Learning Objectives

In this lesson, students will generalize their knowledge of the circle to the ellipse. The
parametric and implicit equations of an ellipse will be generated, as will two important
properties of the ellipse: that the sum of the distances from any point to the foci is equal to the
major diameter, and the Pythagorean Property of Ellipses.

Math Objectives

• Understand the geometric definition of an ellipse.
• Generate the parametric and implicit equations for an ellipse.
• Locate the foci, given the equation of an ellipse.
• Discover relationships between the parameters of an ellipse.

Technology Objectives

• Use Geometry Expressions to create a more complex locus of points.
• Find evidence for equivalence using Geometry Expressions.

Math Prerequisites

• Pythagorean Theorem
• Translations
• Parametric functions and implicit equations
• Sine and Cosine

Technology Prerequisites

• Knowledge of Geometry Expressions from previous lessons.

Materials

• Computers, with Geometry Expressions.
Overview for the Teacher

1. Diagram 1 represents a typical result for question 1. If students are getting semicircles with center at point F1, they are creating a locus in terms of \( t \) instead of \( d \). Only half an ellipse is shown because the entire ellipse cannot be generated as a function of \( d \). Points below the foci will have the same distances as points above.

An ellipse is like a circle in that it is a curve based on distances from fixed points. It is different in that it “has different radii.”

2. An appropriate result would be

\[
\begin{align*}
X &= a \cos \theta \\
Y &= b \sin \theta
\end{align*}
\]

Transposing the \( \sin \) and \( \cos \) functions will have no effect on the final curve. Note that the transposed version is also the “sample” parametric function given by the software. Using the transposed version yields the same results, graphically. It just changes the starting place and direction that the ellipse is graphed.

Encourage students to change values for \( a \) and \( b \) to get an ellipse that is not just a circle. A sample is shown in Diagram 2.

3. Some assumptions are made here about the symmetrical nature of an ellipse. You may wish to explore these assumptions with the class at this time.

Desired solutions:

a. The distance from F1 to P is \( 2a - m \) or \( 2c + m \)

b. The distance from F2 to P is \( m \)

c. The sum is therefore \( 2a \) or \( 2c + 2m \).

d. The width of the ellipse is \( 2a \)

e. \( t = 2a \). Remind students that they need to type \( 2*a \).

4. The constraint from F2 to P is \( 2a - d \).

Diagram 3 shows expected results.
5. Desired solutions:
   a. The sum of the distances is 2a.
   b. The distance from F2 to P is equal to a, since the three points form an isosceles triangle.
   c. Using the Pythagorean theorem,
      \[ a^2 = b^2 + c^2, \] so \( c = \sqrt{a^2 - b^2} \)
   d. Point P will now appear to be on the ellipse, as shown in Diagram 4.

6. In both instances, the implicit equation will be
   \[ Y^2 \cdot a^2 + X^2 \cdot b^2 - a^2 \cdot b^2 = 0 \]

7. The general parametric function that is generated is
   \[
   \begin{align*}
   X &= u_0 + a \cos(T) \\
   Y &= v_0 + b \sin(T)
   \end{align*}
   \]
   The implicit formula is
   \[
   Y^2 \cdot a^2 + X^2 \cdot b^2 - a^2 \cdot b^2 - 2Xb^2u_0 + b^2u_0^2 - 2Ya^2v_0 + a^2v_0^2 = 0
   \]

8. Steps are as follows:
   \[
   \begin{align*}
   Y^2a^2 + X^2b^2 - a^2b^2 - 2Xb^2u_0 + b^2u_0^2 - 2Ya^2v_0 + a^2v_0^2 &= 0 \\
   \left(X^2b^2 - 2Xb^2u_0 + b^2u_0^2\right) + \left(Y^2a^2 - 2Ya^2v_0 + a^2v_0^2\right) &= a^2b^2 \\
   b^2 \left(X^2 - 2Xu_0 + u_0^2\right) + a^2 \left(Y^2 - 2Yv_0 + v_0^2\right) &= a^2b^2 \\
   \frac{b^2 \left(X - u_0\right)^2 + a^2 \left(Y - v_0\right)^2}{a^2b^2} &= a^2b^2 \\
   \frac{a^2b^2}{a^2b^2} + \frac{a^2 \left(Y - v_0\right)^2}{a^2b^2} &= \frac{a^2b^2}{a^2b^2} \\
   \frac{(X-u_0)^2}{a^2} + \frac{(Y-v_0)^2}{b^2} &= 1
   \end{align*}
   \]

