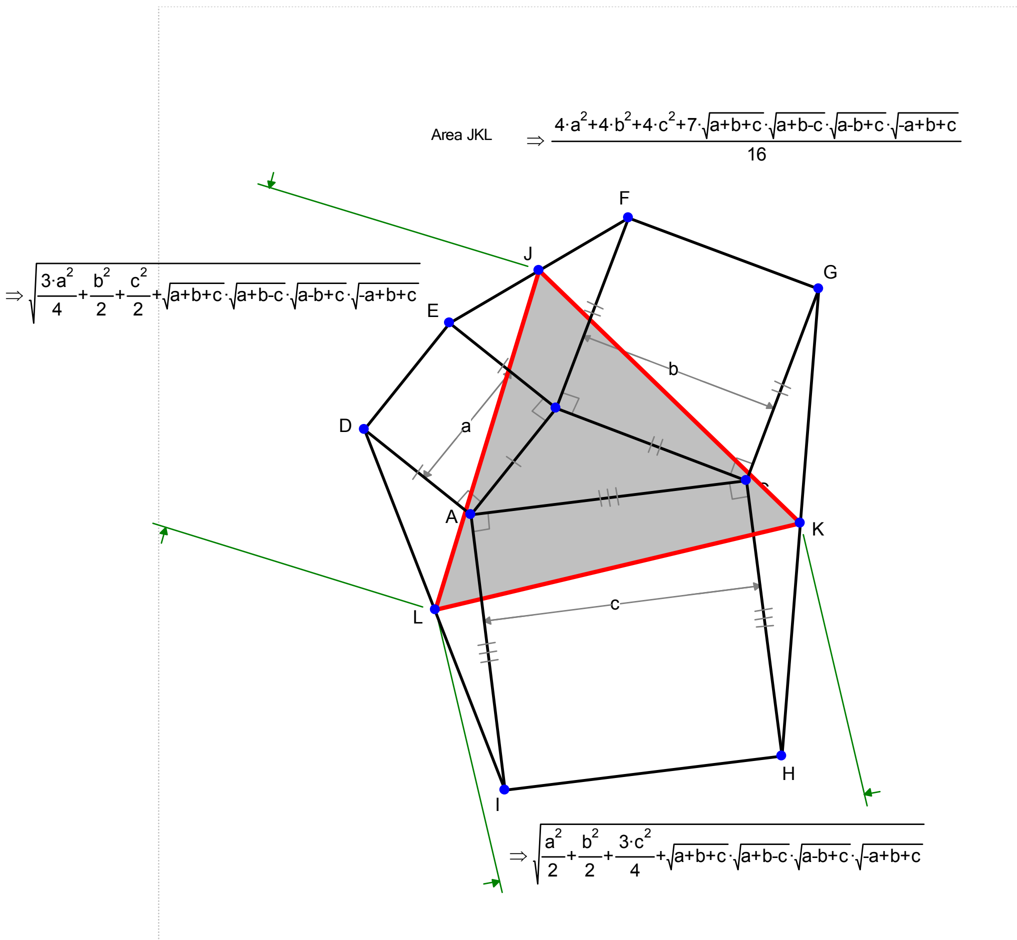
Starting with a triangle whose sides are length a,b,c, we construct squares on each side, join the corners of the squares, then join the midpoints of these lines to create a triangle:



The triangle looks to the naked eye as if it is equilateral. Try dragging the original points and observe that the new triangle still looks equilateral:



