

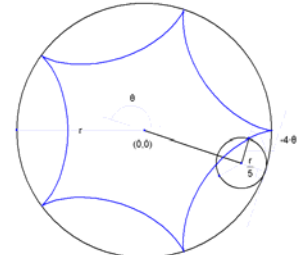
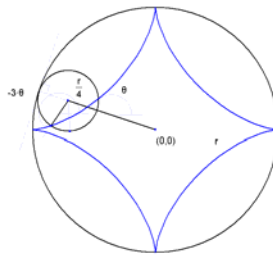
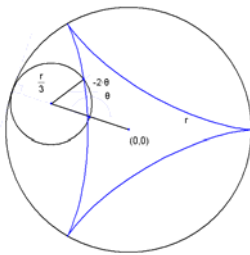
Area Enclosed by a General Hypocycloid

Abstract: In this paper, we investigate the area enclosed by a deltoid, an astroid and a five-cusped hypocycloid to derive a function for the area enclosed by a general hypocycloid. Hypocycloids are plane curves of high degree constructed by drawing the locus of a point on the circumference of a small circle rolling along the inside of a larger circle. We use the symbolic geometry system Geometry Expressions as part of our investigation.

Construction

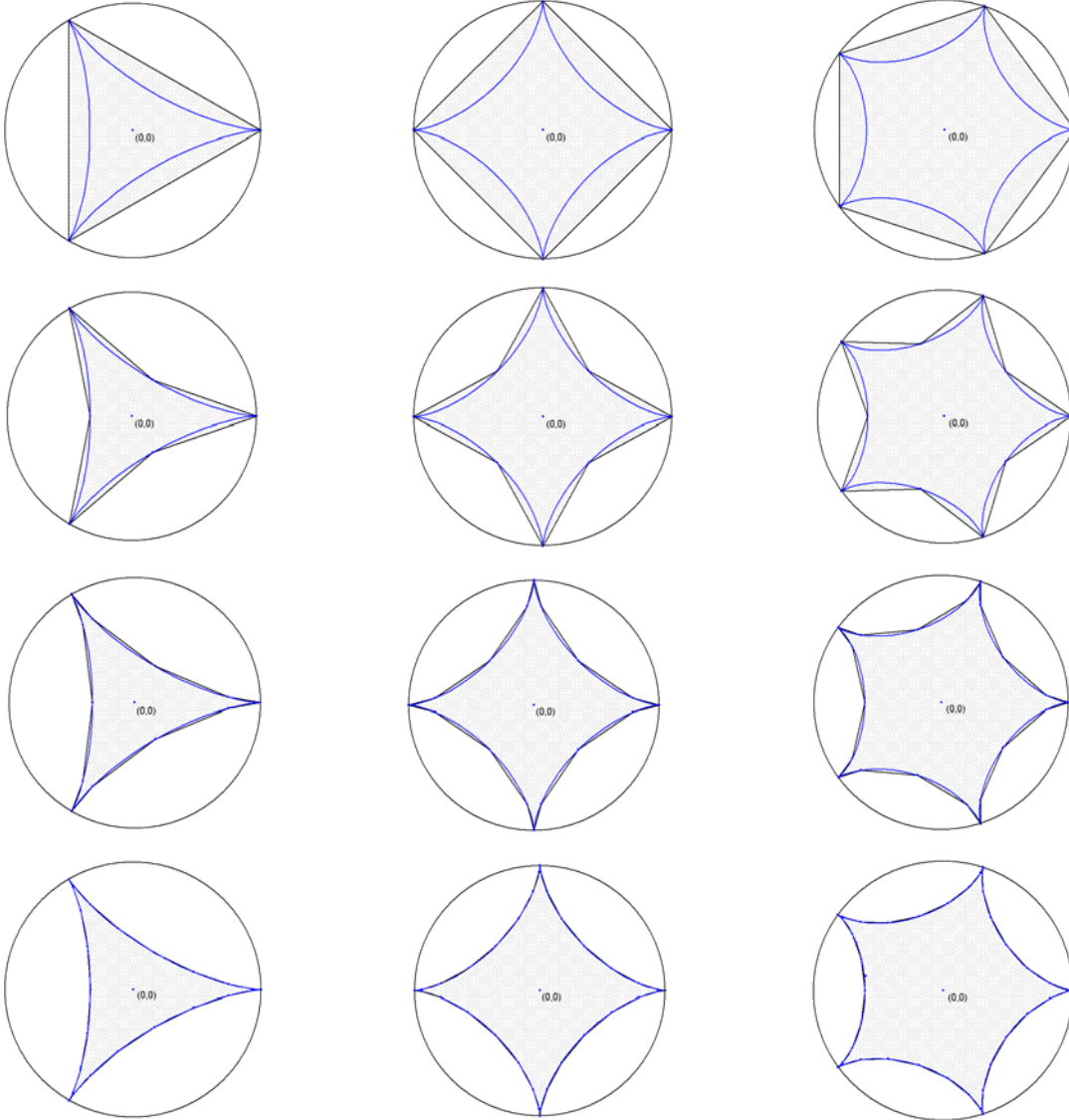
We create three hypocycloids; a deltoid, an astroid and a five-cusped hypocycloid; to integrate before we generalize the findings to a general hypocycloid. A deltoid has three sides; an astroid has four sides, and a five-cusped hypocycloid has five sides.