9. Results as follows:
   a. If \( a = b \), then the result is a circle.
   b. If \( a < b \), then the ellipse is taller than it is wide. The foci lie on a vertical line rather than a horizontal line. The Pythagorean Property would then be \( b^2 = a^2 + c^2 \)
10. Summary:

The general parametric form of the equation of an ellipse is

\[
\begin{align*}
X &= u_0 + a \cos(T) \\
Y &= v_0 + b \sin(T)
\end{align*}
\]

where \((u_0, v_0)\) is the center of the ellipse and \(a\) and \(b\) are the radii of the ellipse.

The general implicit form of the equation of an ellipse is

\[
\left(\frac{x - u_0}{a}\right)^2 + \left(\frac{y - v_0}{b}\right)^2 = 1
\]

where \((u_0, v_0)\) is the center of the ellipse.

If \(a > b\), then \(2a\) is the major diameter and \(2b\) is the minor diameter.

If \(b < a\), then \(2b\) is the major diameter and \(2a\) is the minor diameter.

The sum of the distances from any point on the ellipse to the two foci is \(2a\).

The distance from the center of the ellipse to either focus follows the equation \(c^2 = a^2 + b^2\).
The Ellipse

In the last lesson, you found equations for a circle: the locus of points equidistant from a fixed point.

An ellipse is defined as all the points such that the sum of the distance from two fixed points is a constant. Each of the two points is called a focus (the plural of “focus” is “foci”).

1. Create an ellipse using the definition above.

Open a new Geometry Expressions drawing.

Create two points, and name them F1 and F2. Constrain the coordinates of these points.

Create a third point, and name it P. Constrain the distance from P to F1 to be \(d\).
Constrain the distance from P to F2 to be \(t - d\) (see Diagram 1 to understand why!)
Lock the value of \(t\) in the Variable Panel, but keep \(d\) unlocked.

Find the locus of P with parameter \(d\).

If you drag P around, you can see that the locus forms part of an ellipse.
To get more of the ellipse:
Double click on the curve.
Change the Start Value and End Value for \(d\).

The most you can get is half of the ellipse, because the software assumes you know which side P is on – you drew it there! To get the rest of the ellipse:
Draw a line segment from F1 to F2.
Select the locus curve.

Click on construct reflection and then click on the segment.
Some of the ellipse may still be missing.

How is an ellipse like a circle? How is it different?
Before continuing, make sure Geometry Express is set to Radians.  
In the Edit Menu

- Select preferences.
- Click on the Math icon at the left.
- Under Math, change Angle Mode to radians.

2. Recall that the general parametric equation for a circle with the center at the origin is 
\[
\begin{align*}
X &= r \cos(T) \\
Y &= r \sin(T)
\end{align*}
\] . Predict the general parametric equation for an ellipse.

Test your prediction with Geometry Expressions

- Open a new Geometry Expressions drawing.
- Click on the Function tool in the Draw tool panel.
- Type in your prediction to see if you are right. Make additional guesses if you need to.

3. The curve you created in part 2 looks like an ellipse, but is it really an ellipse? If it is, we will be able to find its foci, and the constant sum of distances from the foci.

   a. In Diagram 2, how far is it from focus F1 to point P?

   b. How far is it from focus F2 to point P?

   c. What is the sum of distances from P to the two foci?

   d. What is the horizontal width of the ellipse?

   e. Write an expression for your answer to part c, in terms of distance \(a\).

