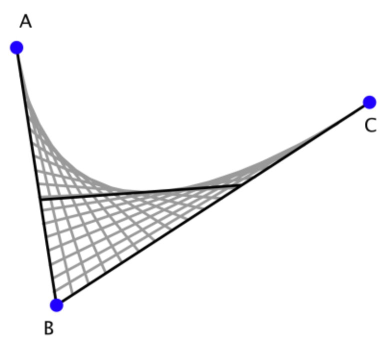
String Art Mathematics:
An Introduction to Geometry Expressions and Math Illustrations

Stephen Arnold
Compass Learning Technologies

Introduction

How do you create string art on a computer? In this lesson sequence, suitable for students at all levels of high school, we learn to create some beautiful designs based on practical principles using Geometry Expressions (GX) and Math Illustrations (MI). In the first lesson, the focus is upon the geometry: constructing string art envelopes using geometric tools. In later lessons, we explore the geometry and algebra behind these designs. Along the way, we will be introduced to proportions, loci and envelopes and, finally, parametric and implicit forms for conics.

Beginning with three points, A, B and C, joined by line segments AB and BC, we construct a third segment which traces out an envelope of lines as it is dragged. By placing the endpoints on the picture of the sail boat, we see that the curve of the forward sail may be modeled nicely by the curve of the envelope of lines.



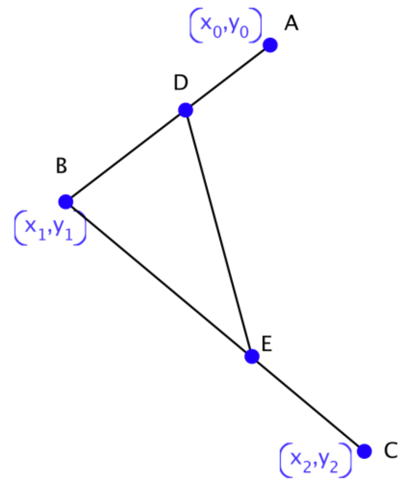
Q. Is there more than one way that the string curve formed by A, B and C may be placed to match the sail?

Teacher Note: *Point B can only be placed in one way: it must lie at the very front (prow) of the sail boat. Points A and C may be placed in either of the other end points of the sail.*

Once we learn how to construct such curves, we may be able to create all sorts of interesting and beautiful patterns.

Lesson 1: Constructing String Art Curves

We begin with three points, A, B and C. At present, points A, B and C are free points and tend to move around a little as we do other constructions. We can use the **Coordinate** tool from the **Constrain** menu to give A, B and C general coordinates (just use the defaults for each one). We can still move A, B and C around if we want, but this constrains their movement and lets us focus on the constructions ahead. Use the **Segment** tool to create two segments joined at a common point, B. Draw the first segment from A to B, then the second from B to C - order does matter here. Use the **Segment** tool again to place another segment joining a new point D on AB and another point E on BC.



Q. See what happens as you drag point D along the segment AB. Is this what you expected?

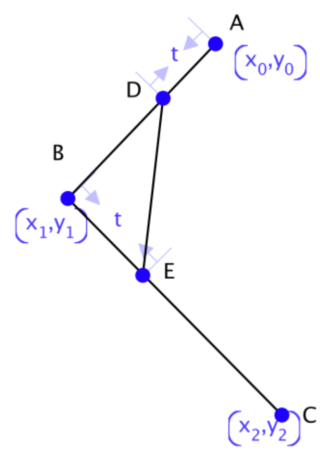
***Teacher Note:** As you drag point D along the segment AB, it moves independently of point E.*

What we would like would be that, as D moves along AB, the point E will move along BC *proportionately*.

Click on point D to select it. Hold down the Shift key and also select the segment AB on which D lies. From the **Constrain** menu, choose **Proportional** and press ENTER to select the default variable t. Now, Shift-click to select both point E and the segment BC and again choose **Proportional** from the **Constrain** menu. This time, change the default ("s") to "t" so that E moves in the same proportion as D.

Drag point D and observe the effect.

