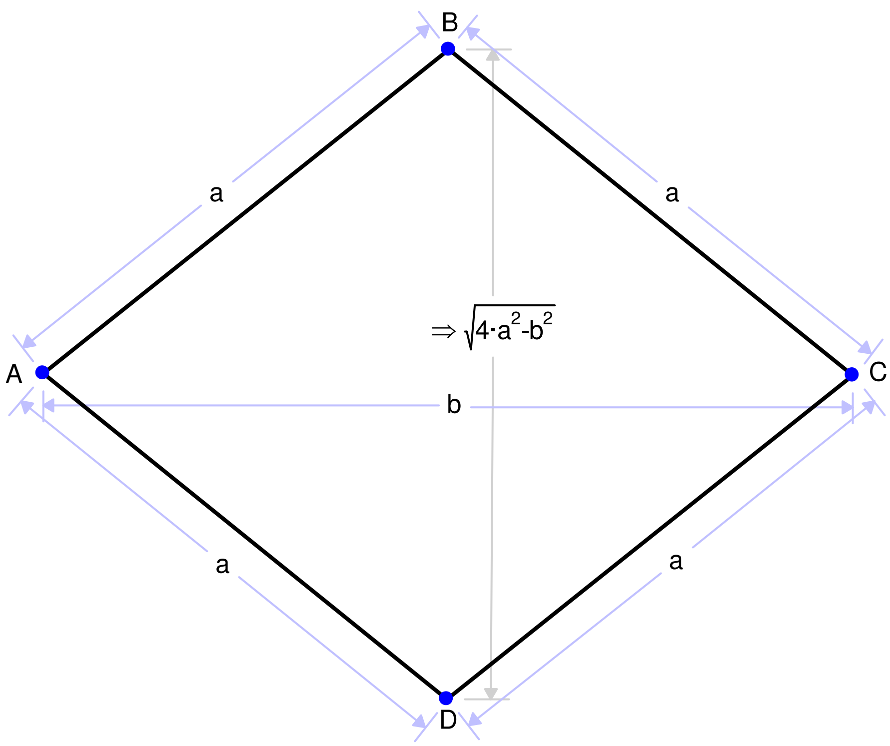
Quadrilaterals

Here are some examples using quadrilaterals

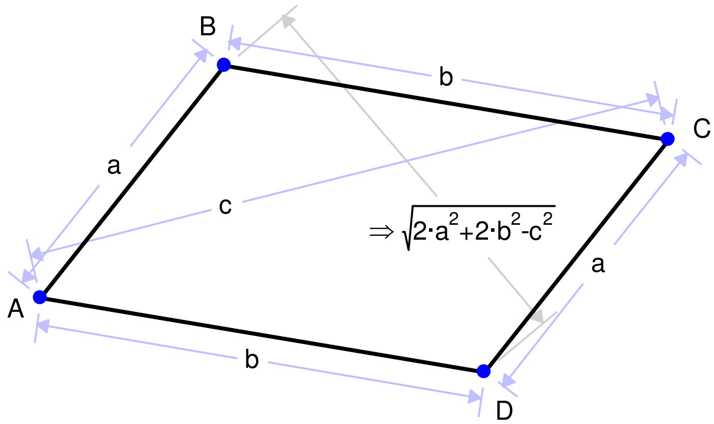
Example 30: Diagonals of a rhombus

A rhombus has sides length a and one diagonal length b, what is the length of the other diagonal?



Example 31: Diagonals of a parallelogram

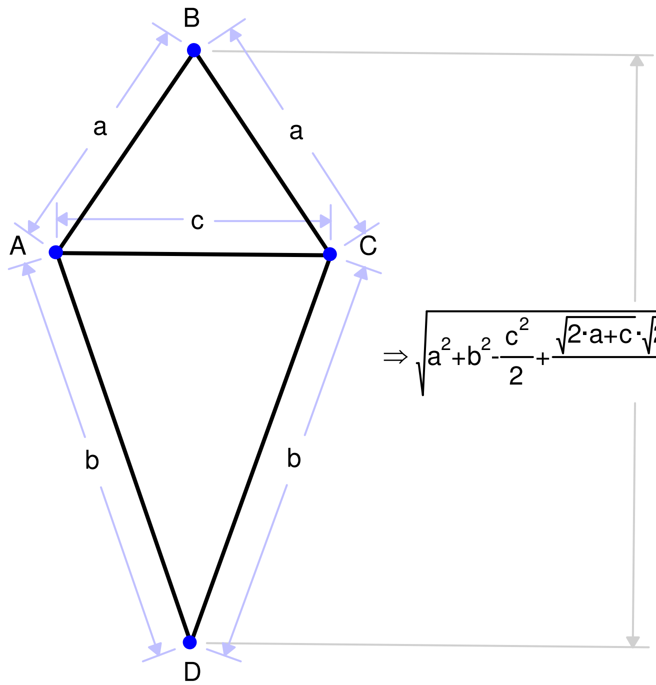
Given a parallelogram whose sides measure a and b, and one diagonal is c, what is the length of the other diagonal?



A simple enough result, but can you derive it? I used the cosine rule, but can you do it by Pythagoras alone?

Example 32: Diagonals of a Kite

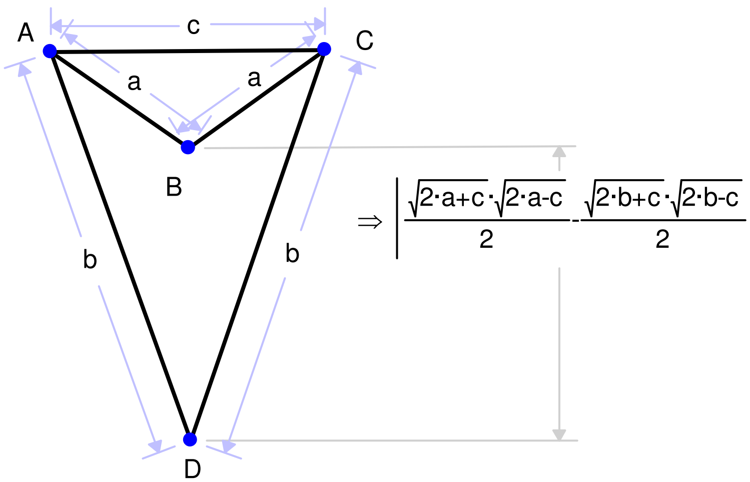
Continuing in this theme, if we have a kite whose non-axis diagonal is length c, and whose sides are length a and b, what is the length of the other diagonal?



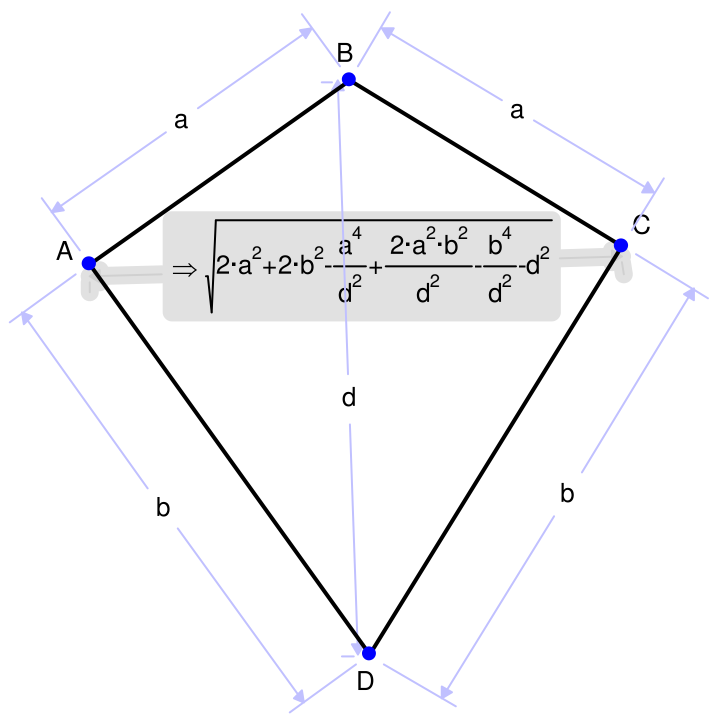
$$\Rightarrow \sqrt{a^2+b^2-\frac{c^2}{2}+\frac{\sqrt{2\cdot a+c}\cdot\sqrt{2\cdot a-c}\cdot\sqrt{2\cdot b+c}\cdot\sqrt{2\cdot b-c}}{2}}$$

Can you see the similarity to the parallelogram?

What about the non-convex kite with the same side lengths and diagonal?



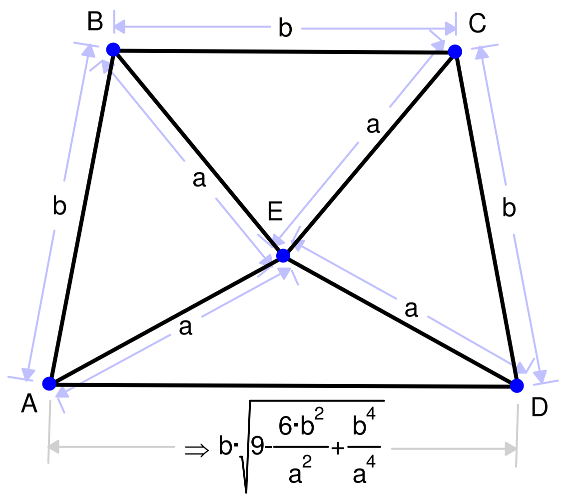
How about if we are specified the axis, what is the non-axis diagonal?



This is very similar to the equation of the altitude of a triangle. Why?

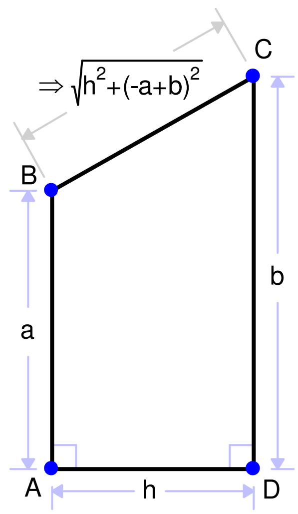
Example 33: Cyclic Quadrilateral

A quadrilateral is inscribed in a circle of radius a. 3 sides of the quadrilateral have length b. What is the length of the fourth side?



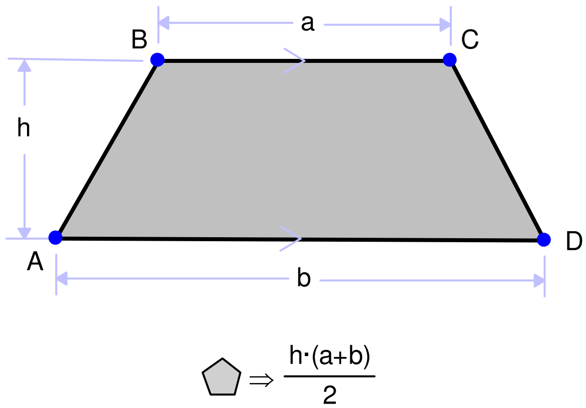
What is the relationship between a and b when the fourth side has length b?

Example 34: A Right Trapezoid

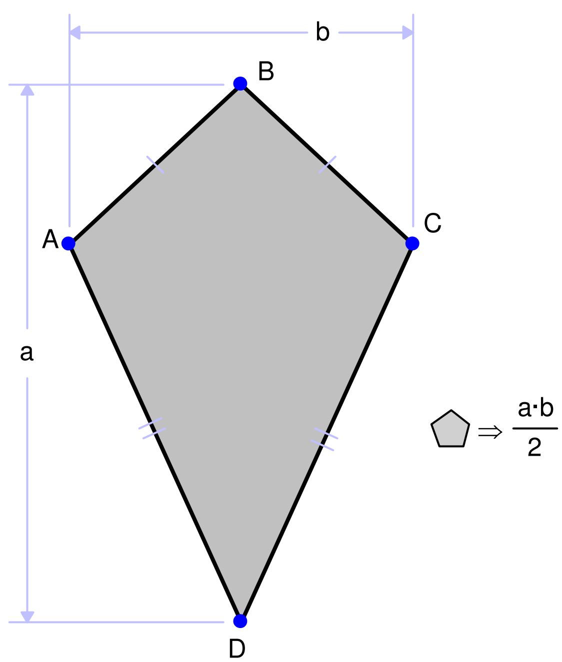


Example 35: **Areas of Quadrilaterals**

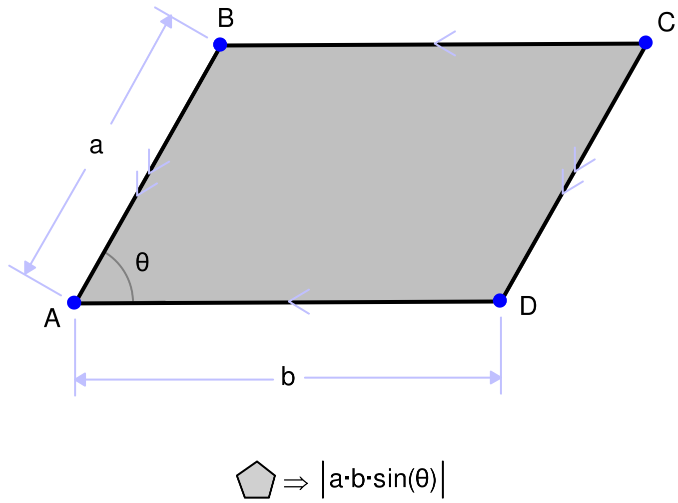
Trapezium:



Kite:

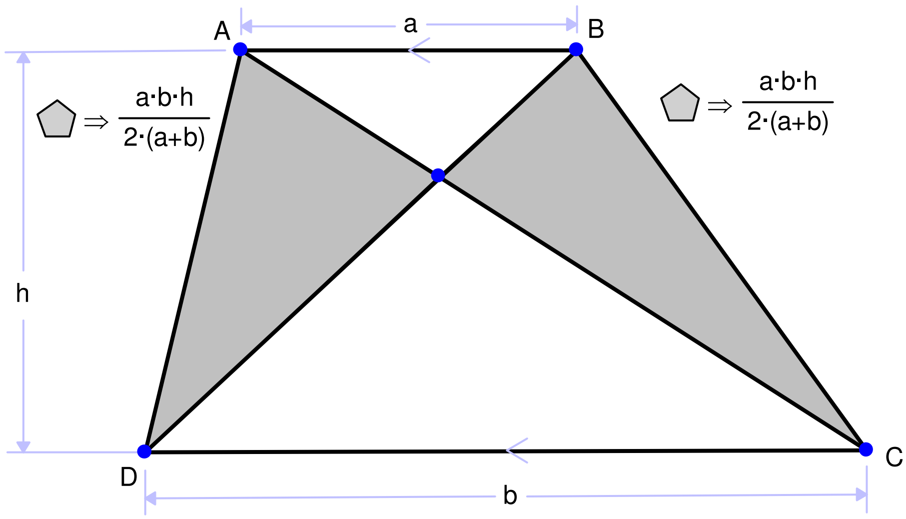


Parallelogram:

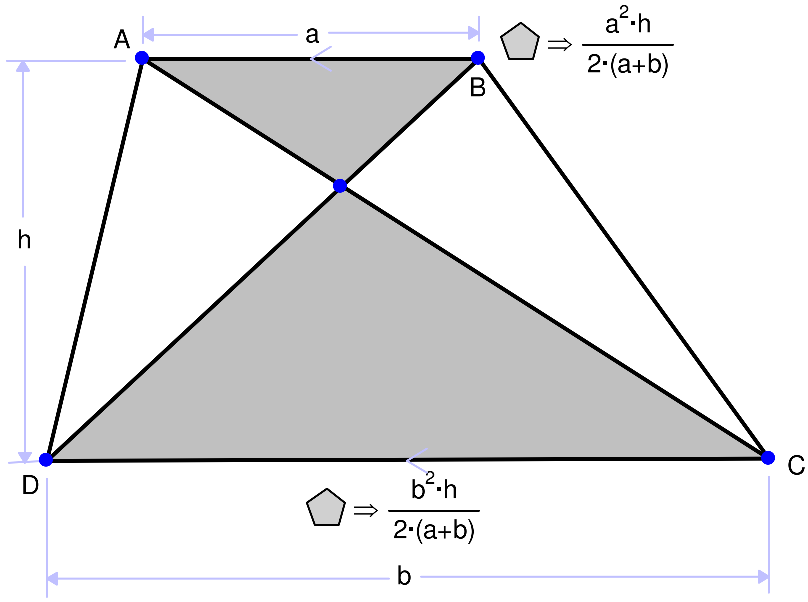


Example 36: Areas of triangles in a trapezoid

One pair of triangles formed by the diagonals of a trapezoid are equal in area.

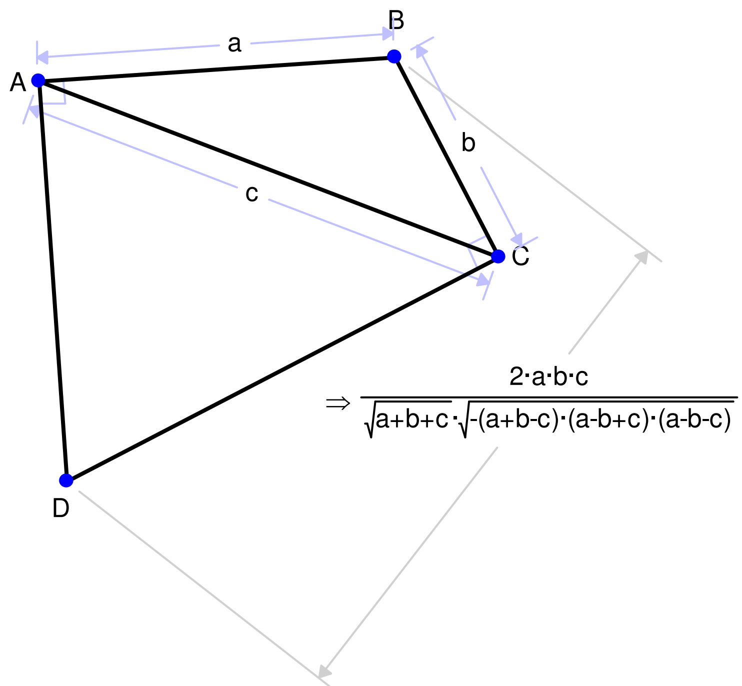


The other pair are not:



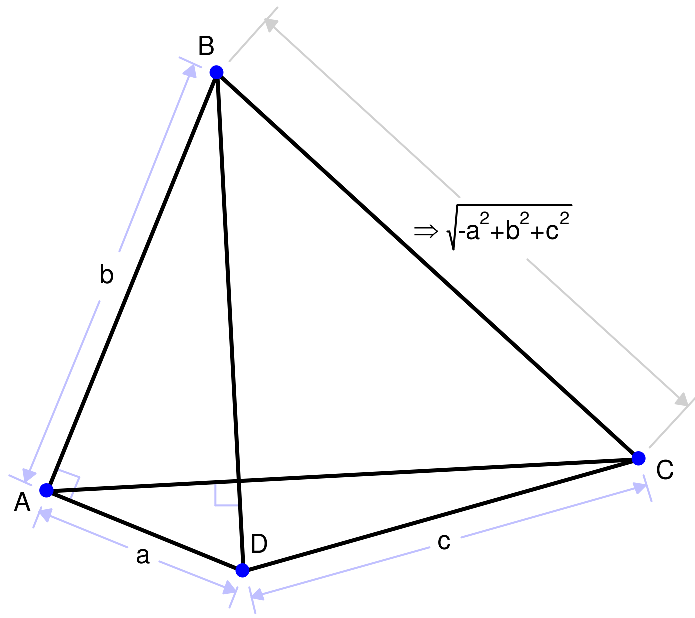
Example 37: Diameter of the circumcircle

Here is a diagram which allows us to find the diameter of the circumcircle of the triangle whose sides have lengths a, b and c. Why?



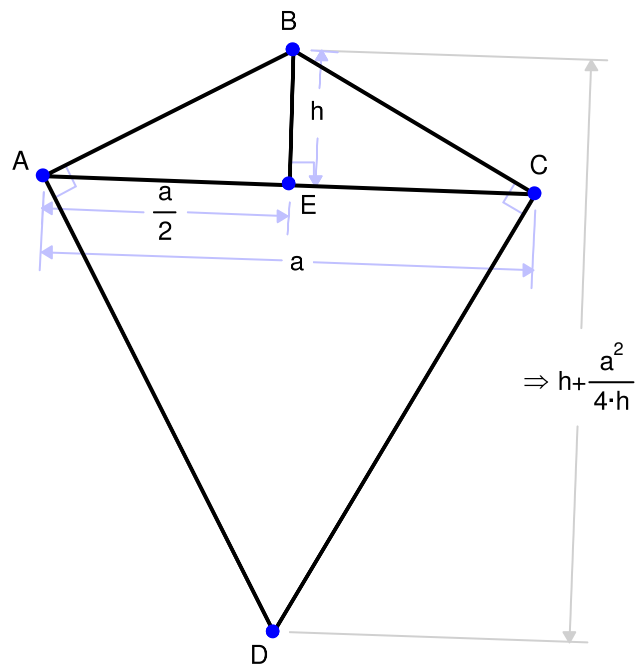
Example 38: Quadrilateral with perpendicular diagonals and one right angle

A Quadrilateral has 3 sides length a,b, c and a right angle. It's diagonals are perpendicular. What is the length of the remaining side?



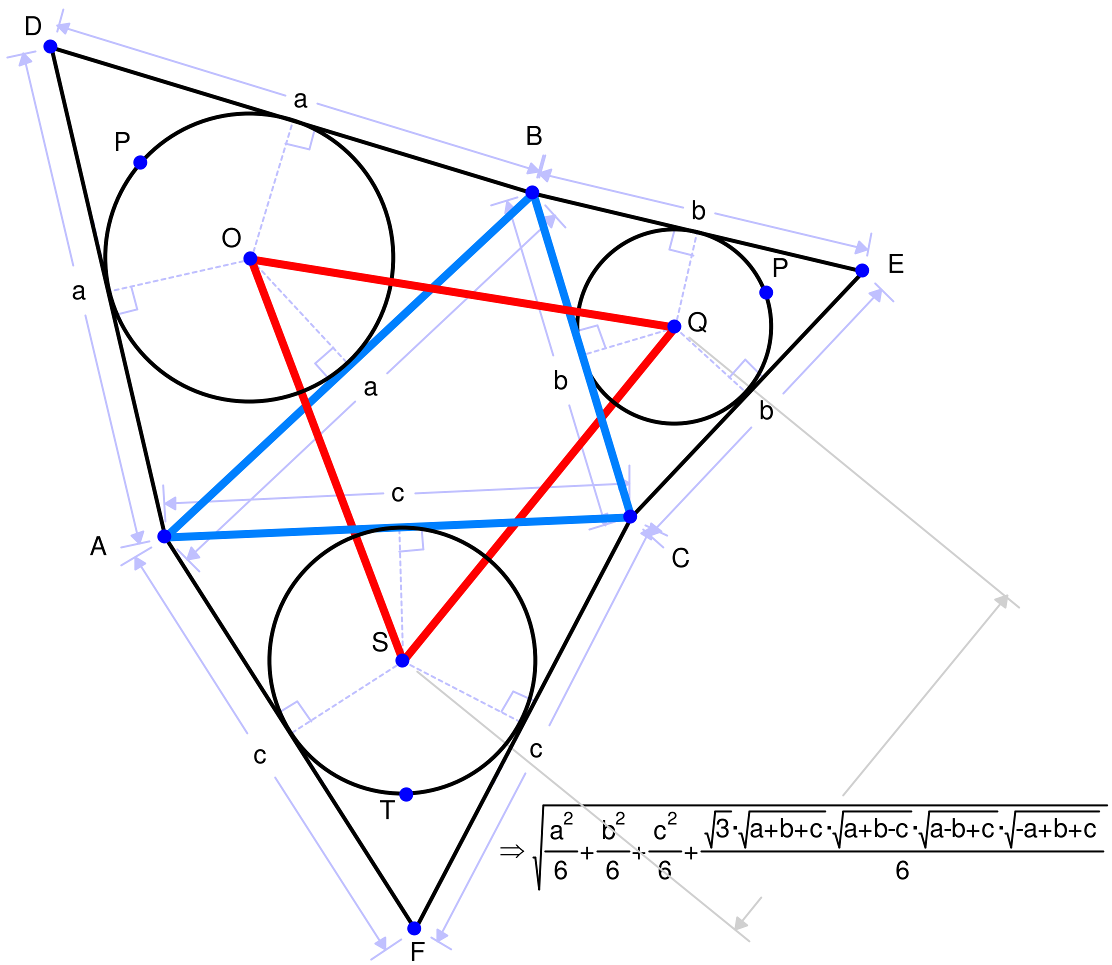
Example 39: Finding the diameter of an arc given the perpendicular offset from the chord.

Here is the diagram. Verify that it is correct:



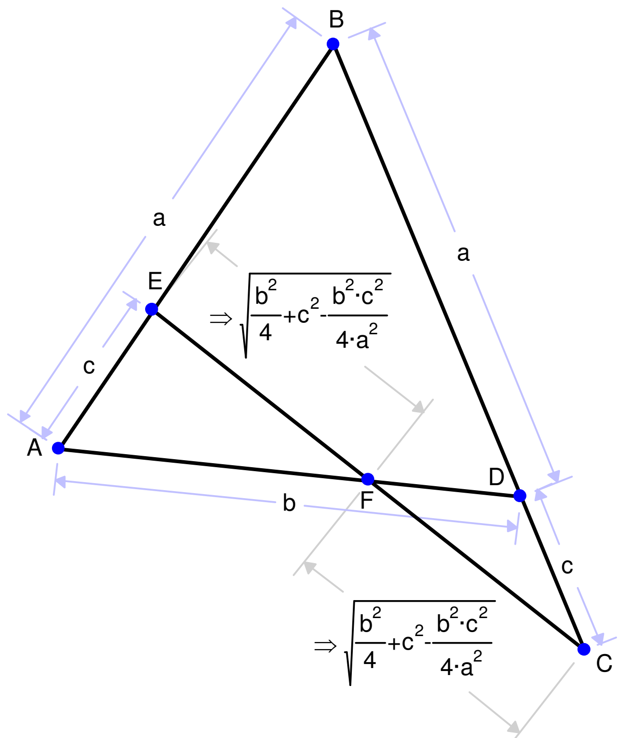
Example 40: Napoleon's Theorem

Napoleon's Theorem states that if you take a general triangle and draw an equilateral triangle on each side, then the triangle formed by joining the incenters of these new triangles is equilateral. You can see that the length is symmetrical in a,b,c and hence identical for the three sides of the triangle.



