Learning Objectives

Students are now acquainted with the idea of "locus," and how Geometry Expressions can be used to explore loci. Now, they will look more closely at the circle, defined as the locus of points on a plane equidistant from a fixed point. In particular, they will find the parametric and implicit equations of circles.

The approach is to make sense of the parametric equation in terms of the unit circle, and the implicit equation in terms of the distance formula. Review of these two concepts may be beneficial before the start of the lesson.

Math Objectives

- Learn or re-enforce the general parametric and implicit equations of a circle.
- Connect translations and dilations to the general equations.
- Connect center and radius to the general equations.

Technology Objectives

- Use Geometry Expressions to find equations of curves.
- Use Geometry Expressions to translate and dilate figures.

Math Prerequisites

- Distance formula
- Algebra, including factoring parts of an expression and completing the square.
- Unit Circle
- Pythagorean identity
- Translations and dilations
- Parametric functions vs. implicit equations

Technology Prerequisites

- Knowledge of Geometry Expressions from previous lessons.

Materials

- Computers with Geometry Expressions.

Overview for the Teacher

1. Question one gets the students back into using Geometry Expressions to find the locus of a point. Diagram 1 exhibits typical results.

   A common error is to use *r* (the default) as the parametric variable instead of changing it to θ.

   Students are asked why they didn't just use the circle tool. Expected results are that the definition of a circle as a locus of points was reinforced.

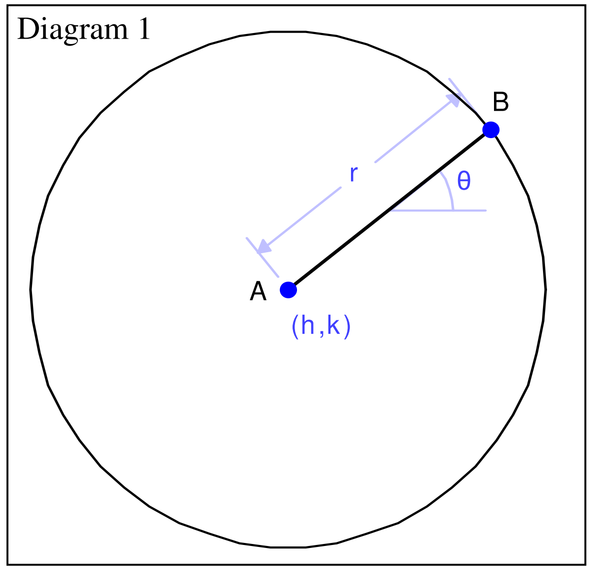


Diagram 1

2. In Diagram 2 you can see the general parametric equation of a circle, where *d* is the radius and (h, k) is the center. If students are getting numerical constants, they are calculating the *real* equation instead of the *symbolic* equation. Help them to select the *symbolic* tab and try again.

   After changing the constraints so that the center is (0,0) and the radius is 1, the parametric equation will be

$$\begin{pmatrix} X = \cos(\theta) \\ Y = \sin(\theta) \end{pmatrix}$$

   If students are not getting this, they may be changing the value of the *d* in the Variable Tool Panel, rather than changing the constraint itself to 1.

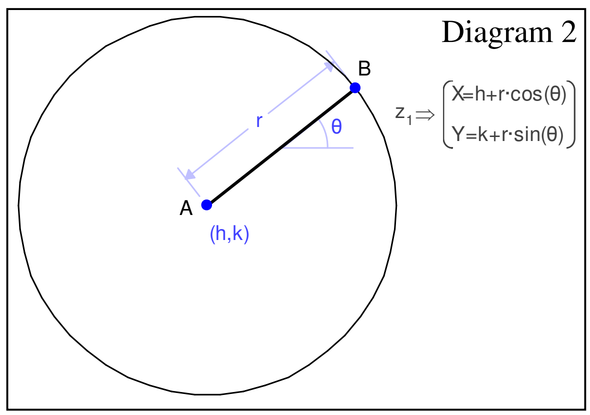


Diagram 2

3. Diagram 3 shows the results of finding the implicit equation for the circle. The simplification process involves some grouping and simple factoring. The end result is the distance formula:
   $r^2 = (X-h)^2 + (Y-k)^2$, which is frequently written $r^2 = (x-h)^2 + (y-k)^2$