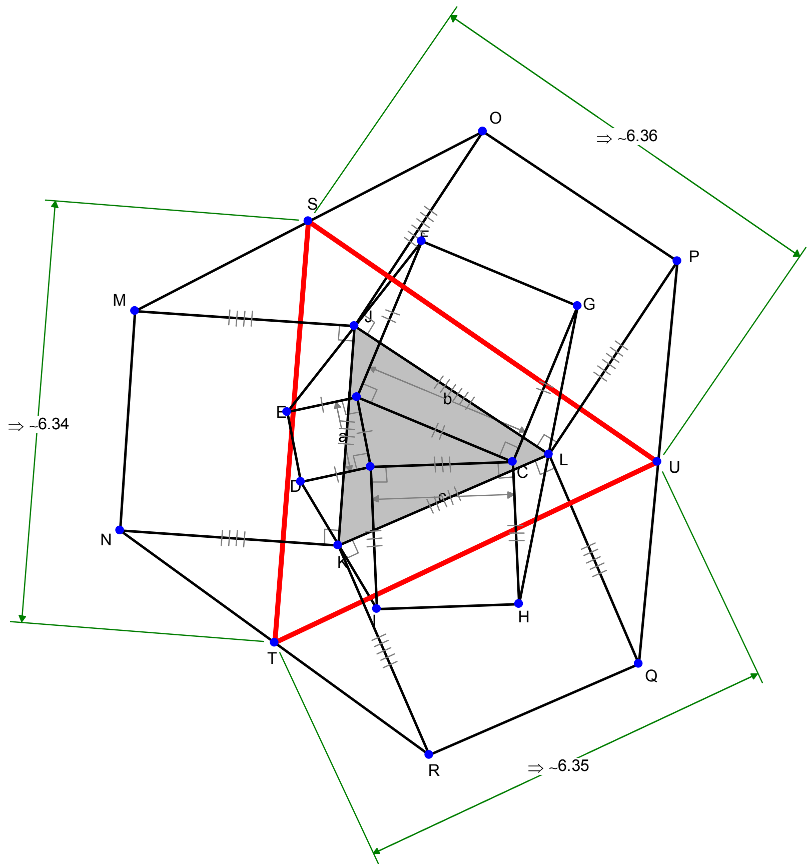
Is it in fact?



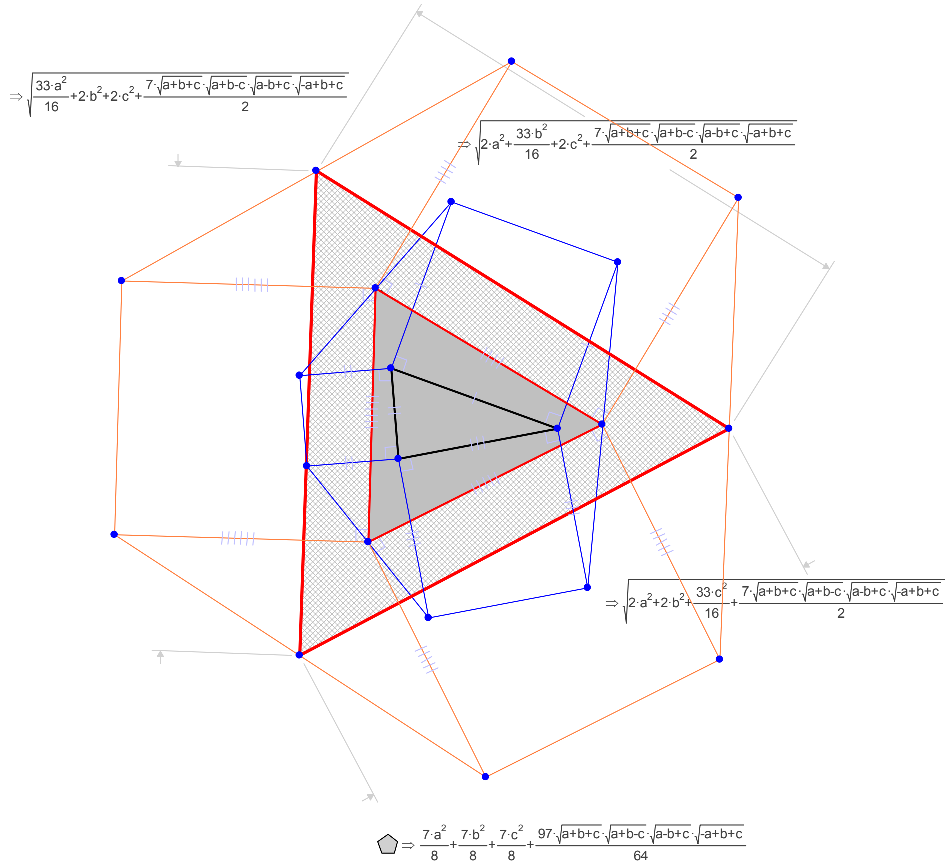
Not quite, we observe the sides are guaranteed to be close in size, but not identical unless the original triangle is isosceles. In fact, the difference in squares of the sides of the new triangle is $\dfrac{a^2}{4}-\dfrac{c^2}{4}$.

Notice we can repeat the process drawing squares on JK, KL and JL. The difference in squares of the ensuing sides will be $\dfrac{a^2}{16}-\dfrac{c^2}{16}$.