For each of the three figures, we draw a circle with radius “ r ” centered on the origin. Inside, we draw another circle tangent to the first and of radius “ r/n ” where “ n ” is the number of sides. We draw a line segment between the centers of the two circles with a direction “ θ ” relative to the axes. Along a radius of the smaller circle, we draw a shorter line segment with direction “ $(1-n)\theta$.” The locus of the endpoint on the circumference is the curve.



Area Expressed as the Limit of a Polygon

Before we determine an exact area, we estimate the value using polygons. We draw points on the curves and connect them with line segments to form polygons that represent the curves*. We approximate the limit of the polygons' area as the number of sides approaches infinity.



Deltoid Approximations:

$$\begin{aligned} n=3: & A=(1.299)r^2 \\ n=6: & A=(0.866)r^2 \\ n=12: & A=(0.756)r^2 \\ n=24: & A=(0.714)r^2 \\ n \rightarrow \infty: & A \approx (0.7)r^2 \end{aligned}$$

Astroid Approximations:

$$\begin{aligned} n=4: & A=(2.000)r^2 \\ n=8: & A=(1.414)r^2 \\ n=16: & A=(1.260)r^2 \\ n=32: & A=(1.200)r^2 \\ n \rightarrow \infty: & A \approx (1.2)r^2 \end{aligned}$$

5-Cusp Approximation:

$$\begin{aligned} n=5: & A=(2.378)r^2 \\ n=10: & A=(1.763)r^2 \\ n=20: & A=(1.597)r^2 \\ n=40: & A=(1.532)r^2 \\ n \rightarrow \infty: & A \approx (1.5)r^2 \end{aligned}$$

Generalizing the Parametric Equations

The symbolic geometry system reveals the follow equations:
For the deltoid:

$$\left[X = \frac{1}{3} r (2 \cos(\theta) + \cos(2 \theta)), Y = \frac{1}{3} r (2 \sin(\theta) - \sin(2 \theta)) \right]$$

For the astroid:

$$\left[X = \frac{1}{4} r (3 \cos(\theta) + \cos(3 \theta)), Y = \frac{1}{4} r (3 \sin(\theta) - \sin(3 \theta)) \right]$$

For the five-cusped hypocycloid:

$$\left[X = \frac{1}{5} r (4 \cos(\theta) + \cos(4 \theta)), Y = \frac{1}{5} r (4 \sin(\theta) - \sin(4 \theta)) \right]$$

When we use “n” in our geometric construction, the symbolic geometry system provides a general set of equations. We note that, in general, hypocycloids have the equations

$$\left[X = \frac{1 r (\cos(\theta) (n - 1) + \cos(\theta (n - 1)))}{n}, Y = \frac{1 r (\sin(\theta) (n - 1) + \sin(\theta (n - 1)))}{n} \right]$$

Area Expressed as an Integral

We take the integral of the first three functions and verify the results with the approximations made above. Then, we integrate the general function. Because each curve is constructed via a full circle, we integrate from “ $\theta=0$ ” to “ $\theta=2\pi$.”