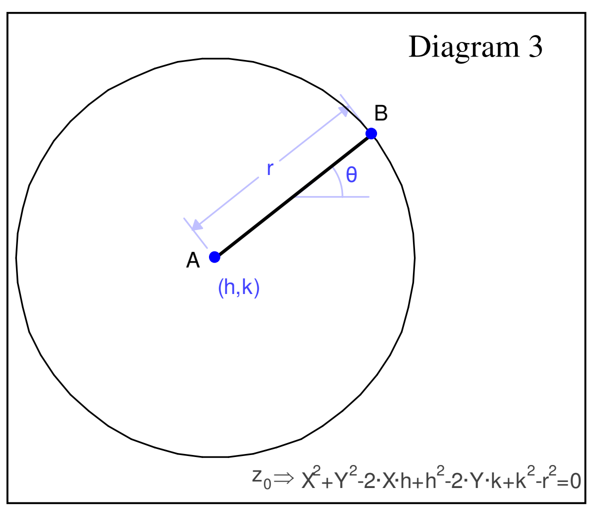


Diagram 3

Changing the center to (0,0) and the radius to 1 results in $-1+X^2+Y^2=0$ or $X^2+Y^2=1$

Exceptional students will begin making connections between the equation of a circle, the distance formula, and the Pythagorean Theorem at this point. Most will likely have been exposed to these connections in a previous course. You may wish to highlight these relationships at this time.

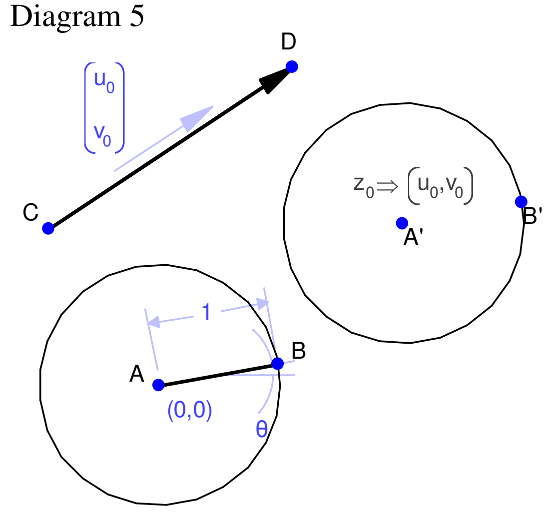
4. The result is the Pythagorean Identity: $\cos^2\theta+\sin^2\theta=1$

5. Translating and dilating the unit circle will restore us to the general form. Note that the center will be (u_0, v_0) instead of (h, k). If students find this confusing, they can change the variables used in the vector to (h, k).

   Encourage students to change the vector by dragging D, especially if they cannot see the whole picture on the screen.

   If your students are not comfortable with completing the square, then the generalization of the implicit equation may be a demonstration.

Diagram 5



6. If students get incorrect results for their dilations, check to see that they have chosen the center of the translated circle as their center of dilation. If they choose some other point, their dilated circle will not be concentric with the translated circle, and their equations will be wrong. See diagram 6 for typical correct results.

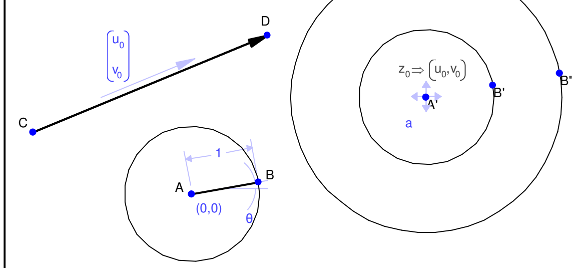
7. In summary:
   The general parametric form of the equation of a circle is

$$\begin{pmatrix} X=h+r\cos\theta \\ Y=h+r\sin\theta \end{pmatrix}$$

   (x_0, y_0) is the center of the circle and *d* is the radius.