By repeating this process, we can create a triangle as close as we like to an equilateral triangle, but still not exactly one.

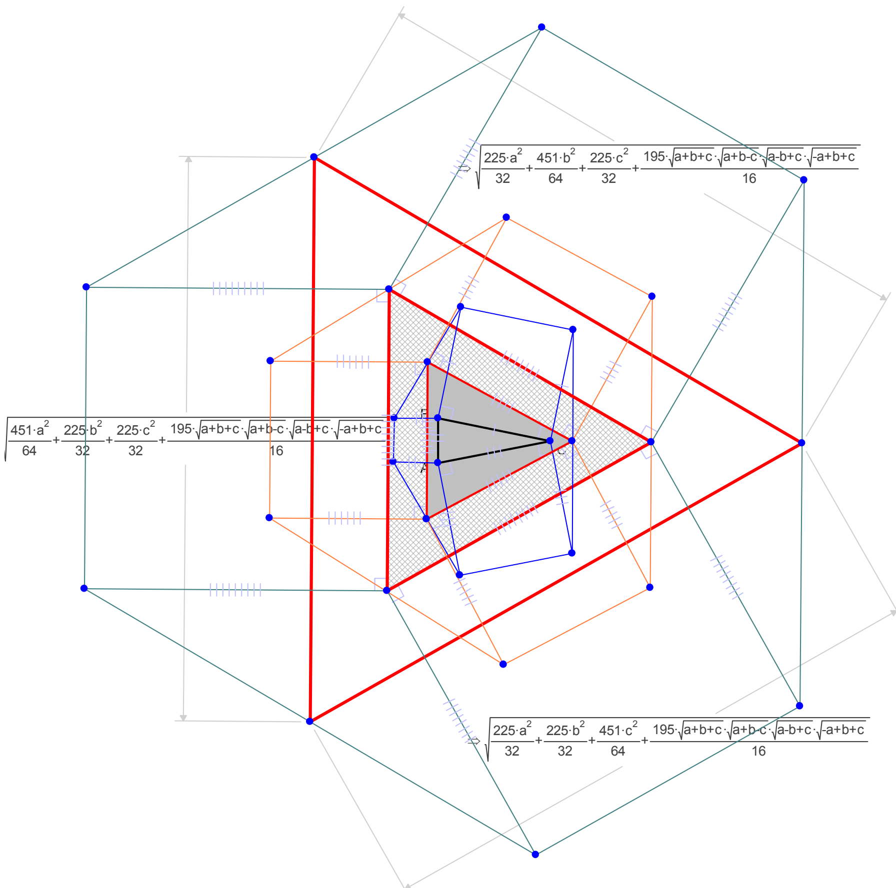


Symbolically:



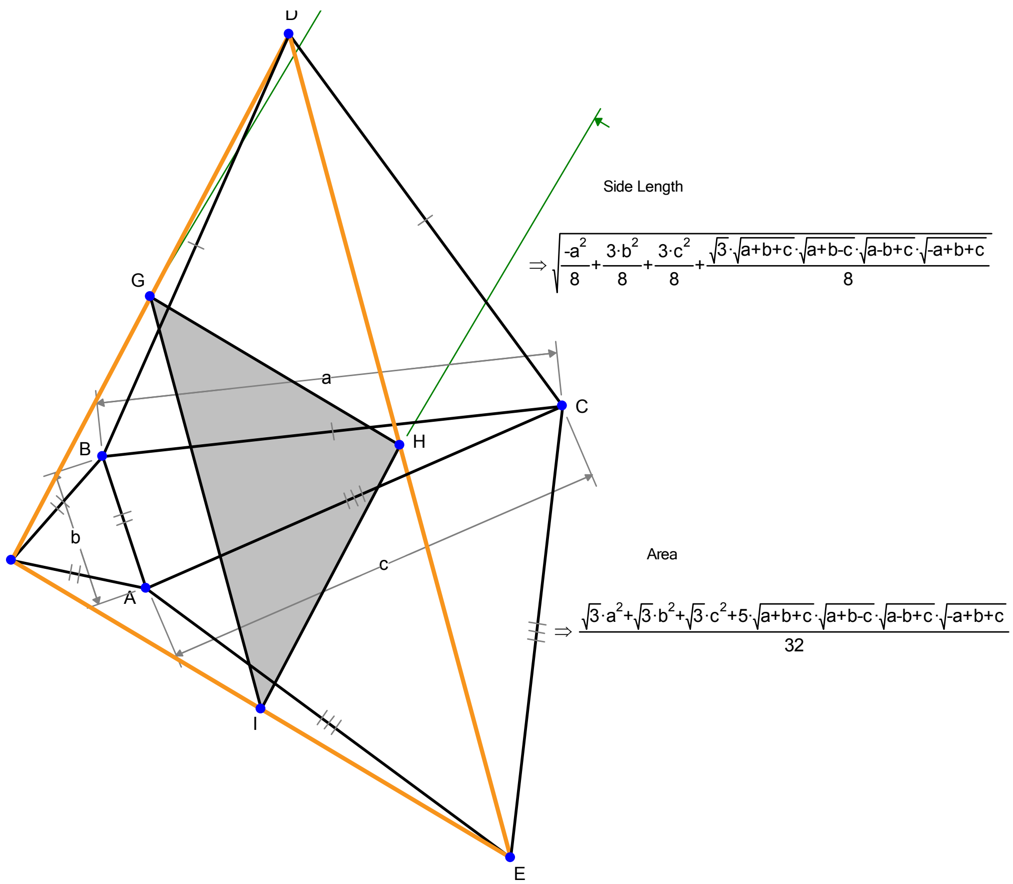
Can you find a recurrence relation for the side lengths and the area?