Is the **Variables** panel visible? If not, go to the **View** menu and choose it from the **Tool Panels** menu. You will see each of the variables you have defined so far, including our value for "t".



Q. Watch the values for t change as you drag point D: can you explain what is happening here?

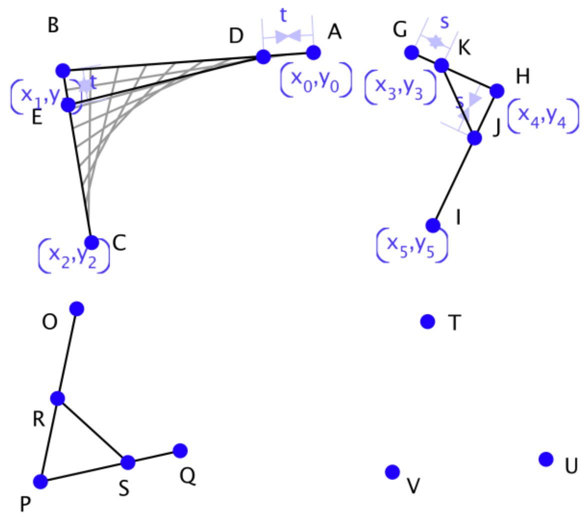
***Teacher Note:** Points D and E are now moving proportionately.*

Consider what is happening here.

When D is halfway along AB, then E will be halfway along BC. When D lies at the start of AB (at point A), then E will lie over point B.

As the value of t varies between 0 and 1, then when t = 0, D will lie on A and E on B; when t = 1, then D will lie on B and E on C.

This is proportional movement.

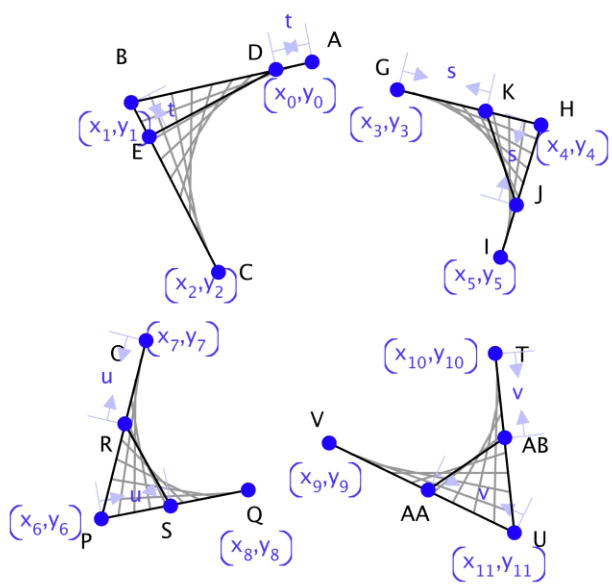


Now observe the way the third segment, DE, moves as you drag D. Can you see that it seems to trace out a curve? Click to select the segment DE and choose **Trace** from the **Construct** menu. Accept the defaults.

What we see here is the "trace" of the segment - a series of segment images as the control point D is dragged along segment AB, i.e. as the parameter t varies between 0 and 1 in small steps.

You should now try to make some of your own string art patterns. You will see that, using this method, you have a start and an end point and what we might call a "shape" point.

In the top left example, it is easy to see what points A and C do. Drag point B to see how this point affects the shape of the envelope.



Complete the sets partly created for you on page 1.6 to design your own works of string art. Add more if you wish.