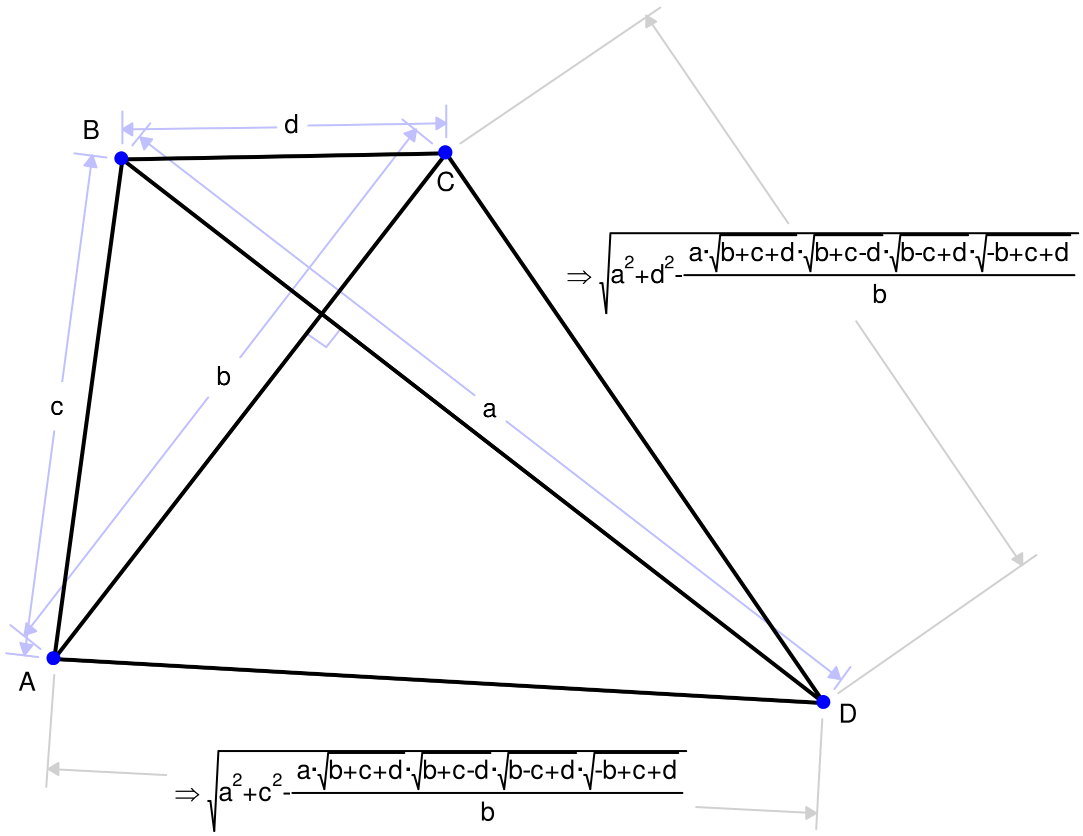
Example 41: An Isosceles Triangle Theorem

ABC is isosceles. AE=DC. We show that EF=FC.



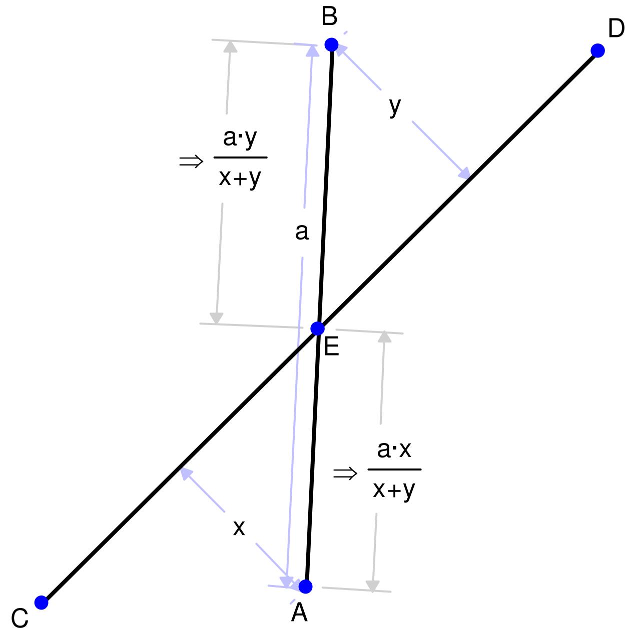
Example 42: A Quadrilateral with Perpendicular Diagonals

Given two sides, the lengths of the diagonals and the fact that they are perpendicular, what are the lengths of the other two sides of a quadrilateral?

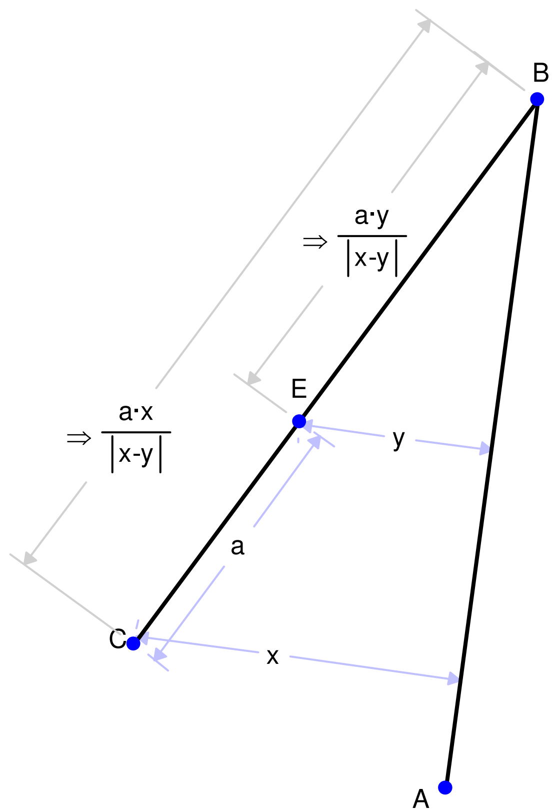


Example 43: Intersection of Common Tangent with Axis of Symmetry of Two Circles

Line AB has length a. A is perpendicular distance x from line CD, and B is perpendicular distance y from CD. Find the distance of the intersection point between AB and CD from A and B:

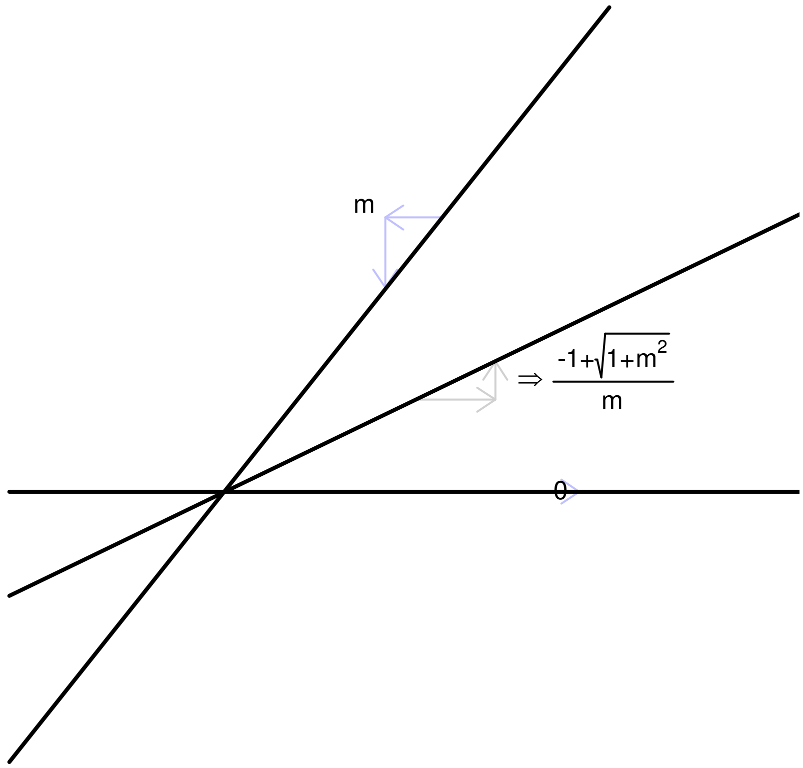


What if we change the diagram slightly so that E is external to AB:



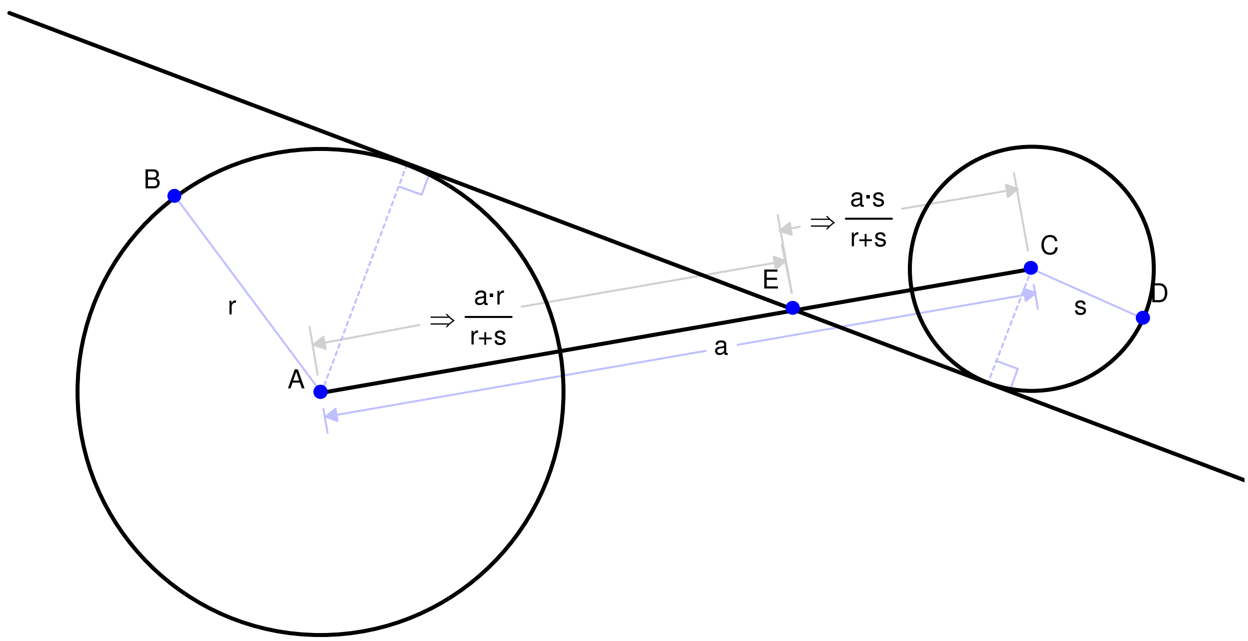
Example 44: Slope of the Angle Bisector

What is the slope of the angle bisector of a line with slope 0 and a line with slope m?

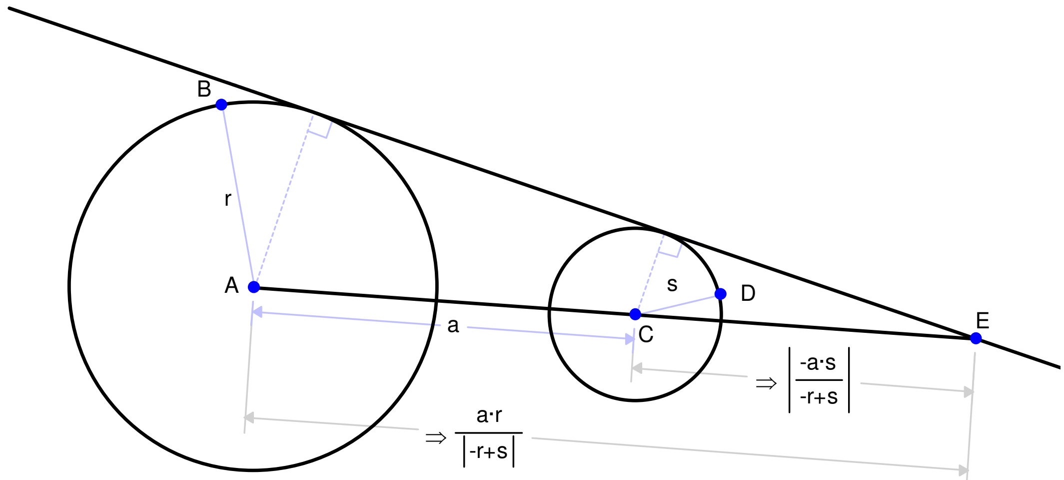


Example 45: Location of intersection of common tangents

Circles AB and CD have radii r and s respectively. If the centers of the circles are a apart, and E is the intersection of the interior common tangent with the line joining the two centers, what are the lengths AE and CE?

