Sectors Centered at Vertices of a Triangle

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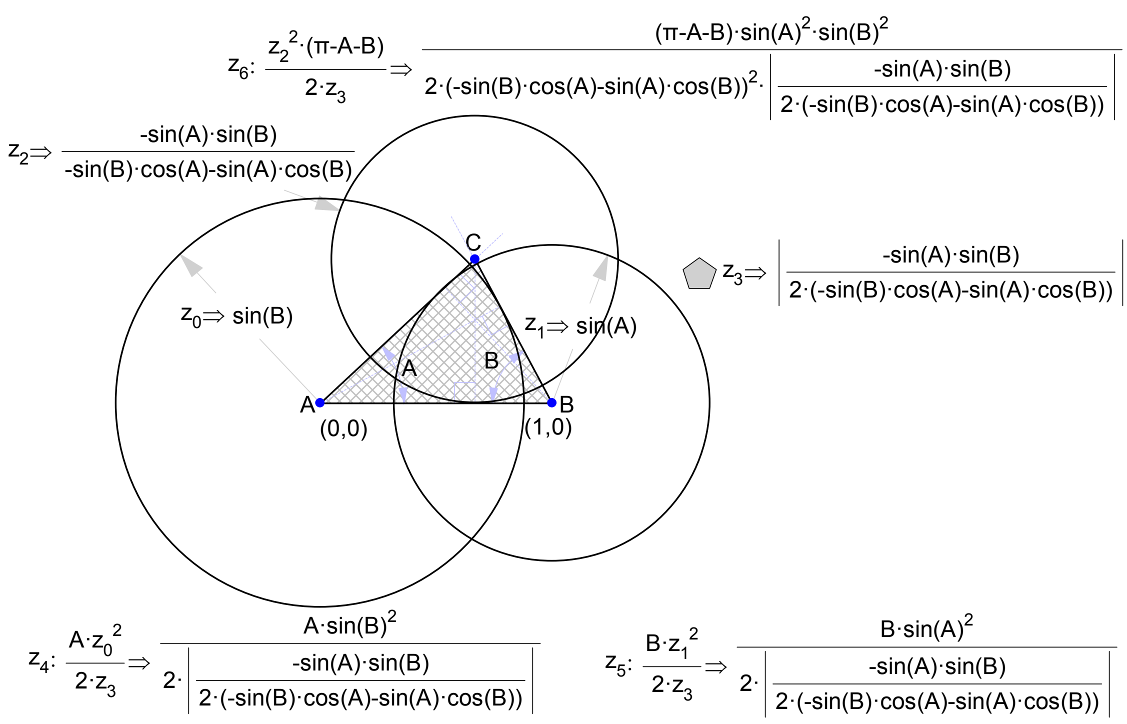
1. Introduction

In a triangle with angles A, B, and $\pi - A - B$, we draw three sectors centered at each vertex, each with maximal angle, and then maximal radius. Using Geometry Expressions, we look for the largest of the three sectors for all possible triangles.

2. Solution

Acute triangles have the sector tangent to the opposite side, while obtuse triangles do not. Thus, their area formulas will be different, so we break this problem into two cases, acute triangles and obtuse triangles. Using Geometry Expressions, we constrain the side opposite the largest angle to have endpoints at $(0,0)$ and $(1,0)$. We then construct the model and calculate expressions for the radii and angles. Additionally, we calculate the area of the triangle so we can determine the percentage of the triangle that is covered by the sector.

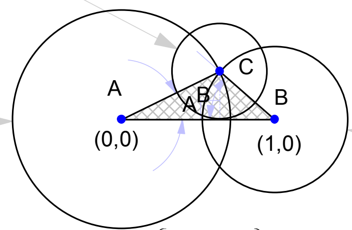


$$z_6: \frac{z_2^2\cdot(\pi-A-B)}{2\cdot z_8} \Rightarrow \frac{(\pi-A-B)\cdot\sin(A)^2\cdot\sin(B)^2}{2\cdot(-\sin(B)\cdot\cos(A)-\sin(A)\cdot\cos(B))^2\cdot\left|\frac{-\sin(A)\cdot\sin(B)}{2\cdot(-\sin(B)\cdot\cos(A)-\sin(A)\cdot\cos(B))}\right|}$$

$$z_2 \Rightarrow \frac{-\sin(A)\cdot\sin(B)}{-\sin(B)\cdot\cos(A)-\sin(A)\cdot\cos(B)}$$

$$z_8 \Rightarrow \left|\frac{-\sin(A)\cdot\sin(B)}{2\cdot(-\sin(B)\cdot\cos(A)-\sin(A)\cdot\cos(B))}\right|$$

$$z_5 \Rightarrow \frac{\sqrt{1-\cos(2\cdot B)}}{\sqrt{1-\cos(2\cdot(A+B))}}$$



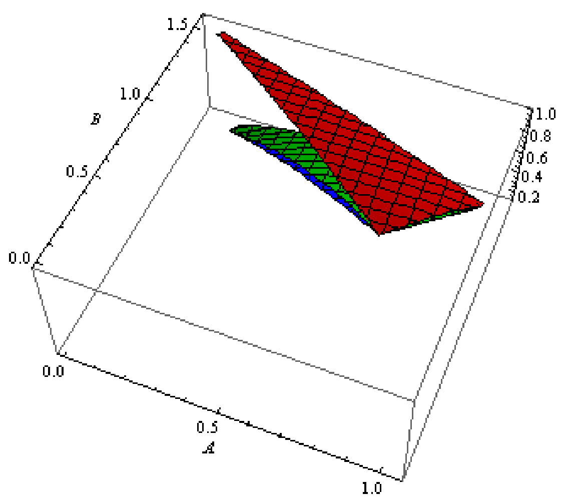
$$z_1 \Rightarrow \frac{\sin(A)}{|-\sin(A+B)|}$$

$$z_4: \frac{A\cdot z_5^2}{2\cdot z_8} \Rightarrow \frac{A\cdot\left(1-\cos(B)^2\right)}{2\cdot(-\sin(B)\cdot\cos(A)-\sin(A)\cdot\cos(B))^2\cdot\left|\frac{-\sin(A)\cdot\sin(B)}{2\cdot(-\sin(B)\cdot\cos(A)-\sin(A)\cdot\cos(B))}\right|}$$