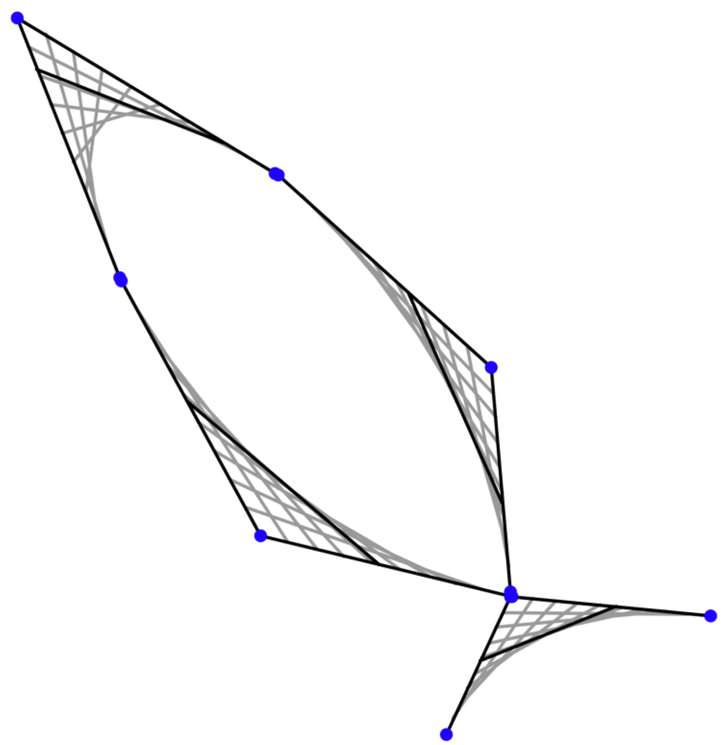
On page 1.7, four sets are all done for you.

Q. Are there any shapes you cannot make in this way? Could you form a perfect circle? Could you make a curve which loops over itself?

One of these is possible but the other is not: which do you think?

Teacher Note: *Perhaps surprisingly, it is NOT possible to form a perfect circle using such curves, but by joining together two or more such shape sets, it IS possible to build curves that loop and cross over themselves.*

Once you start to play, who knows what you might come up with?



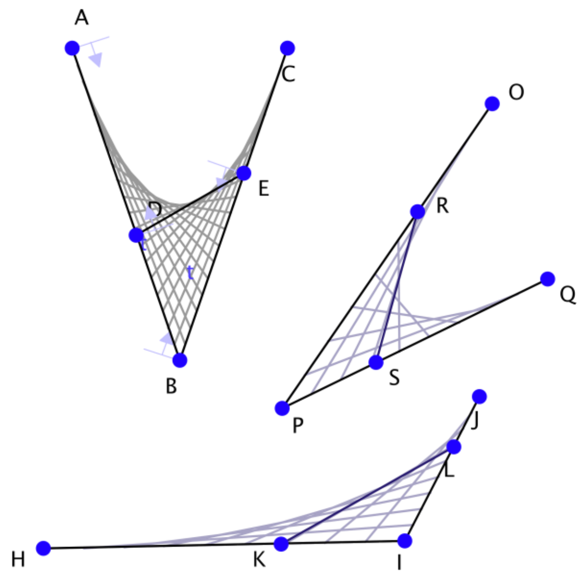
Lesson 2: String Art and Proportionality

In Lesson 1, we learned how to create interesting patterns using *Geometry Expressions* and *Math Illustrations*. In this lesson, we focus, not on the string art itself, but on the curve described by the "strings".

Teacher Note: *We might expect students to see a curve and assume that it is a parabola, since this is the curve with which they are likely to be most familiar. In fact, the curve IS a parabola, but we will need to be convinced of that!*

Q. What sort of curve do you think this is?

At the moment, we appear to see a curve traced out by virtual strings, but there is no actual curve there. Although we are looking at a set of straight lines, our mind joins the lines into a curve. What sort of curve is it?

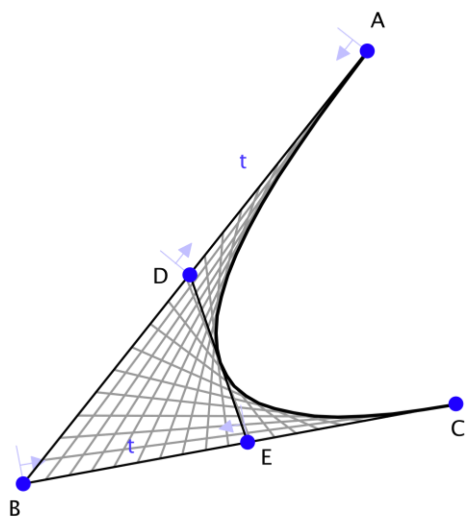


The curve produced by points A, B and C certainly looks like a parabola, but what about the others?

To create the curve which joins each of the strings, we use **Locus** from the **Construct** menu. Just click to select the segment DE which traces out the envelope, and choose **Locus** from the **Construct** menu. Choose the default values for t.

Q. One way to think about locus is as a path traced out by a point in motion. We can see the segment DE in motion on page 2.3, tracing out an envelope of lines which form a curved shape.

But can you see where a point might lie that will trace the curve shown which bounds the envelope?



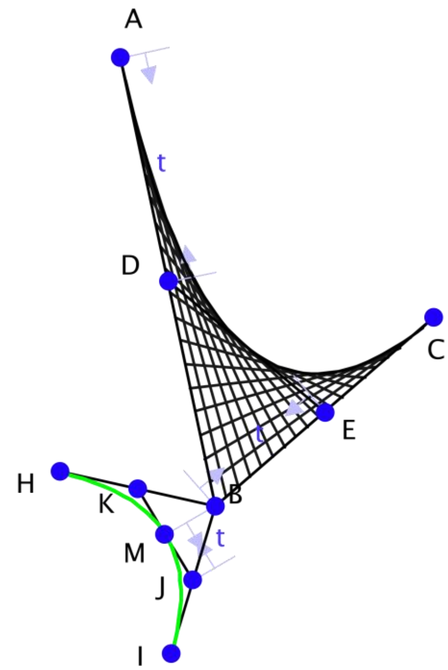
Teacher Note: *By dividing segment DE proportionately just as we divided AB and BC, we produce a point which will trace out the curve.*

The point where the segment DE appears to meet the locus curve does not seem to be fixed. How do you think it moves?

Could it be moving in the same proportion as our segment points D and E?

We can test this idea by constructing another segment pair, BH and BI with segment KJ joining these, as shown. As we did previously, we constrain these segments to move proportionally with parameter "t", then similarly constrain a point M which lies on the new segment, as shown.

The locus of point M as t varies from 0 to 1 is shown here. To verify that this is the same curve as we have produced, drag points H and I onto our original points A and C on page 2.4.



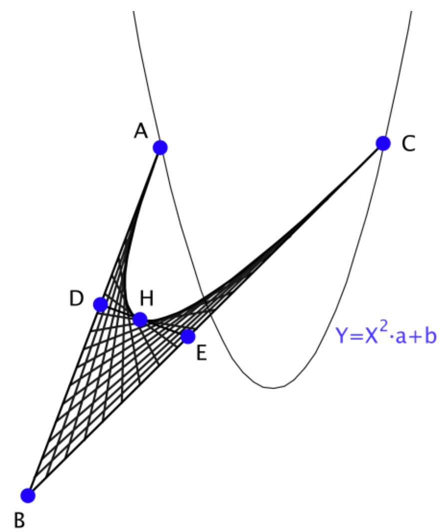
More formally, in Lesson 3, we will learn how we might use the algebraic forms for these curves to establish their equality.

Our locus curve does look like a parabola - but we need to check that this is true.

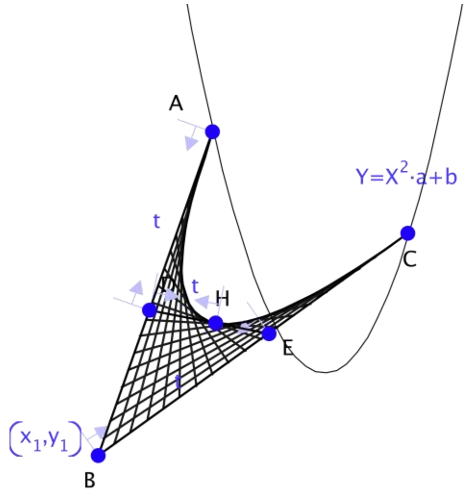
Points A and C have been attached to a general parabola shown on page 2.5.

Q. Can you move point B so that the locus curve exactly matches the parabola? If so, will this work for anywhere you choose to drag A and C?

Teacher Note: *Students should be encouraged to try a variety of placements for A and C - especially asymmetrical positions.*



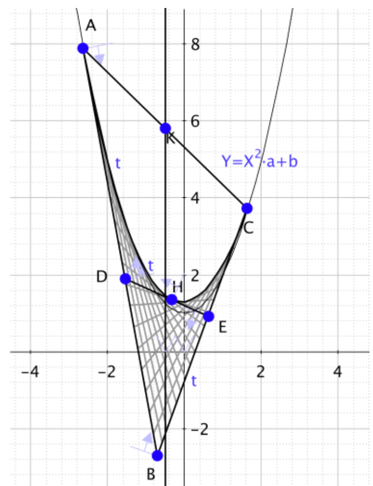
Think about the things you know about parabolas: one important property is their **symmetry**.



Construct the **midpoint** of AC and draw a line through this point and the point B: what do you notice about this median line?

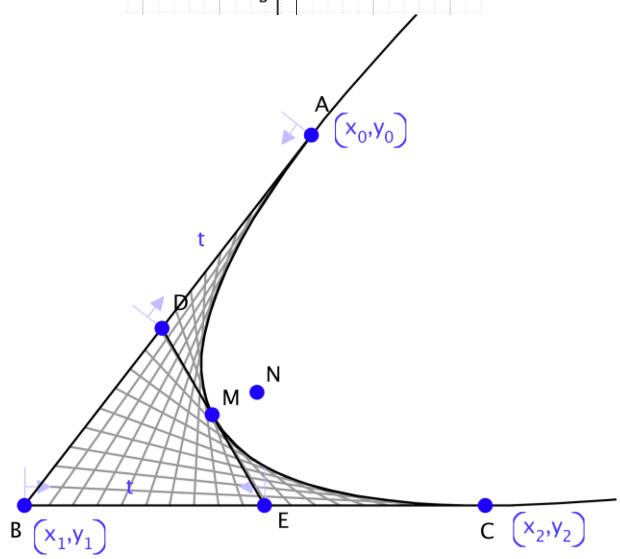
Click the Axis button to display the coordinate axes - compare the median line with the y-axis.

If you knew the axis of symmetry for a parabola, then this might be an easy way to decide where to place the point B!



Teacher Note: *the median line from B through the midpoint of AC is always parallel to the axis of symmetry of the parabola. If we know the axis of symmetry of the parabola, then B should be placed so that the median line lies parallel to the axis.*

The line through the midpoint of AC and the point B should always be parallel to the axis of symmetry of the parabola.

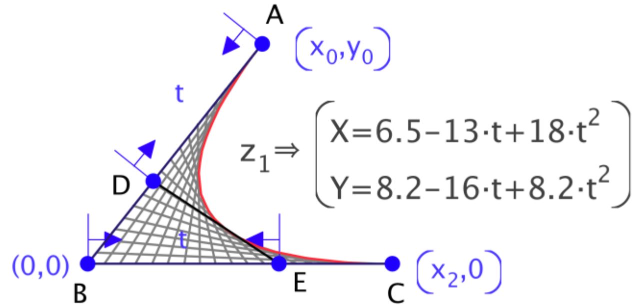


If the locus curve shown here is a parabola, then what happens when it is not pointing directly upwards? Can parabolas face in any direction? If so, what might their equations look like? Is the curve shown here a parabola? What else could it be? Perhaps part of an ellipse or even a hyperbola?

In Lesson 3 we will examine the algebraic forms for these curves and see different ways in which they may be represented.

Lesson 3: String Art and Algebra

It is time to look at the algebraic forms for the curves we are studying here. The two forms we will use here are called *parametric* and *implicit*. Parametric equations consider the x and y coordinates separately, traced out by a parameter, t. The implicit form is our more usual x and y form.



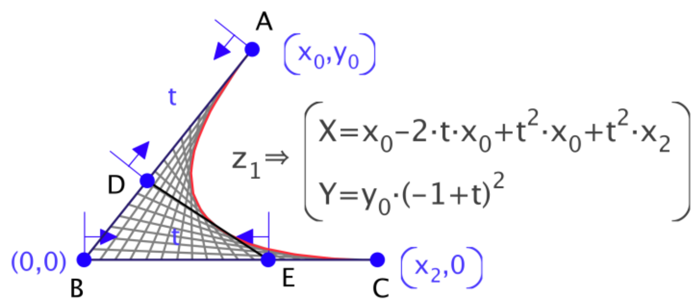
We can make our lives a little easier by carefully selecting coordinates for some of our points. In particular, change the coordinates for point B to (0,0). On page 3.1, click on the curve which defines our envelope. Select Calculate > Real Measurement and choose Parametric Equation.

Q. What do you notice about the two equations that make up this form?

The parametric form consists of two equations in the variable "t" - one for the x values and the other for the y values. You should observe that both these equations are of degree 2 - they are quadratic.

If we wished to further simplify things (without making our observations any less general) we could tie the segment BC to the x-axis. Discuss with a partner what change you would need to make to establish this condition and what effect it would have upon the parametric form for the curve.

We see that both equations in the parametric form are quadratics, but if we want to understand what we see, then we should examine the algebraic forms: in *Geometry Expressions*, choose **Calculate > Parametric Equation** (shown). Study the x-function for the parametric form: if we gathered the x_0 terms together how could we simplify the expression? Using *Geometry Expressions*, you may copy this equation and use a computer algebra system (CAS) to help if you have one available.



Q. Try and describe in your own words how this expression relates to the point D moving between A and B.

What would happen if x_0 was equal and opposite to x_2? What is the geometrical significance of this relationship for our parabola?

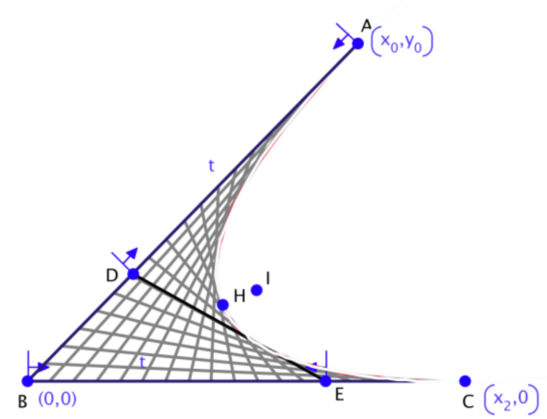
You can hide or delete the parametric form and choose to **Calculate** the **Real Measurement** for the **Implicit Equation** - this is displayed here.

This form appears much more complicated than the parametric form, but if you focus on the variables X and Y you should see a series of terms and their coefficients. For example, study the coefficient of the Y^2 term - what does this suggest?

Now delete the real valued equation and Calculate the symbolic form using *Geometry Expressions*.

Teacher Note: *The x-function may be simplified to* $x_0 \cdot (1-t)^2 + x_2 \cdot t$.

When t is 0, the value is x_0; when t = 1, the expression equals x_2, just as the point D moves between A and B as the value of t varies between 0 and 1. If $x_0 = -x_2$ then the X equation would become linear, making everything much simpler!



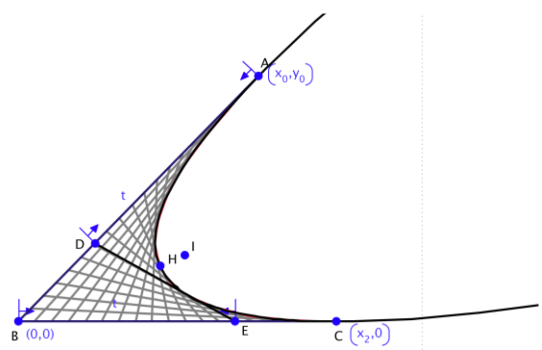
$z_0 \Rightarrow 1.4-0.16\cdot X+0.0046\cdot X^2-0.051\cdot Y-0.021\cdot X\cdot Y+0.024\cdot Y^2=0$

Q. Look at the XY coefficient of the symbolic form shown below - what value of x_0 could make this disappear? What about the Y term?

Teacher Note: *The coefficient of the Y^2 can be factored to $(x_0 + x_2)^2$ which suggests that Y^2 will be zero when x_0 and x_2 are opposites.*

Finally, we can use the **Parabola** tool to fit a parabola to our envelope curve.

Copy the implicit form of our equation, then choose the **Parabola** tool and click once for the vertex and once for the focus to draw a parabola. (BE CAREFUL: make sure the vertex and focus don't snap to any existing geometry when you create the parabola).

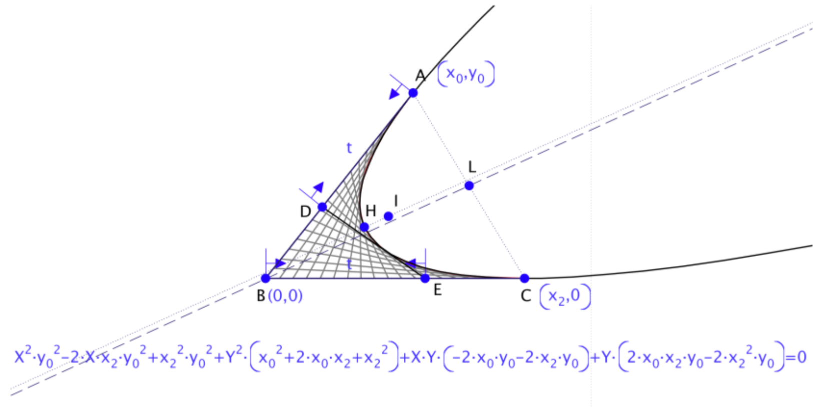


$X^2\cdot y_0^2-2\cdot X\cdot x_2\cdot y_0^2+x_2^2\cdot y_0^2+Y^2\cdot(x_0^2+2\cdot x_0\cdot x_2+x_2^2)+X\cdot Y\cdot(-2\cdot x_0\cdot y_0-2\cdot x_2\cdot y_0)+Y\cdot(2\cdot x_0\cdot x_2\cdot y_0-2\cdot x_2^2\cdot y_0)=0$

Constrain the equation of the new parabola and paste the equation for our curve to see the two curves match. Check the new focus point using our construction method from lesson 2 for placing the axis of symmetry and focus.

Teacher Note: *To find the axis of symmetry, join the midpoint of AC to point B - it will be parallel to the axis of symmetry for the parabola, which will pass through the vertex and focus.*

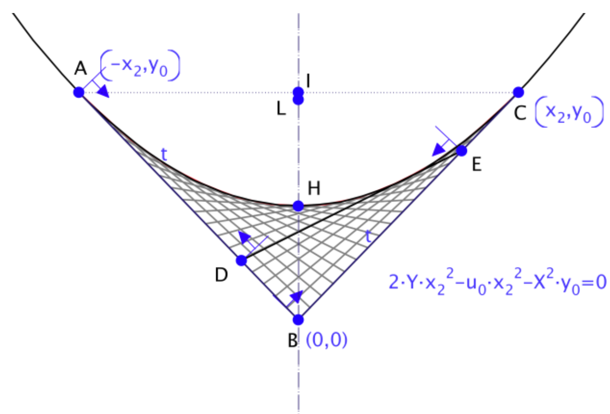
Calculate the slopes of the lines BL (median of triangle ABC) and HI (axis of symmetry of the parabola, passing through focus I and vertex H). What do you observe?



Could this property be used to make it easier to fit our string art to an existing parabola?

Q. The equation we got for the curve depended on our choice for lining our string art with the coordinate system. Did we make the best possible choice? What other choice could we have made? How would it have affected the equation?

It should hardly be surprising at this stage that if we set points A and C to be symmetrical around the Y-axis, then our implicit equation will have a much simpler form.



Our question remains: when we construct a string art design in this way, are you convinced yet that the curve traced out by the envelope will always be a parabola - even when not facing upwards and symmetrical around the y-axis?

Teacher Note: *Given a string art triangle, rotate it so that its median lies on the y-axis. The x coordinate of A will be the negative of the x coordinate of C. We have shown that this generates a parabola.*

The general question: for any triangle ABC, can we find envelope curves and establish algebraically that these curves are always parabolic?

Continue your exploration...