How about a further recursion (this may take some time to compute):



Example 4: Another Penequilateral Triangle

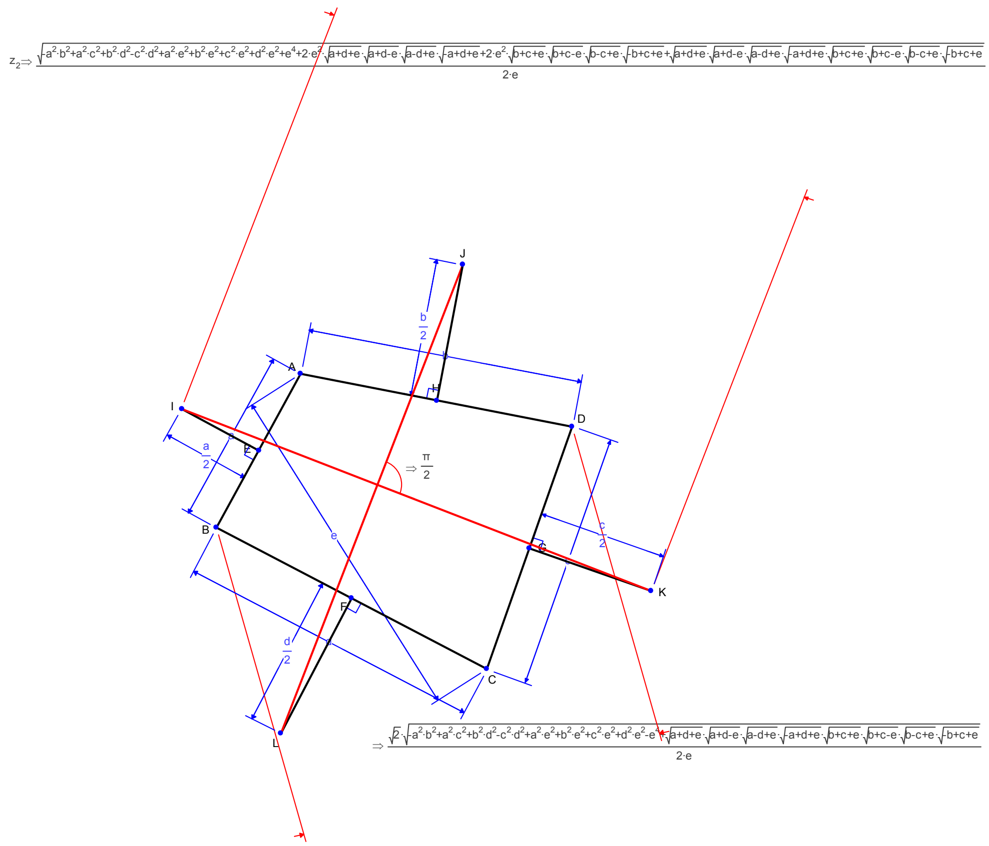
We can do a similar construction based on equilateral triangles drawn on the sides of an original triangle:



We see that the difference in squares of the sides of the new triangle corresponding to a and b is $\frac{a^2}{2}-\frac{c^2}{2}$. If we repeat the process, this triangle will eventually become equilateral, but not as quickly as the previous triangle.