For each:

$$A = \int_{X(0)}^{X(2\pi)} Y(\theta) dX$$

$$A = \int_0^{2\pi} Y(\theta) \left(\frac{\partial}{\partial \theta} X(\theta) \right) d\theta$$

For the deltoid:

$$A = \int_0^{2\pi} r \frac{2 \sin(\theta) - \sin(2\theta)}{3} \left(\frac{\partial}{\partial \theta} \frac{r(2 \cos(\theta) + \cos(2\theta))}{3} \right) d\theta$$

$$A = \int_0^{2\pi} \frac{2 r^2 (-2 \sin(\theta) + \sin(2\theta)) (\sin(\theta) + \sin(2\theta))}{9} d\theta$$

$$A = \frac{r^2 (24\pi - 12 \sin(4\pi) + 12 \sin(2\pi) - 4 \sin(6\pi) + 3 \sin(8\pi))}{108}$$

$$A = \frac{2}{9} \pi r^2$$

$$A \approx (0.698)r^2 \approx (0.7)r^2 \text{ (from above)}$$

For the astroid:

$$A = \int_0^{2\pi} r \frac{3 \sin(\theta) - \sin(3\theta)}{4} \left(\frac{\partial}{\partial \theta} \frac{r(3 \cos(\theta) + \cos(3\theta))}{4} \right) d\theta$$

$$A = \int_0^{2\pi} \frac{3 r^2 (-3 \sin(\theta) + \sin(3\theta)) (\sin(\theta) + \sin(3\theta))}{16} d\theta$$

$$A = \frac{r^2 (24\pi - 3 \sin(4\pi) - 3 \sin(8\pi) + \sin(12\pi))}{64}$$

$$A = \frac{3}{8} \pi r^2$$

$$A \approx (1.178)r^2 \approx (1.2)r^2 \text{ (from above)}$$

For the five-cusped hypocycloid:

$$A = \int_0^{2\pi} r \frac{4 \sin(\theta) - \sin(4\theta)}{5} \left(\frac{\partial}{\partial \theta} \frac{r(4 \cos(\theta) + \cos(4\theta))}{5} \right) d\theta$$

$$A = \int_0^{2\pi} \frac{4 r^2 (-4 \sin(\theta) + \sin(4\theta)) (\sin(\theta) + \sin(4\theta))}{25} d\theta$$

$$A = \frac{r^2 (240\pi - 80 \sin(4\pi) + 40 \sin(6\pi) - 24 \sin(10\pi) + 5 \sin(16\pi))}{500}$$

$$A = \frac{12}{25} \pi r^2$$

$$A \approx (1.508)r^2 \approx (1.5)r^2 \text{ (from above)}$$

For the general hypocycloid:

$$A = \int_0^{2\pi} r \frac{(n-1) \sin(\theta) - \sin((n-1)\theta)}{n} \left(\frac{\partial}{\partial \theta} r \frac{((n-1) \cos(\theta) + \cos((n-1)\theta))}{n} \right) d\theta$$

$$A = \int_0^{2\pi} \frac{r^2 ((n-1) \sin(\theta) - \sin(\theta (n-1))) (- (n-1) \sin(\theta) - \sin(\theta (n-1)) (n-1))}{n^2} d\theta$$

$$A = \frac{1}{4} r^2 (-\sin(4\pi) n^3 + 4 n^3 \pi + 2 \sin(4\pi) n^2 - 12 n^2 \pi + 2 \sin(2(-2+n)\pi) n^2 - 2 \sin(2\pi n) n^2 + \sin(4(n-1)\pi) n - 2 \sin(2(-2+n)\pi) n - \sin(4\pi) n + 6 \sin(2\pi n) n + 8 \pi n - 4 \sin(2\pi n)) / n^3$$

$$A = \frac{\pi r^2 (n^2 - 3n + 2)}{n^2}$$

$$A = \frac{\pi r^2 (n-1)(n-2)}{n^2}$$

Substituting “3,” “4” and “5” for “n,” we confirm

$$A(3) = \frac{2}{9} \pi r^2$$

$$A(4) = \frac{3}{8} \pi r^2$$

$$A(5) = \frac{12}{25} \pi r^2$$

We conclude that that the area of a general hypocycloid is equal to the expression

$$\frac{\pi r^2 (n-1)(n-2)}{n^2}$$

Notes:

* Points are equidistant in terms of θ .