   The general implicit formula of a circle is
   $r^2=(X-h)^2+(Y-k)^2$
   Again, (h, k) is the center of the circle and *d* is the radius.

Diagram 6



It is important that students have these three concepts, as they will be repeated for the rest of the conic sections:

1. Description of the curve as a locus of points.
2. Parametric equation of the curve.
3. Implicit equation of the curve.

Subsequent lessons will search for further attributes of the curve being studied. For example, the study of ellipses will include the focus, major axis and minor axis.

Name: ___________________________________

Date: ___________________________________

Loci: Circles

In the last lesson, you were introduced to the idea of a "locus." A locus is a collection of points that meet a particular description. For example, the locus of points equidistant from a fixed point is a circle.

1. Reconstruct the locus of points equidistant from a fixed point.

   Open Geometry Expressions.
   Create point A, and constrain its location to (h, k).
   Create point B, and constrain its distance from point A to be *r.*
   Create line segment $\overline{AB}$ – be sure to draw starting at point A and ending at point B – and constrain its direction to be θ.

   Create the locus of points *d* units from point A. Use θ as your parameter.

   You could have just used the circle tool if all you wanted to do was to draw a circle. What did you learn about circles by doing it this way?

Sketch the locus of points equidistant from a fixed point.

Write the general parametric equation of a circle

2. Often, a locus can be described with equations describing its x and y coordinates. This type of algebraic description is called a Parametric Equation. X and Y are described as functions of a third variable, called a parameter.

   Click on the circle you created in step one.

   Click on Calculate Symbolic Parametric equation.

   Geometry Expressions has just given you the General Parametric Equation of a circle. Write it in the box to the right.

   Recall the Unit Circle. How are the coordinates of points on the unit circle defined in terms of cosine and sine?

   Change the constraint on the center to (0,0) and change the radius to 1. What does this do to the Parametric equation?

3. You can use CTRL-Z to undo your changes to the constraints on your drawing. Do so repeatedly, until the radius is *r* and the center is (*h, k).* Then use Geometry Expressions to create the implicit equation.
   Select the circle.
   Click on Calculate Implicit equation .

   The result looks confusing at first, but you can clean it up:

   Add d^2 to both sides.
   Group the X and x_0 terms.
   Group the Y and y_0 terms.
   What formula is beginning to emerge?

   Factor the X and x_0 terms.
   Factor the Y and y_0 terms.
   The result is the General equation of a circle.

   Change the constraints as you did in part 2. The result is the implicit equation for the unit circle. Write it here:

Simplify the Implicit Equation here, to get the General Equation of a circle.

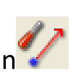
4. Using the equations for the unit circle, substitute the two parametric equations into the implicit equation. What relationship do you get?

5. We are going to subject our unit circle to some transformations. First, translate the unit circle.

   Create a vector. 

   Constrain the components of the vector to $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$.

   Select the circle and its center (use shift and click on each).

   Click Construct Translation 
   Click the vector.

Translation of a circle

What are the coordinates of the center of the translated circle?

Click on the center.

Click on Calculate Symbolic Coordinates. 

Find the parametric equation of the translated circle.

Where do the coordinates of the center appear in the equations?

Find the implicit equation of the translated circle.

Use the method of *completing the square* to simplify the implicit equation.

6. Now, dilate the circle that you translated.

   Click on the circle.

   Click on Construct Dilation. 
   Click on the center of the translated circle.
   Type *r* for your scale factor.

   What is the radius of the dilated circle?

   Find the parametric equation of the translated/dilated circle.

   Where does *a*, the length of the radius appear in the equations?

Sketch the translation and dilation of the unit circle.

   Find the implicit equation of the translated/dilated circle.

   Simplify with completing the square, as you did in part 5.

7. Summary

The general parametric form of the equation of a circle is:

Where ______ is the center of the circle and ____ is the radius of the circle.

The general implicit form of the equation of a circle is:

Where ______ is the center of the circle and ____ is the radius of the circle.