Example 5: Quadralateral Theorem

This theorem states that if you draw a square on each side of a quadralateral, then connect the center of opposite sides, the resulting lines have the same length, and are perpendicular. Here is the result in Geometry Expressions



If we create the length of the other side we can by careful examination see that the lengths are identical. Alternatively, we can do some simplification. Our constraints are necessarily asymetric – Geometry Expressions will not let you over-constrain the diagram, and one diagonal is sufficient to define the quadralateral. However, we might expect the formula to be simpler if expressed in terms of both diagonals.

Close inspection of the formula for the length shows that it incorporates the square of the other diagonal of the figure, as well as Heron's formula for the areas of the triangles ABC and ACD. The following Mathematica worksheet contains the formulas from Geometry Expressions for L the length of the desired line, and f the length of the other diagonal. A little manipulation gives a simple formula for L^2-f^2/2. This can be simplified further by noting that the remaining terms are e^2/2 and twice the area of the quadralateral:

In[6]:=
$$L=\frac{1}{2\cdot e}\left(\sqrt{\left(-a^2\cdot b^2+a^2\cdot c^2+b^2\cdot d^2-c^2\cdot d^2+a^2\cdot e^2+b^2\cdot e^2+c^2\cdot e^2+d^2\cdot e^2+e^4+2\cdot e^2\cdot\sqrt{a+d+e}\cdot\sqrt{a+d-e}\cdot\sqrt{a-d+e}\cdot\sqrt{-a+d+e}+2\cdot e^2\cdot\sqrt{b+c+e}\cdot\sqrt{b+c-e}\cdot\sqrt{b-c+e}\cdot\sqrt{-b+c+e}+\sqrt{a+d+e}\cdot\sqrt{a+d-e}\cdot\sqrt{a-d+e}\cdot\sqrt{-a+d+e}\cdot\sqrt{b+c+e}\cdot\sqrt{b+c-e}\cdot\sqrt{b-c+e}\cdot\sqrt{-b+c+e}\right)}\right)$$

Out[6]=
$$\frac{1}{2e}\left(\sqrt{\left(-a^2b^2+a^2c^2+b^2d^2-c^2d^2+a^2e^2+b^2e^2+c^2e^2+d^2e^2+e^4+2\sqrt{b+c-e}\,e^2\sqrt{b-c+e}\sqrt{-b+c+e}\sqrt{b+c+e}+2\sqrt{a+d-e}\,e^2\sqrt{a-d+e}\sqrt{-a+d+e}\sqrt{a+d+e}+\sqrt{b+c-e}\sqrt{a+d-e}\sqrt{b-c+e}\sqrt{-b+c+e}\sqrt{b+c+e}\sqrt{a-d+e}\sqrt{-a+d+e}\sqrt{a+d+e}\right)}\right)$$

In[4]:=
$$f=\frac{1}{2\cdot e}\left(\sqrt{2}\cdot\sqrt{\left(-a^2\cdot b^2+a^2\cdot c^2+b^2\cdot d^2-c^2\cdot d^2+a^2\cdot e^2+b^2\cdot e^2+c^2\cdot e^2+d^2\cdot e^2-e^4+\sqrt{a+d+e}\cdot\sqrt{a+d-e}\cdot\sqrt{a-d+e}\cdot\sqrt{-a+d+e}\cdot\sqrt{b+c+e}\cdot\sqrt{b+c-e}\cdot\sqrt{b-c+e}\cdot\sqrt{-b+c+e}\right)}\right)$$

Out[4]=
$$\frac{1}{\sqrt{2}\,e}\left(\sqrt{\left(-a^2b^2+a^2c^2+b^2d^2-c^2d^2+a^2e^2+b^2e^2+c^2e^2+d^2e^2-e^4+\sqrt{b+c-e}\sqrt{a+d-e}\sqrt{b-c+e}\sqrt{-b+c+e}\sqrt{b+c+e}\sqrt{a-d+e}\sqrt{-a+d+e}\sqrt{a+d+e}\right)}\right)$$