4. Open a new Geometry Expressions drawing, and create this parametric function:
\[
\begin{align*}
X &= a \cos(T) \\
Y &= b \sin(T)
\end{align*}
\]

Use the Variable Tool Panel to change the values of \(a\) and \(b\) so that \(a\) is greater than \(b\). Lock variable \(a\) and \(b\).
Add three points to your Geometry Expressions drawing.
First, turn on the axes.
Draw F1 and F2 on the x-axis.
Draw P so that it is not on either axis, nor is it on the curve.
Constrain the distance from F1 to P to be \( d \).

What should the constraint from F2 to P be? Review the results from part 3 to help you decide.

5. Refer to Diagram 3 to find the positions of F1 and F2. The triangle shown is an isosceles triangle, with P at the vertex.

a. If P is on the ellipse, what is the sum of the distances from F1 to P and from F2 to P (your solution to 3c)?

b. Given that the triangle is isosceles, what is the distance from F1 to point P?

c. Use the Pythagorean Theorem to write an expression for the distance from the origin to point F2.

d. In your Geometry Expressions drawing, constrain the distance from F1 to the origin to your answer to 5c. Do the same for the distance from F2 to the origin. (You will need to draw a point at the origin first). NOTE: If you want to type a square root in to Geometry Expressions, use sqrt, and if you want to type in an exponent, use ^. For example, \( \sqrt{a^2 + b^2} \) can be typed: sqrt(a^2 + b^2)

e. Does P appear to fall on the ellipse? Drag it around. Does it stay on the ellipse?
6. If the curve in your Geometry Expressions drawing is truly an ellipse, then its implicit equation will match the locus of point P.

Select the curve, and click on the Calculate Implicit Equation icon.
Now, hide the curve.
Select the curve.
Right click on the curve.
Choose hide.

Create the locus of point P with respect to $d$, and calculate its implicit equation.

Restore the original curve by clicking on Show All in the View menu.

Are the two implicit equations the same? How do they differ, if at all?

7. How does a translation affect the equation of an ellipse?
Open a new drawing and use Draw Function to create a new ellipse.

Choose Parametric for the type and enter \[
\begin{align*}
X &= a \cos(T) \\
Y &= b \sin(T)
\end{align*}
\]

Create a vector, and constrain it to its default values, \[
\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}
\]

Translate the ellipse.
Select the ellipse
Click on Construct Translation
Click on the vector.

Calculate the parametric equation of the new ellipse, and record it in the box.

Calculate the implicit equation of the new ellipse.
8. The General form for the implicit equation of an ellipse is \( \frac{(x-u_0)^2}{a^2} + \frac{(y-v_0)^2}{b^2} = 1 \). Verify that your implicit equation is equivalent to the general form.

9. In part 6, you found a relationship known as “The Pythagorean Property for Ellipses”

\[ a^2 = b^2 + c^2 \]

- \( a \) is half the horizontal axis of the ellipse
- \( b \) is half the vertical axis of the ellipse
- \( c \) is the distance from the center of the ellipse to each focus

a. What happens if \( a = b \)?

b. Is it possible for \( a < b \)? How would you need to modify the Pythagorean Property for the ellipse in Diagram 4?

In any ellipse, the larger of \( 2a \) and \( 2b \) is called the **major axis**. The smaller of \( 2a \) and \( 2b \) is called the **minor axis**. If \( a > b \), then the foci lie on a horizontal line. If \( a < b \), then the foci lie on a vertical line. If \( a = b \), then the ellipse is actually a circle.

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10. Summary:

The general parametric form of the equation of an ellipse is:

where ______ is the center of the ellipse, is the horizontal radius of the ellipse, and ___ and ____ are the radii of the ellipse.

The general implicit form of the equation of an ellipse is:

where ______ is the center of the ellipse.

If _______, then ___ is the major diameter and ___ is the minor diameter.

If _______, then ___ is the major diameter and ___ is the minor diameter.

The sum of the distances from any point on the ellipse to the two foci is _______.

The distance from the center of the ellipse to either focus follows the equation _______.

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