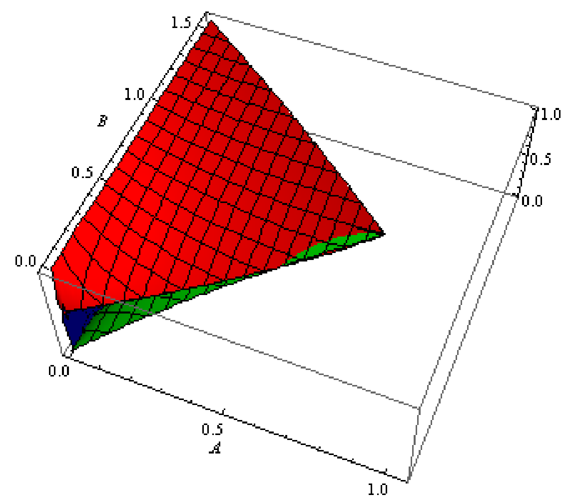
$$z_0: \frac{B\cdot z_1^2}{2\cdot z_8} \Rightarrow \frac{B\cdot\left(1-\sin(B)^2\cdot\cos(A)^2+\left(-1+\sin(A)^2\right)\cdot\cos(B)^2\right)}{2\cdot(-\sin(B)\cdot\cos(A)-\sin(A)\cdot\cos(B))^2\cdot\left|\frac{-\sin(A)\cdot\sin(B)}{2\cdot(-\sin(B)\cdot\cos(A)-\sin(A)\cdot\cos(B))}\right|}$$

With equations for the areas of the sectors, we plot them in 3-D. Red represents the sector at the smallest angle, blue the middle angle, and green the largest angle.

Acute Triangle

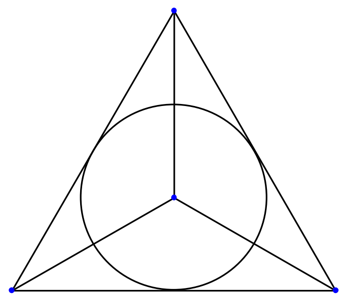


Obtuse Triangle

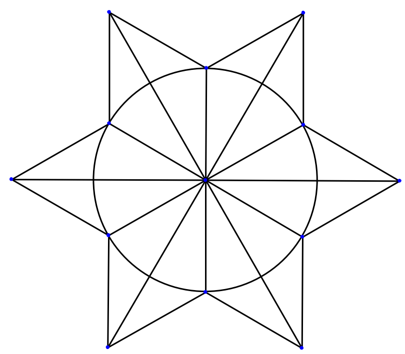


We see that for the acute triangle, the red is always on top, so the sector at the smallest angle is always largest. However, this is not always the case for obtuse triangles. There is a small region where the sector at the largest angle is bigger.

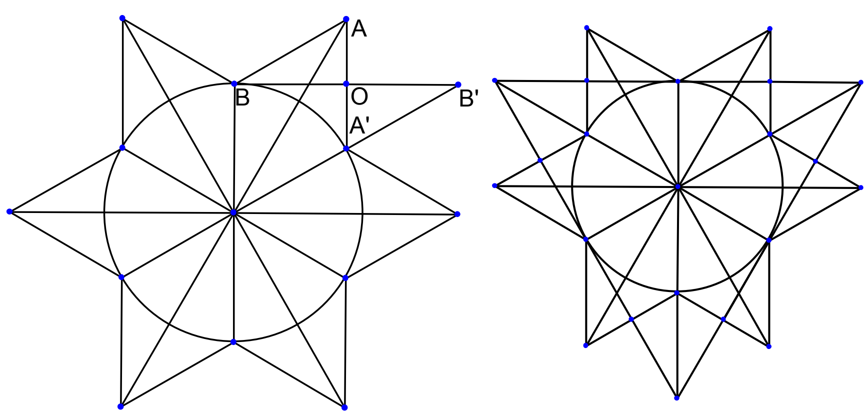
We examine the bottom boundary, where $A = B$, and the triangle is isosceles. It can be shown that the crossover point between the sector at the greater angle being larger and the sector at the smaller angle being larger is at $A = B = \frac{\pi}{6}$ and $A = B = \frac{\pi}{4}$. We offer a geometric proof of the former case below. Below, simple rotations show that the percent coverage of the sector at the larger angle is the same as that of a circle inscribed in an equilateral triangle.



Now, we show the same thing for a sector at the small angle. With appropriate reflections and rotations, we have a six-pointed star.



Now, we apply another set of transformations to turn the star into an equilateral triangle. We construct the midpoint O' of AA', and rotate AOB so that BOB' is a straight line. Now, we can repeat this process around the star so that it becomes an equilateral triangle, as desired.



A similar argument holds for the crossover at $A = B = \frac{\pi}{4}$. The percent coverage is equal to that of a circle inscribed in square.

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