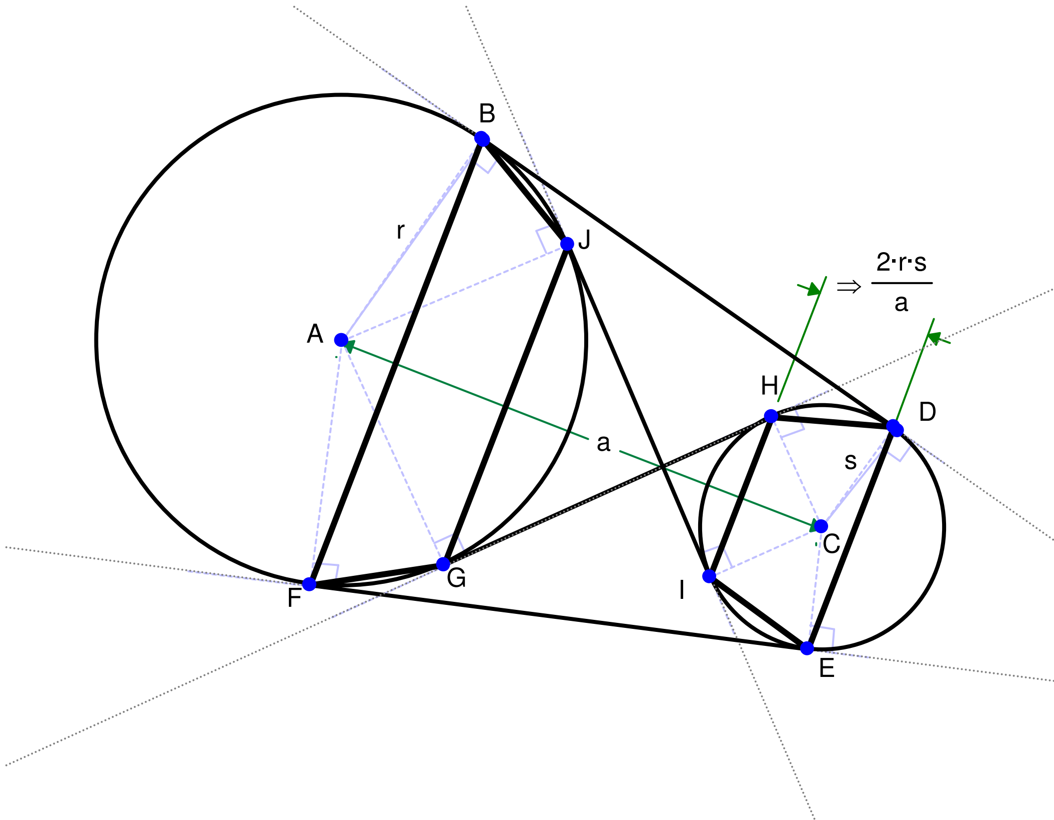


How about the exterior common tangent?



Example 46: Altitude of Cyclic Trapezium defined by common tangents of 2 circles

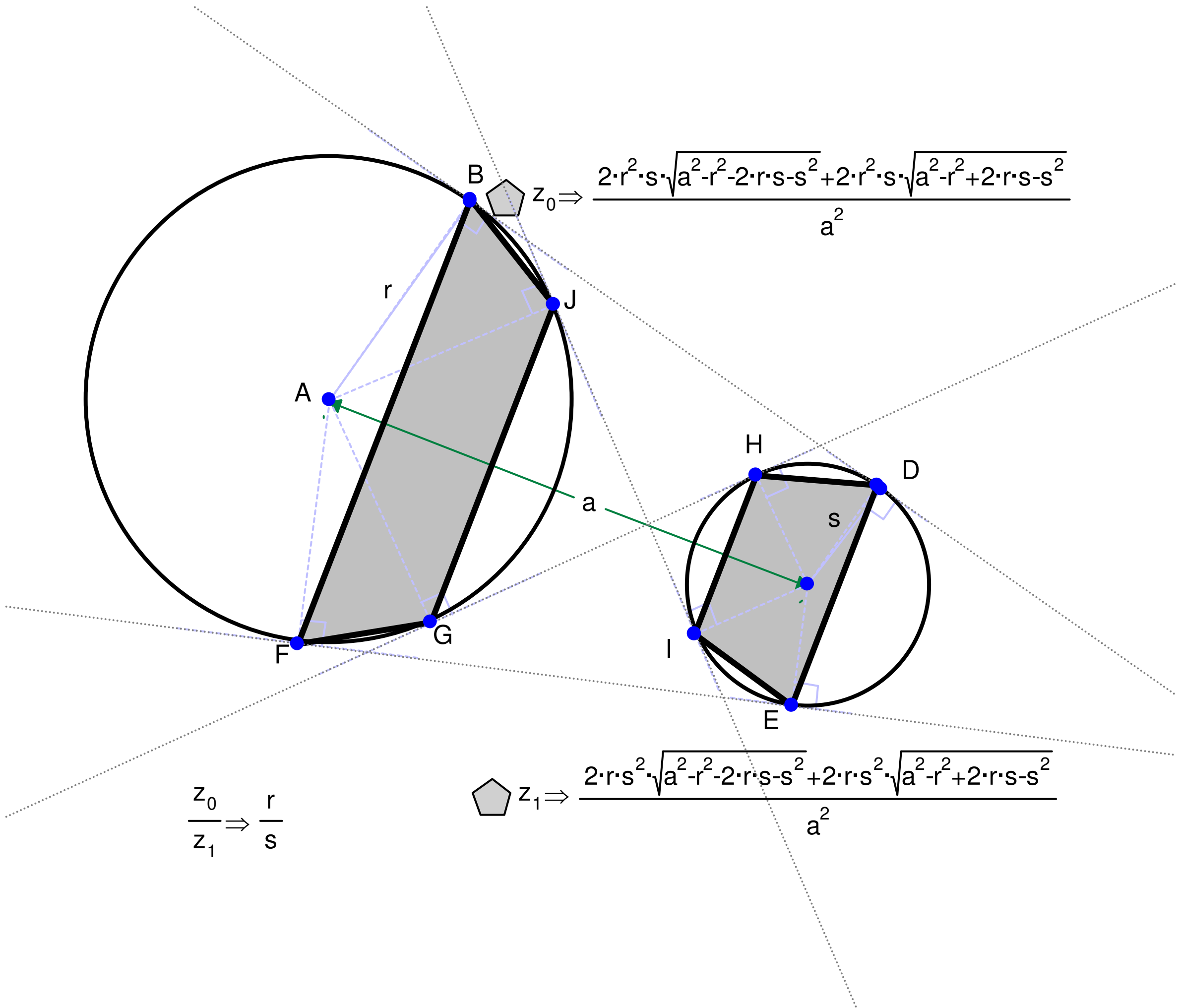
Given circles radii r and s and distance a apart, what is the altitude of the trapezium formed by joining the intersections of the 4 common tangents with one of the circles?



Notice that this is symmetrical in r and s, and hence the trapezium in circle AB has the same altitude.

Example 47: Areas Cyclic Trapezia defined by common tangents of 2 circles

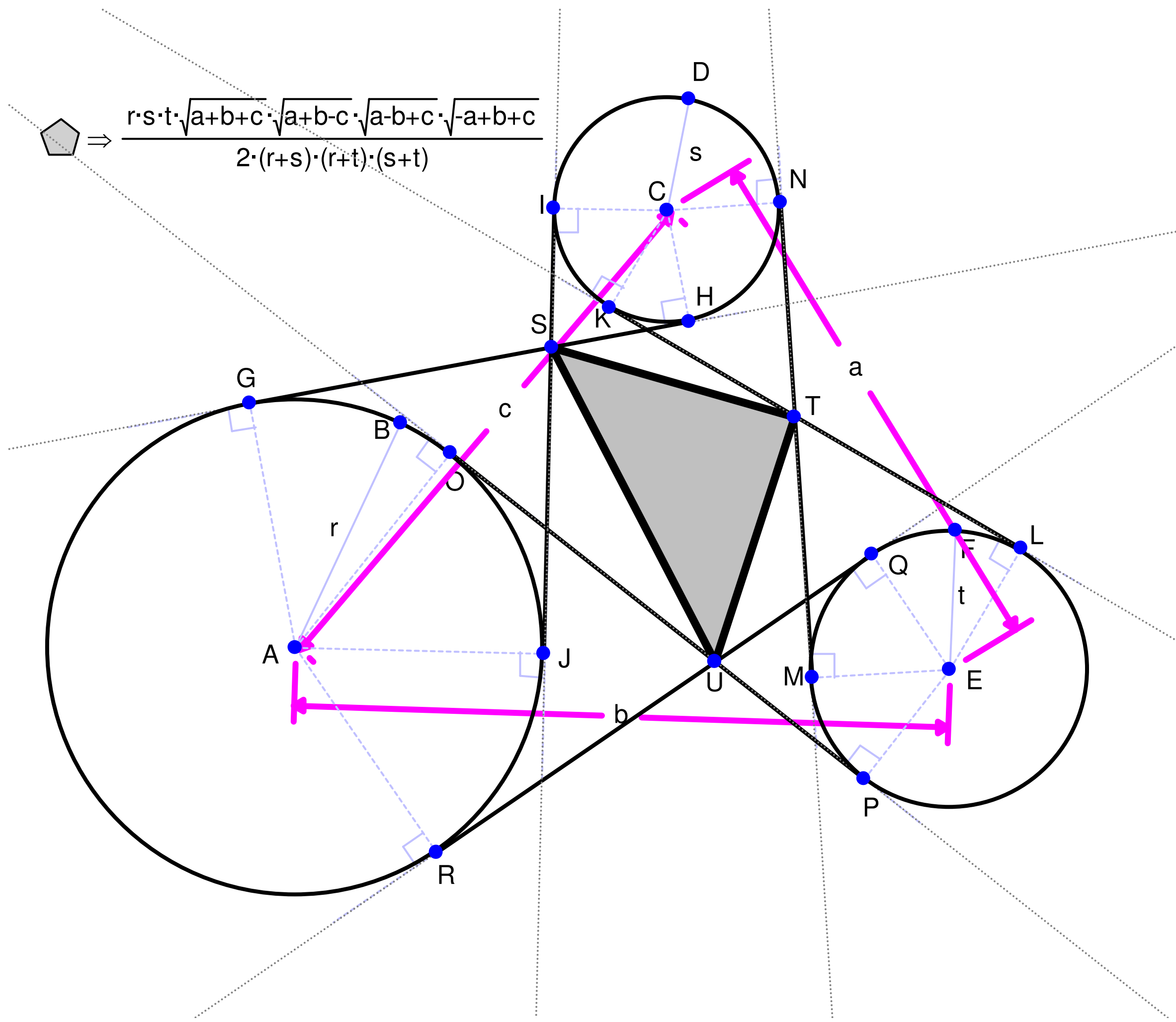
Look at the ratio of the areas of the trapezia in the previous example:



Example 48: Triangle formed by the intersection of the interior common tangents of three circles

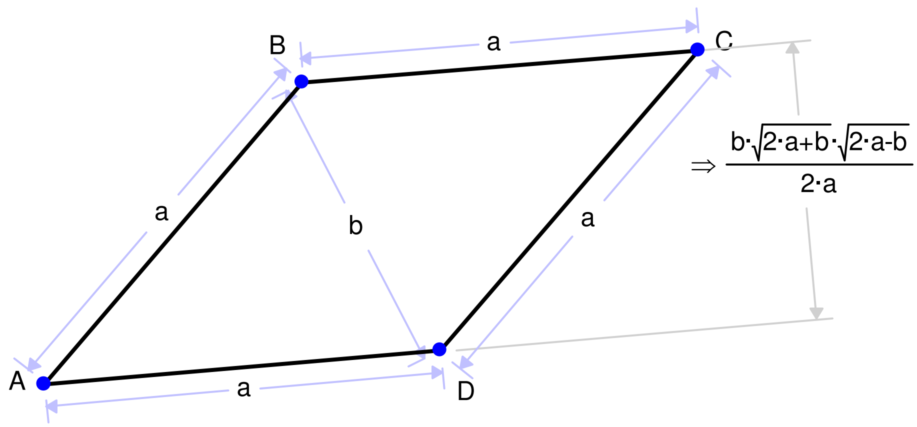
Notice that if A is the area of the triangle formed by the centers of the circles, then area STU is:

$$\frac{2rstA}{(r+s)(s+t)(r+t)}$$



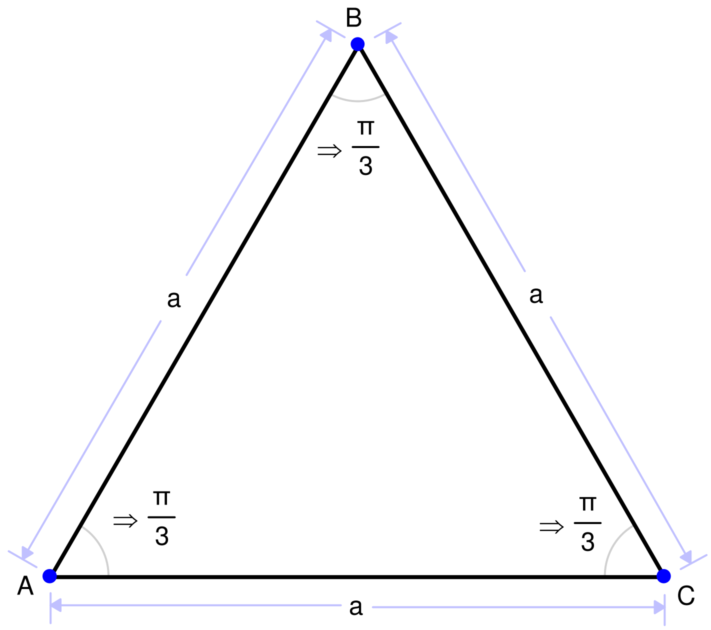
Example 49: Distance between sides of a rhombus

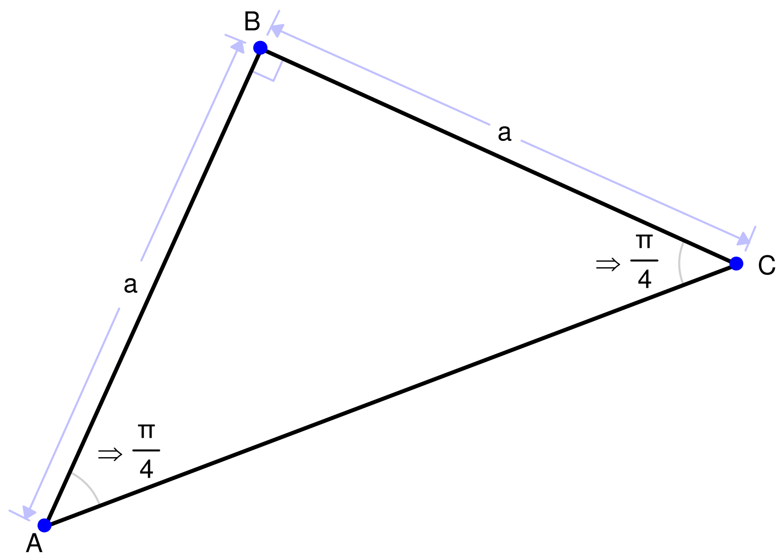
Given a rhombus with side length a and diagonal b, what is the perpendicular distance between opposite sides?



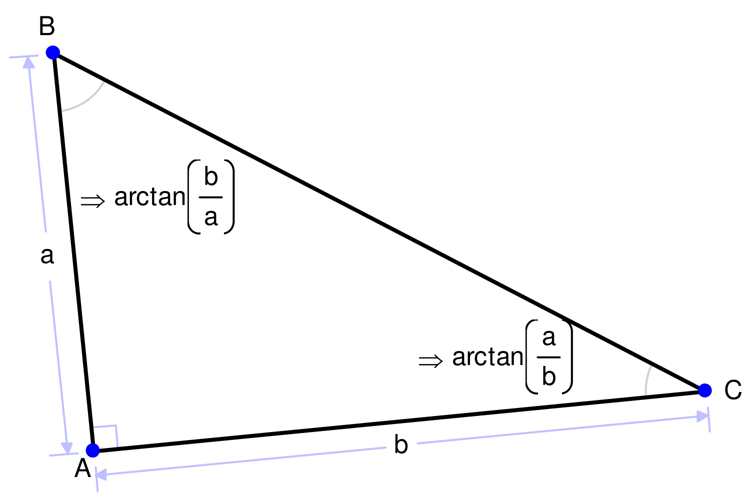
Example 50: Angles of Specific Triangles

Here are some triangles, with their angles displayed





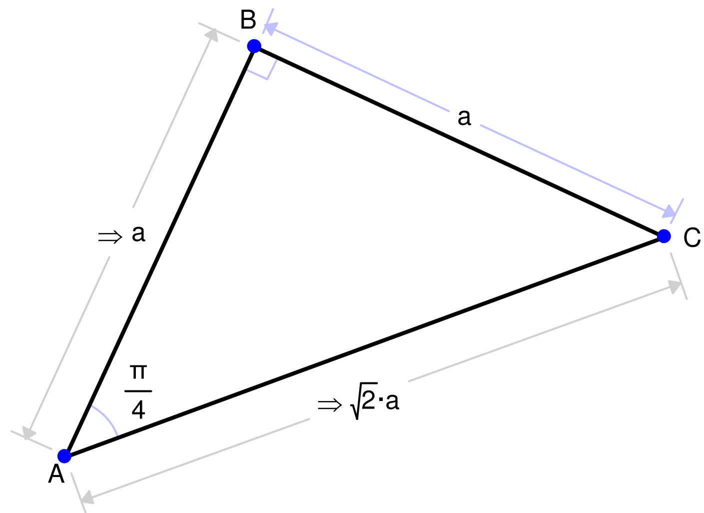
B
a
a
$\Rightarrow \frac{\pi}{4}$
C
$\Rightarrow \frac{\pi}{4}$
A

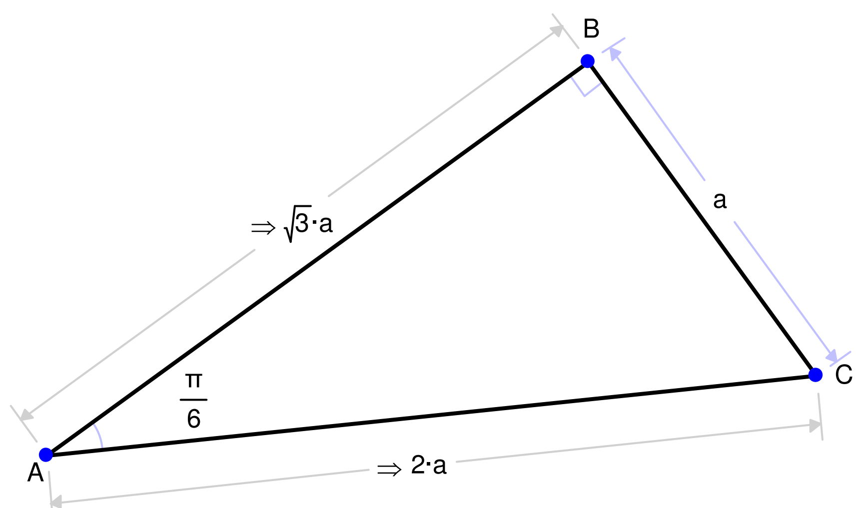


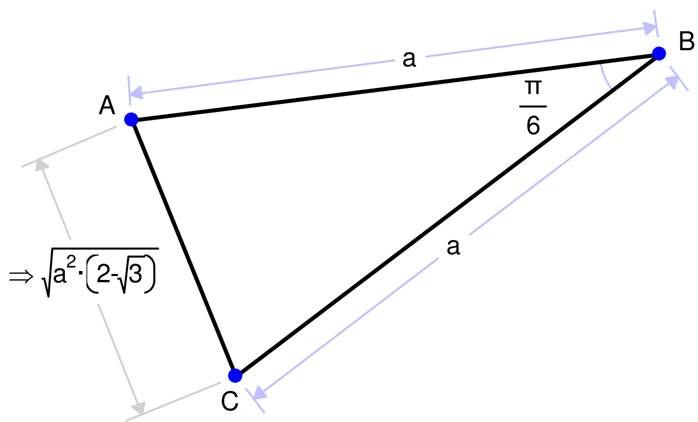
B
$\Rightarrow \arctan\left(\frac{b}{a}\right)$
a
$\Rightarrow \arctan\left(\frac{a}{b}\right)$
C
A
b

Example 51: Sides of Specific Triangles

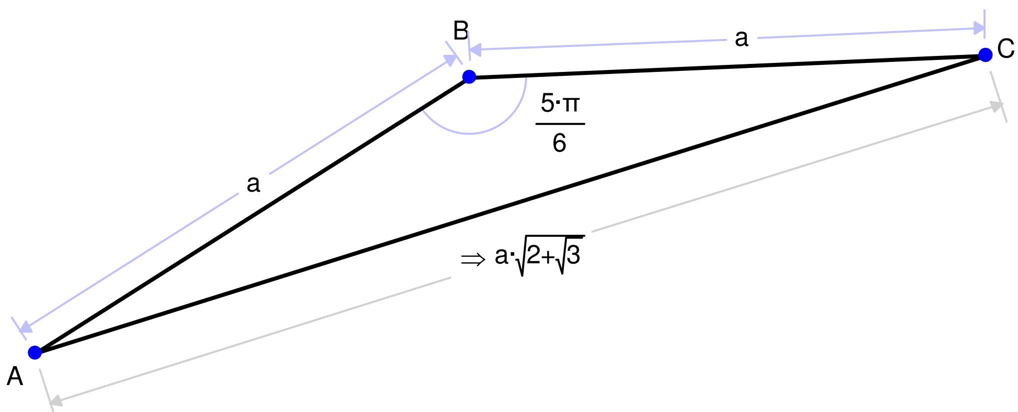
Here are some specific angle-defined triangles:







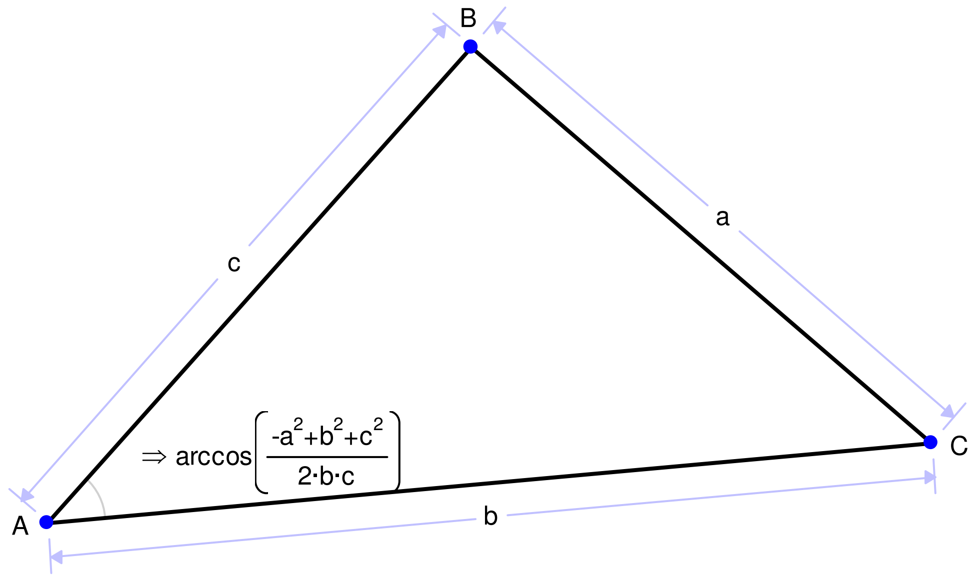
A
B
C
a
a
$\frac{\pi}{6}$
$\Rightarrow \sqrt{a^2 \cdot (2-\sqrt{3})}$



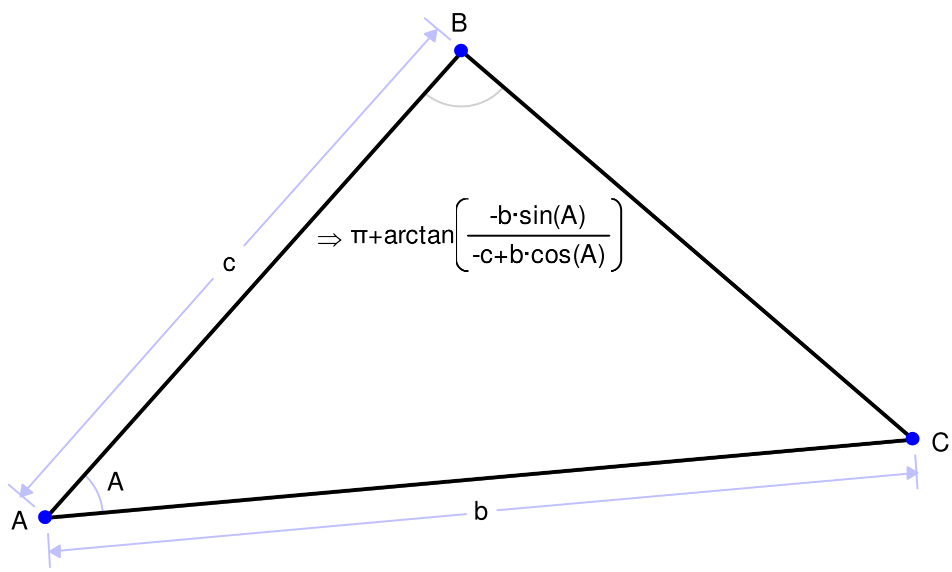
A
B
C
a
a
$\frac{5\cdot\pi}{6}$
$\Rightarrow a\cdot\sqrt{2+\sqrt{3}}$

Example 52: Angles in the general triangle

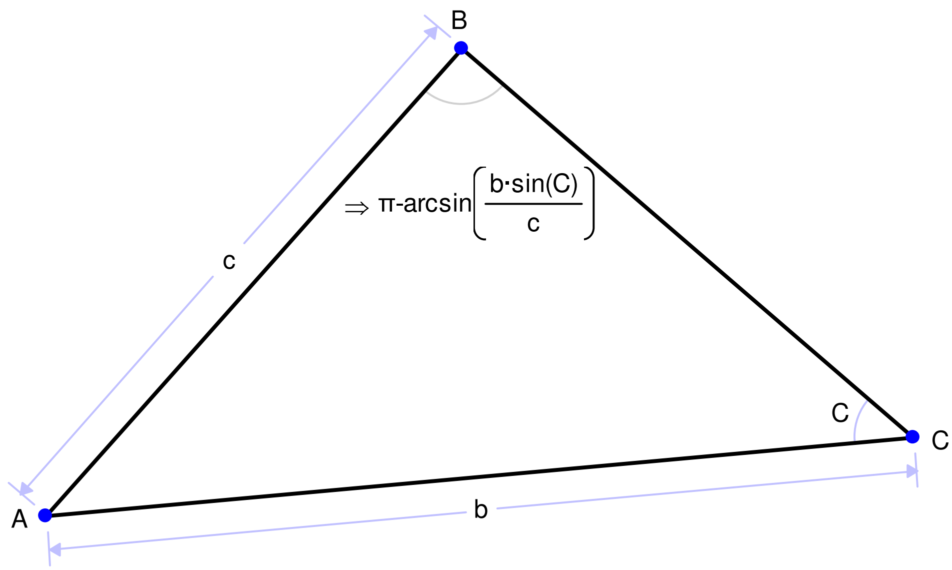
3SIDES DEFINED



TRIANGLE WITH 2 SIDES AND INCLUDED ANGLE

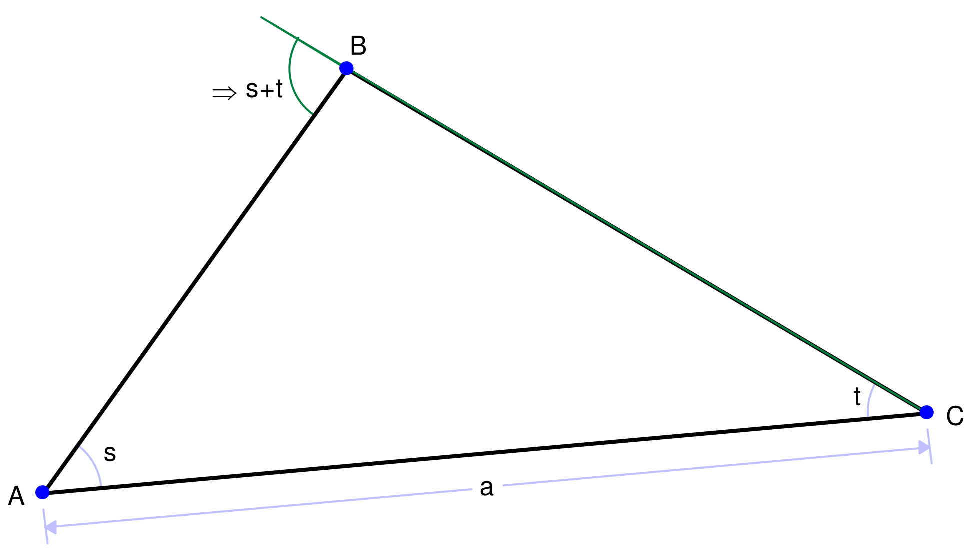


TRIANGLE WITH 2 SIDES AND NON-INCLUDED ANGLE



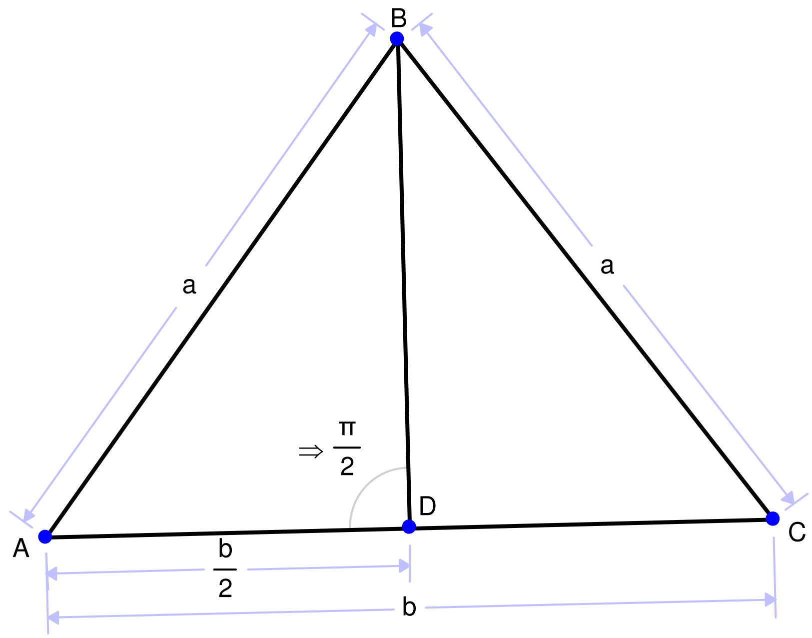
TRIANGLE WITH TWO ANGLES

Here's the exterior angle:

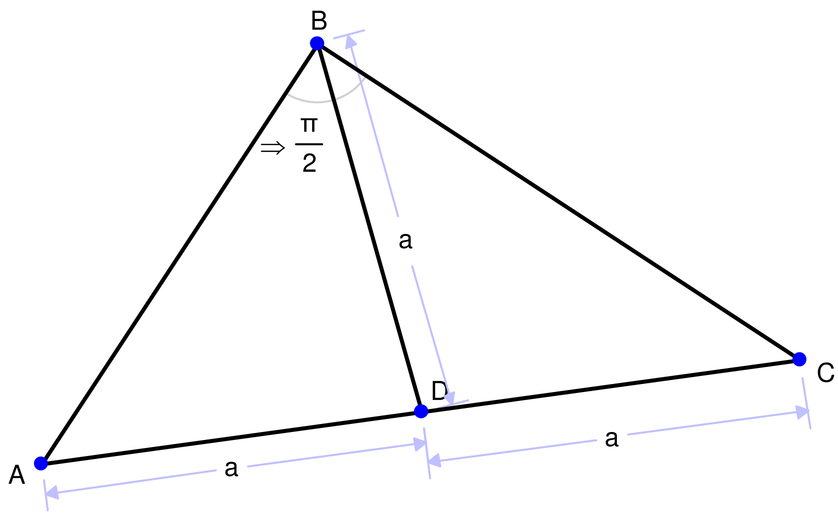


Example 53: Some implied right angles

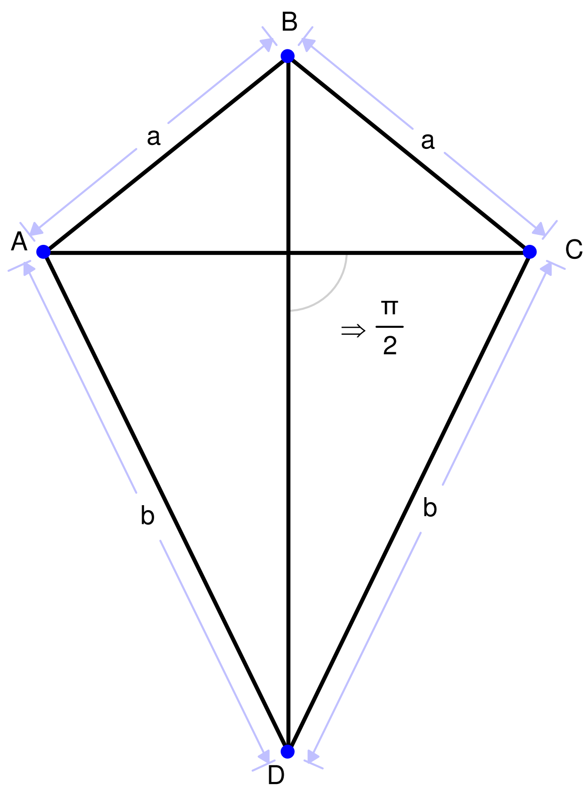
The median of an isosceles triangle is also its perpendicular bisector, and altitude:



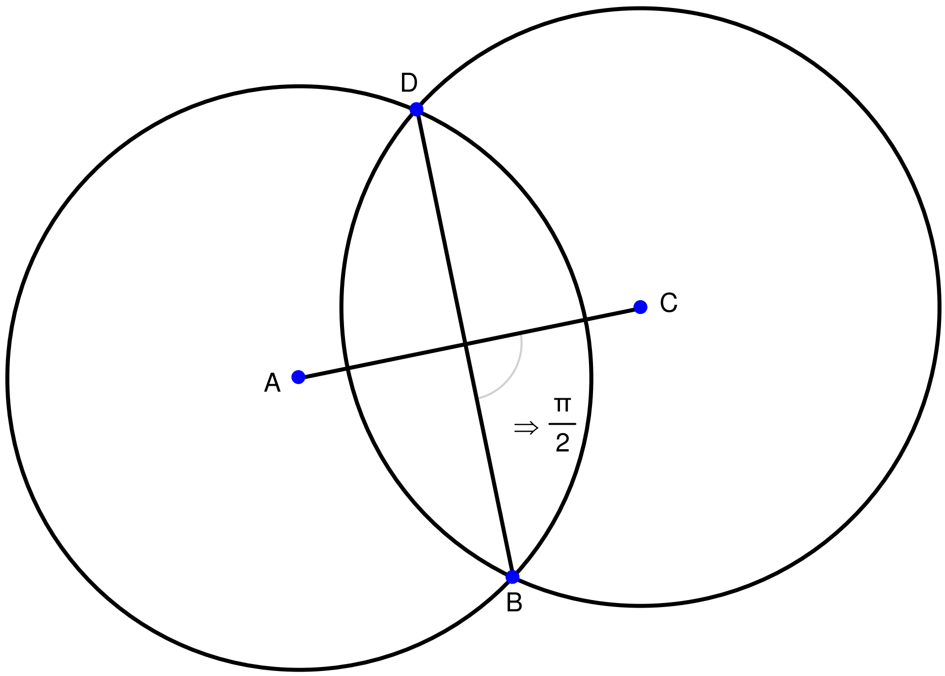
A triangle whose median is the same length as half its base is right angled:



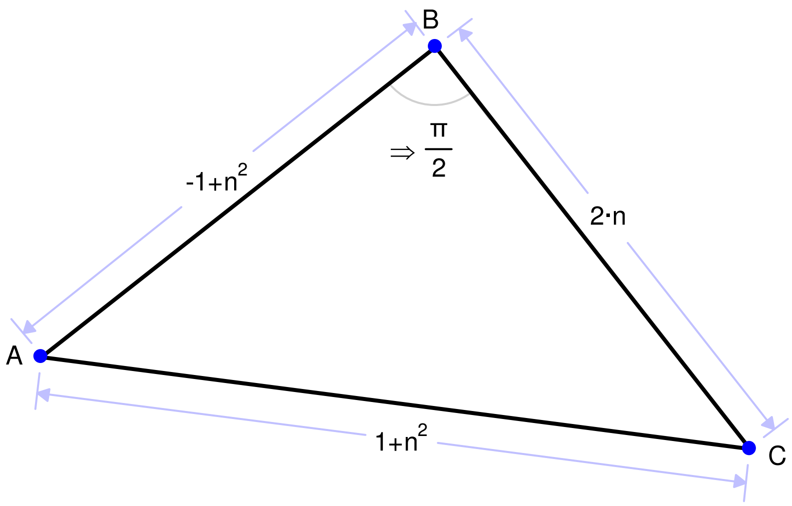
The diagonals of a kite are perpendicular



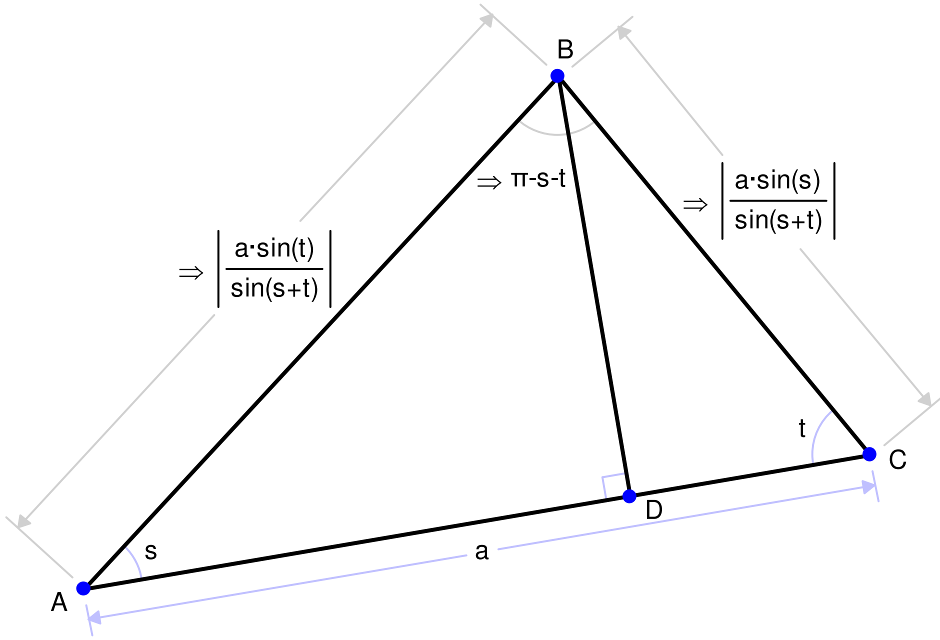
The line joining the intersection points of two circles is perpendicular to the line joining their centers:



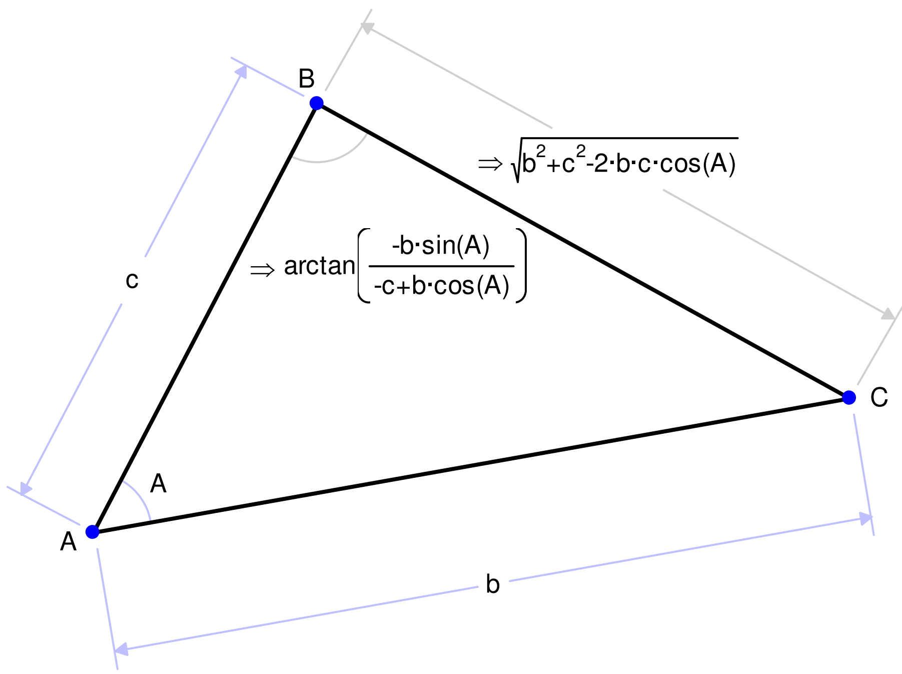
Here is a particular triangle (from the book “The Curious Incident of the Dog in the Night”)



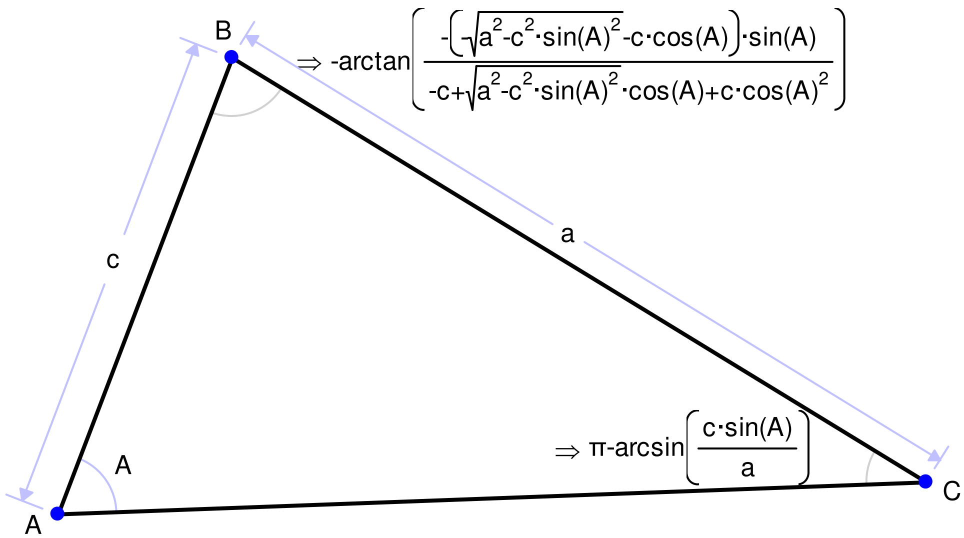
Example 54: **Triangle defined by 2 angles and a side**



Example 55: Triangle defined by two sides and the included angle

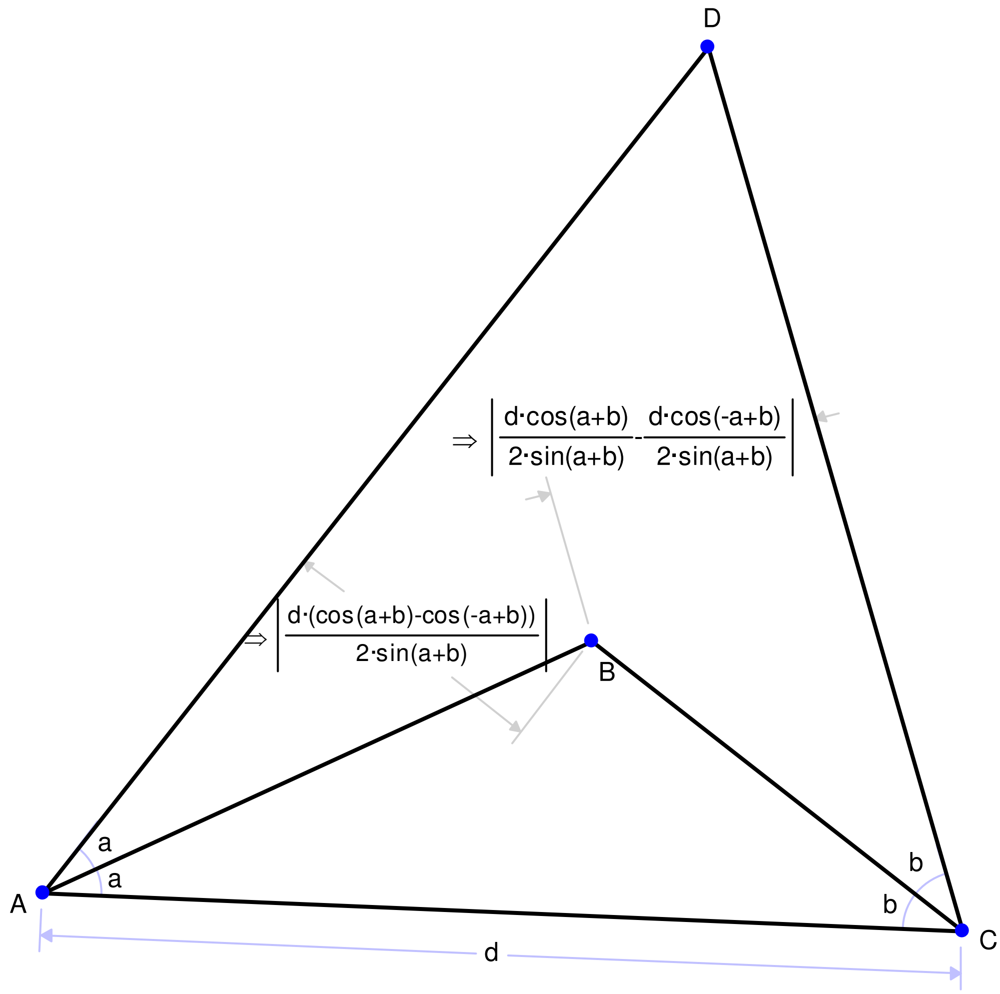


Example 56: Triangle defined by 2 sides and the non-included angle



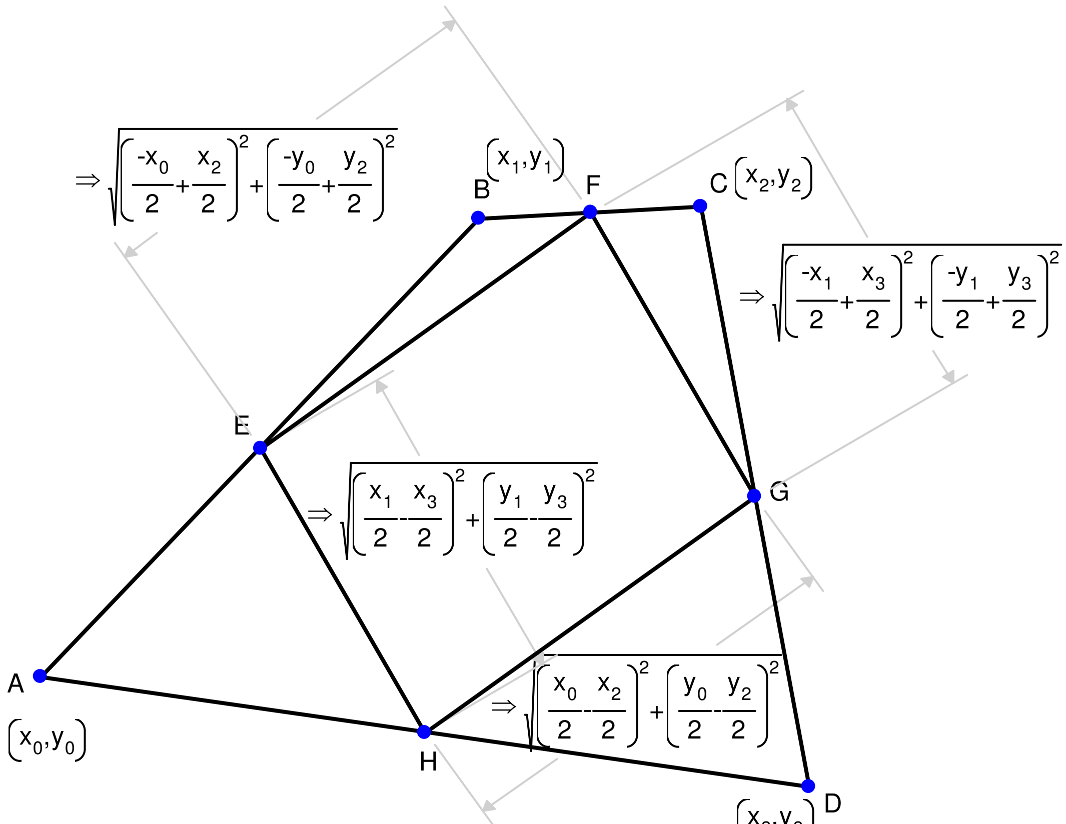
Example 57: Incenter

The incenter is the intersection of the angle bisectors. We have a triangle with angles 2a and 2b and base d:



We see that the perpendicular distance to the other sides is the same. This shows that B is the center of a circle tangent to AD and CD.

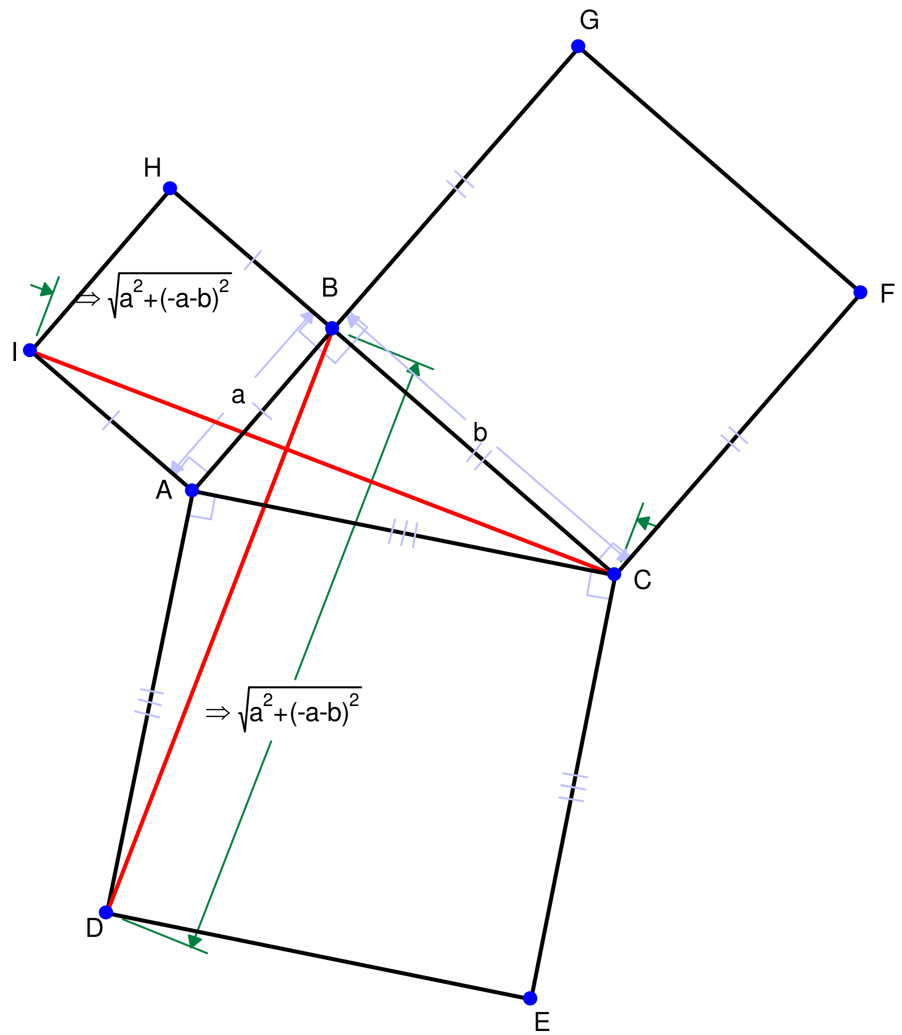
Example 58: Quadrilateral Formed by Joining the Midpoints of the sides of a Quadrilateral



Quick inspection of the side lengths shows that the new figure is a parallelogram

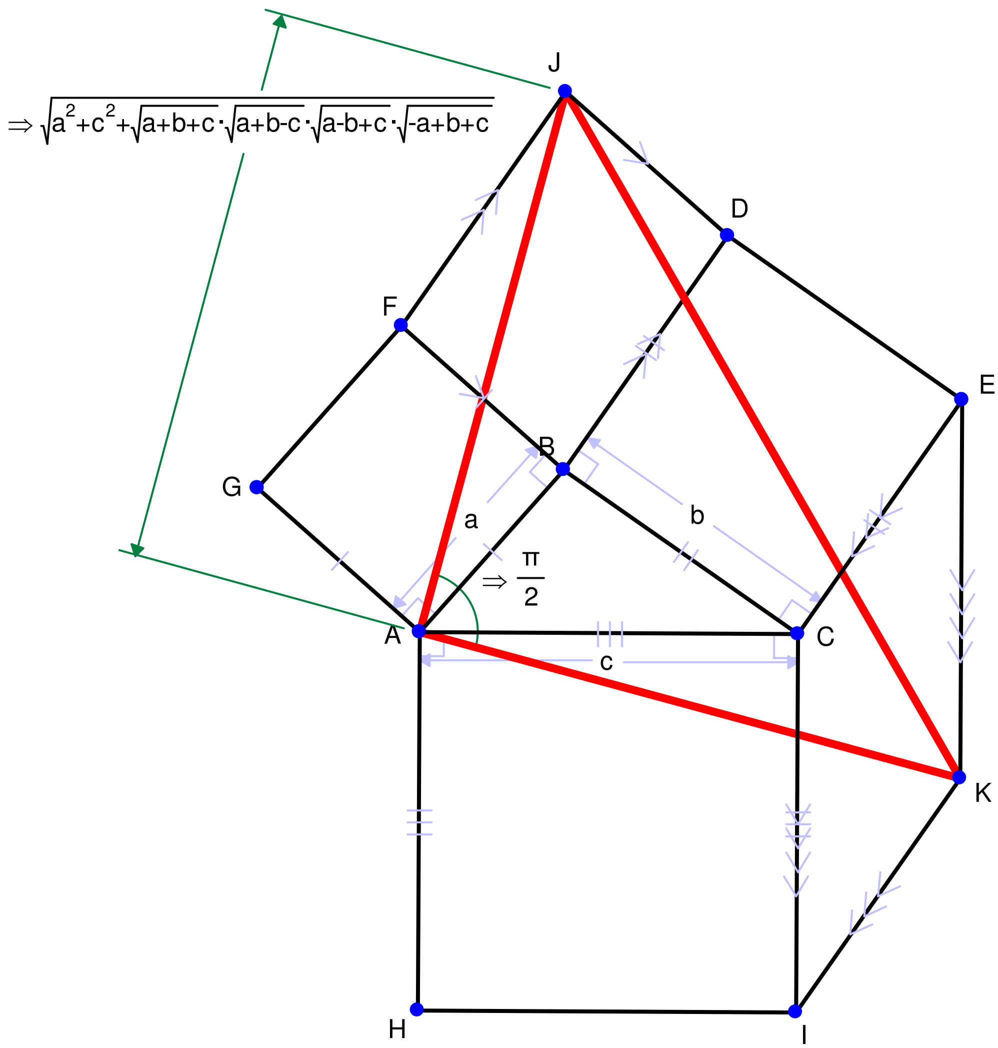
Example 59: Some measurements on the Pythagoras Diagram

Draw a right triangle and subtend a square on each side. The two red lines in the diagram are equal in length.



Example 60: An unexpected triangle from a Pythagoras-like diagram

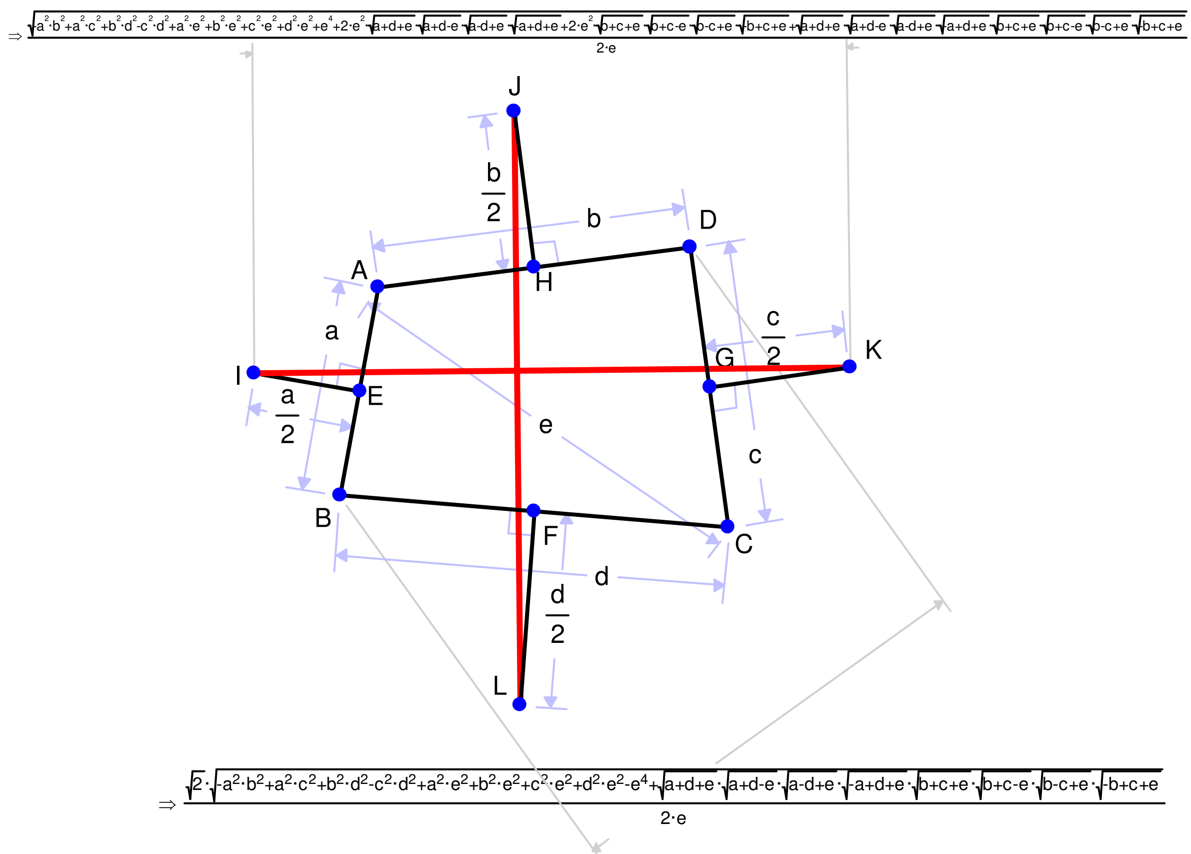
Regardless of the original triangle the resulting triangle from this diagram is a right angled-isosceles:



Examination of the length AJ shows that it is symmetric in a and b, and hence identical to AK.

Example 61: A Theorem on Quadrilaterals

This theorem states that if you draw a square on each side of a quadrilateral, then connect the center of opposite sides, the resulting lines have the same length, and are perpendicular. Here is the result in Geometry Expressions



If we create the length of the other side we can by careful examination see that the lengths are identical. Alternatively, we can do some simplification. Our constraints are necessarily asymetric – Geometry Expressions will not let you over-constrain the diagram, and one diagonal is sufficient to define the quadralateral. However, we might expect the formula to be simpler if expressed in terms of both diagonals.

Close inspection of the formula for the length shows that it incorporates the square of the other diagonal of the figure, as well as Heron's formula for the areas of the triangles ABC and ACD. The following Mathematica worksheet contains the formulas from Geometry Expressions for L the length of the desired line, and f the length of the other diagonal. A little manipulation gives a simple formula for L^2-f^2/2. This can be simplified further by noting that the remaining terms are e^2/2 and twice the area of the quadralateral:

In[6]:= $L =$ [illegible]

Out[6]= [illegible]

In[4]:= $f =$ [illegible]

Out[4]= [illegible]

In[7]:= `L^2`

Out[7]= [illegible]

In[8]:= `f^2`

Out[8]= [illegible]

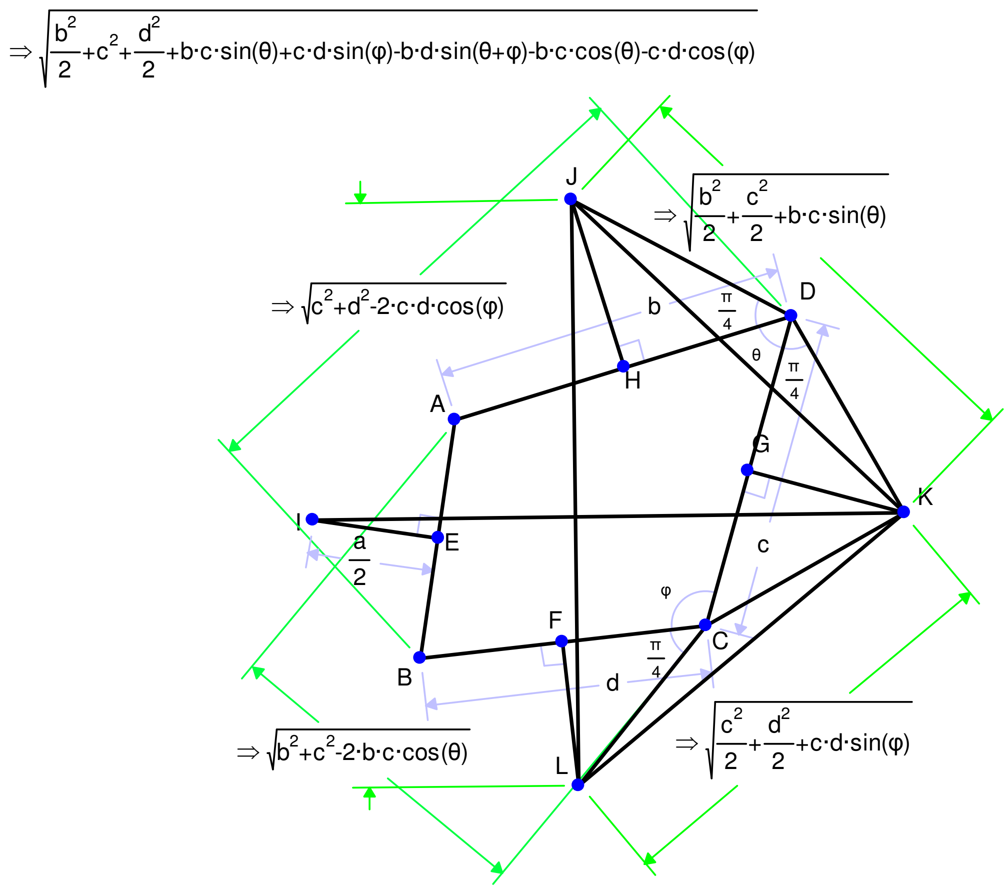
In[11]:= `Simplify` [illegible]

Out[11]= [illegible]

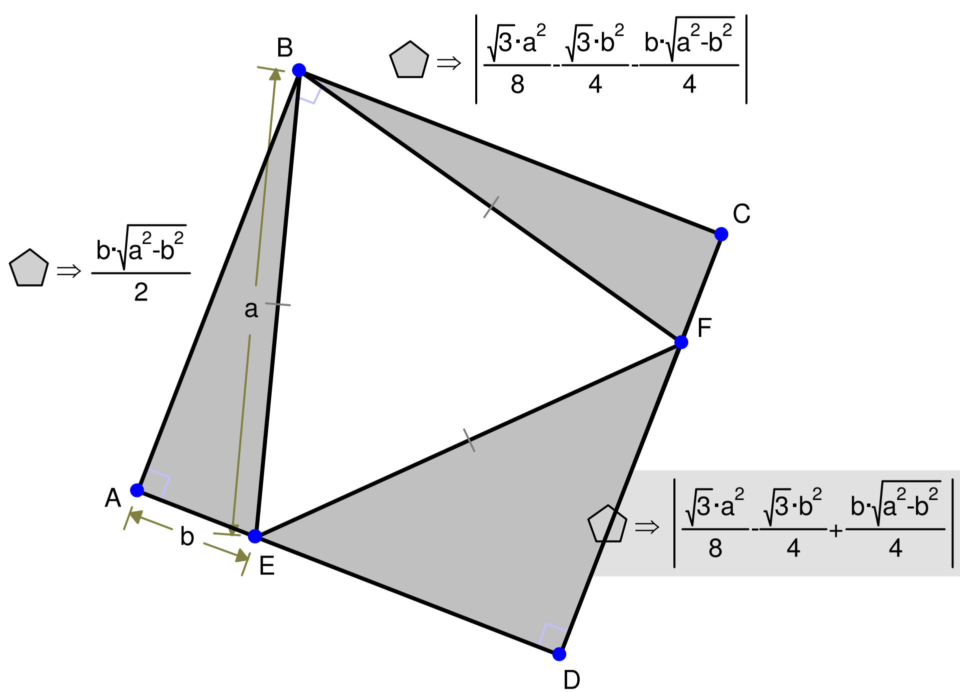
From which we can derive that:

$$L^2 = \frac{e^2}{2} + \frac{f^2}{2} + 2A$$

Areas are simpler when expressed in terms of angles. Here is a revision of the diagram with angles inserted. This gives us more of a clue of how to prove the result:



Example 62: Rectangle Circumscribing an Equilateral Triangle



Directly from the diagram we have the following theorem:

The area of the larger right triangle is the sum of the areas of the smaller two.

This appears in page 19-21 of Mathematical Gems, by Ross Honsberger (and various other places).