In[7]:= **L^2**

Out[7]=
$$\frac{1}{4e^2}\left(-a^2b^2+a^2c^2+b^2d^2-c^2d^2+a^2e^2+b^2e^2+c^2e^2+d^2e^2+e^4+2\sqrt{b+c-e}\,e^2\sqrt{b-c+e}\sqrt{-b+c+e}\sqrt{b+c+e}+2\sqrt{a+d-e}\,e^2\sqrt{a-d+e}\sqrt{-a+d+e}\sqrt{a+d+e}+\sqrt{b+c-e}\sqrt{a+d-e}\sqrt{b-c+e}\sqrt{-b+c+e}\sqrt{b+c+e}\sqrt{a-d+e}\sqrt{-a+d+e}\sqrt{a+d+e}\right)$$

In[8]:= **f^2**

Out[8]=
$$\frac{1}{2e^2}\left(-a^2b^2+a^2c^2+b^2d^2-c^2d^2+a^2e^2+b^2e^2+c^2e^2+d^2e^2-e^4+\sqrt{b+c-e}\sqrt{a+d-e}\sqrt{b-c+e}\sqrt{-b+c+e}\sqrt{b+c+e}\sqrt{a-d+e}\sqrt{-a+d+e}\sqrt{a+d+e}\right)$$

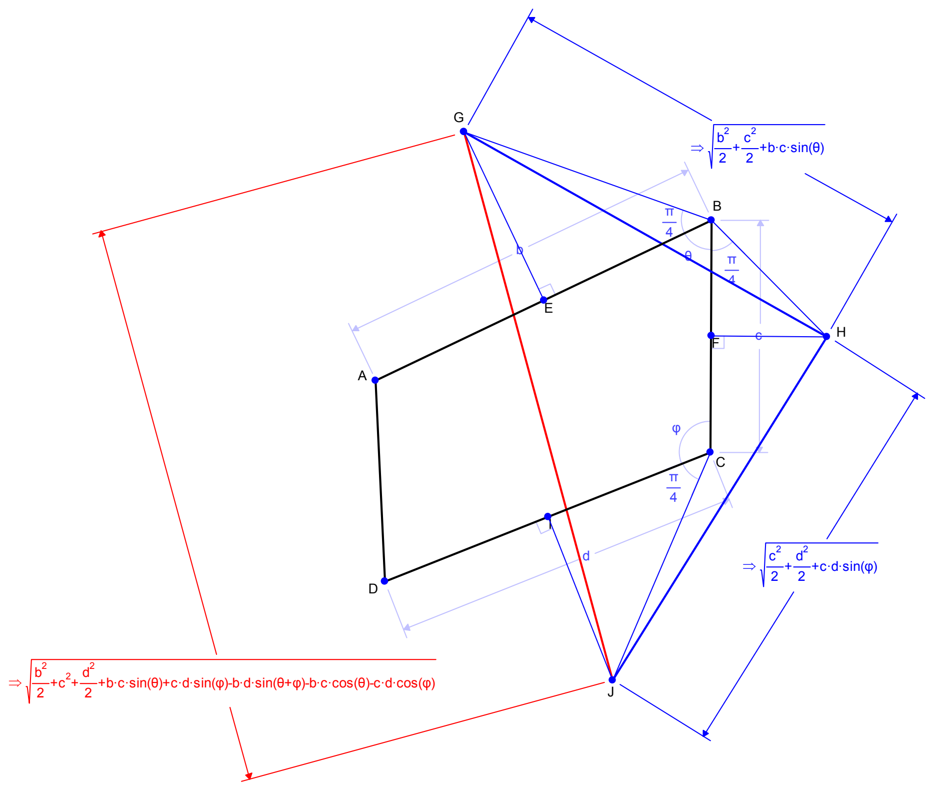
In[11]:= **Simplify[L^2 - f^2/2]**

Out[11]=
$$\frac{1}{2}\left(e^2+\sqrt{b+c-e}\sqrt{b-c+e}\sqrt{-b+c+e}\sqrt{b+c+e}+\sqrt{a+d-e}\sqrt{a-d+e}\sqrt{-a+d+e}\sqrt{a+d+e}\right)$$

From which we can derive that:

$$L^2=\frac{e^2}{2}+\frac{f^2}{2}+2A$$

Areas are simpler when expressed in terms of angles. Here is a revision of the diagram with angles inserted. This gives us more of a clue of how to prove the result:



If we also examine the equations for the diagonals and the missing side in terms of these angles we can see how the expression above is pieced together: