141 theorems proved using GXWeb

GXWeb gives both numeric and symbolic measurements taken from your diagram. The symbolic measurements are exact, computed algebraically, and hence provide a form of automated proof of theorems. We give examples here.

1 Introduction

GXWeb is not designed primarily for geometry theorem proving, however, the fact that it can derive symbolic results from a geometry diagram means that it can be used as a theorem prover with a little creativity. In this document, we use the software to prove 141 different theorems, taken from the book "Machine Proofs in Geometry" by Chou, Gao and Zhang.

Geometry theorems frequently give results which are stated in terms of unexpected properties which hold. For example, while it is unremarkable that two points lie on a straight line, it may be the result of a theorem that three points are collinear. Similarly, a theorem might state that three lines are concurrent, or four points lie on a circle. GXWeb generates symbolic quantities for measurements (such as distance, angle, area) taken from the drawing. Here are some ideas for making symbolic measurements to establish theorem results.

3 Collinear points: To show that points A,B,C are collinear, you could draw the line segment between A and B, then ask GXWeb to measure the distance(C,AB). This will measure the perpendicular distance between C and the line AB. If the distance is 0, then the lines are collinear. Another approach is to evaluate area(A,B,C). This returns the area of the triangle ABC and will be 0 if the points are collinear.

Note the difference if you make the measurement in the numeric pane versus the symbolic pane. If you measure in the numeric pane it tells you that the points are collinear for this specific geometry, but if there are points which can be dragged, or parameters changed, it may not remain 0. If it is 0 in the symbolic pane, it is guaranteed to be 0 for any value of the parameters.

3 concurrent lines: To show that lines L0, L1, L2 are concurrent, first create the intersection point A between L0 and L1. Now evaluate distance(A,L2). If this is 0, then A lies on all 3 lines.

4 concyclic points: To show A,B,C,D lie on a circle, you can create the circumcircle of A,B,C, let's assume its center is called E. now you would need to show distance(D,E)=distance(A,E). In some cases you can do this by inspection. In other cases you might need some simplification done for you, so you can ask for

$$\frac{distance(D,E)}{distance(A,E)}$$

Which should evaluate to 1 if the distances are identical.

Alternatively,

$$distance(D,E) - distance(A,E)$$

should evaluate to 0.

Equilateral triangle: To check that triangle ABC is equilateral, you could ensure all the side lengths are equal, by examining

$$\frac{distance(A,B)}{distance(B,C)} = 1$$

and

$$\frac{distance(A,C)}{distance(B,C)} = 1$$

Alternatively, you could check the angles:

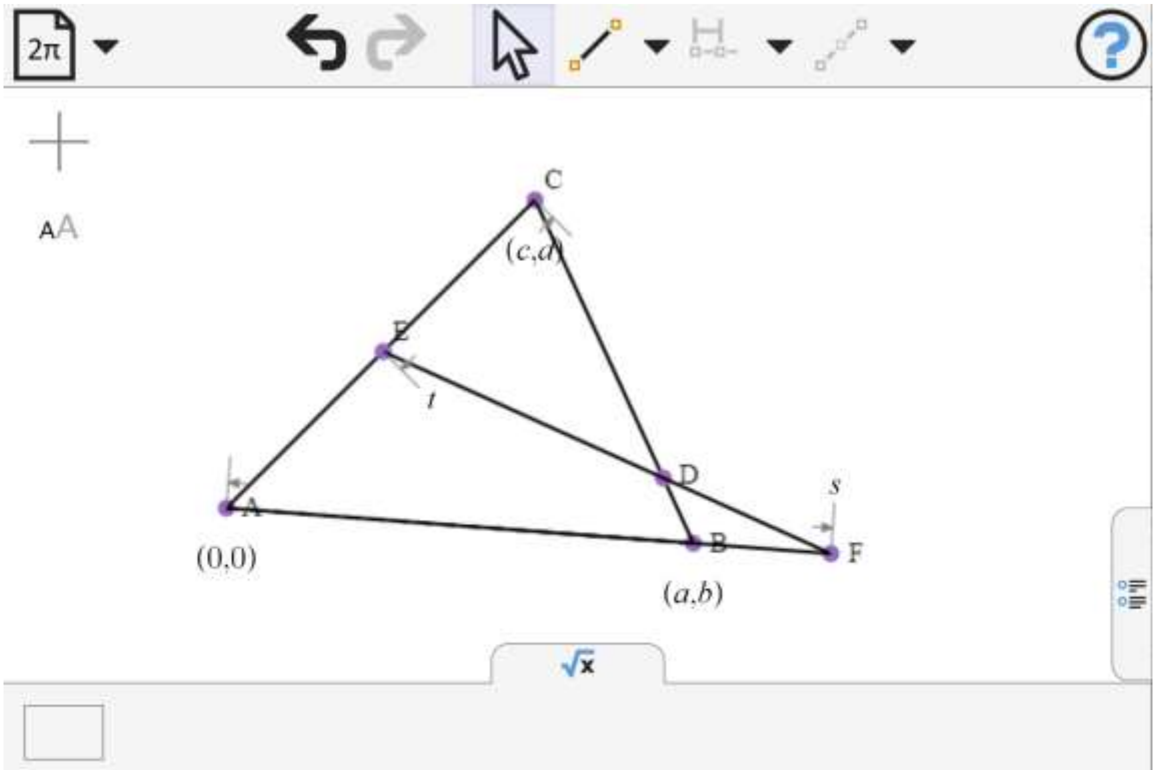
$$angle(A,B,C) = \frac{\pi}{3}$$

and

$$angle(B,C,A) = \frac{\pi}{3}$$

2 Geometry of Incidence

1. Menelaus Theorem

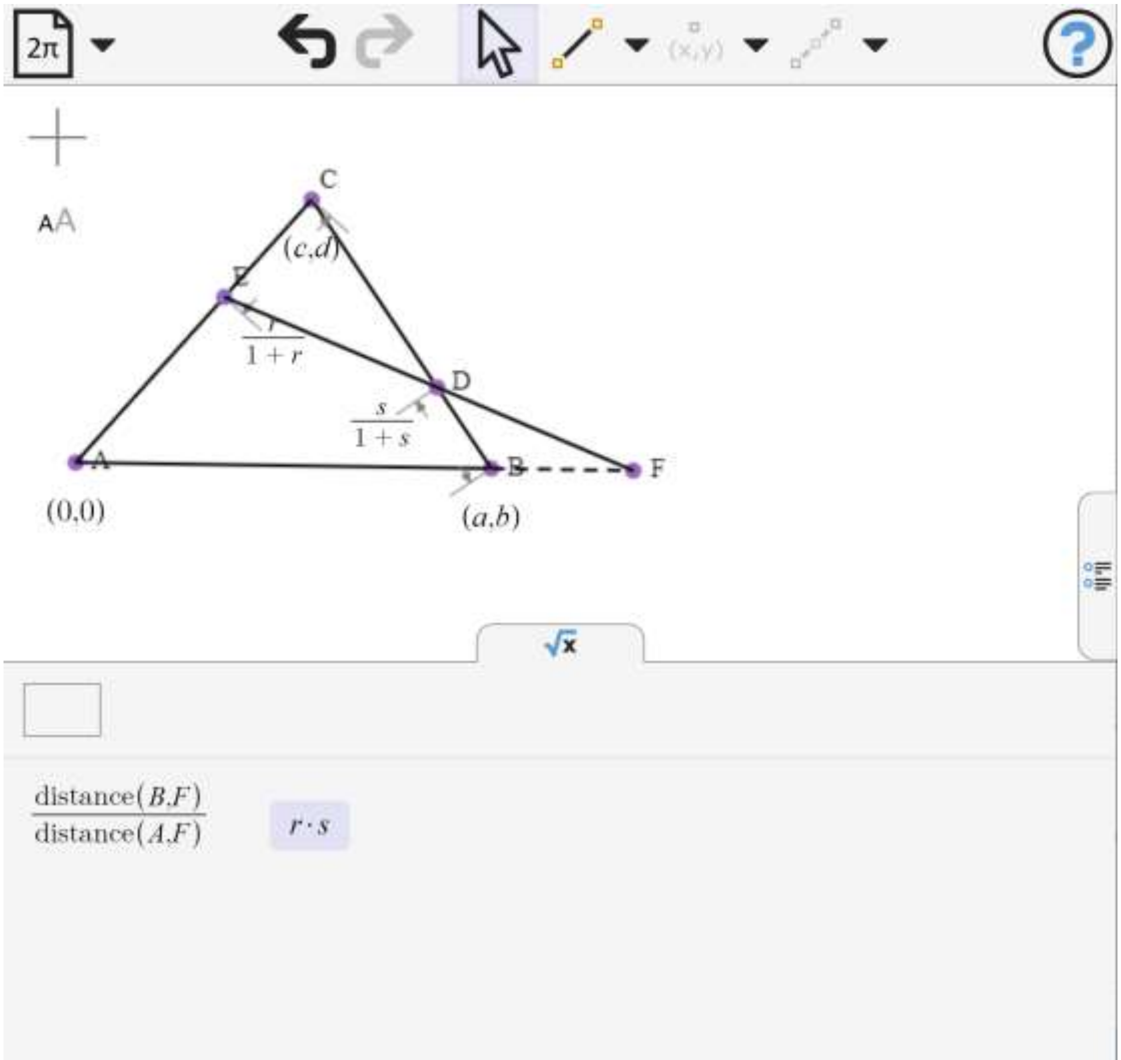


$$\frac{\text{distance}(A,F)}{\text{distance}(B,F)} \cdot \frac{\text{distance}(B,D)}{\text{distance}(D,C)} \cdot \frac{\text{distance}(C,E)}{\text{distance}(E,A)} \quad 1$$

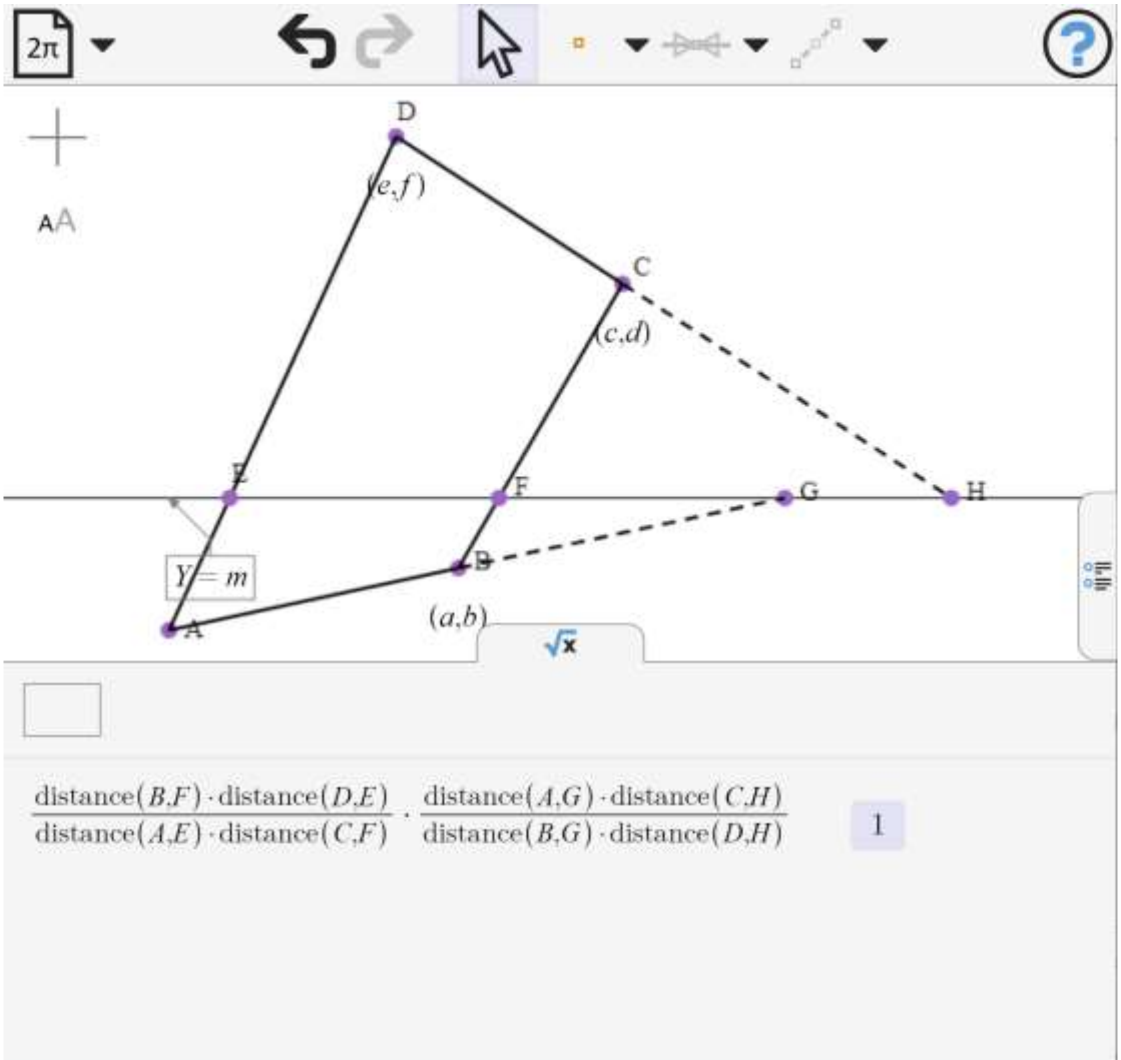
Without loss of generality, we make A be the origin.

2. Converse of Menelaus Theorem

Ratio r_1 is defined to be BD/DC. GX point proportional constraint accepts the ratio BD/BC. So we enter $r_1/(1+ r_1)$.

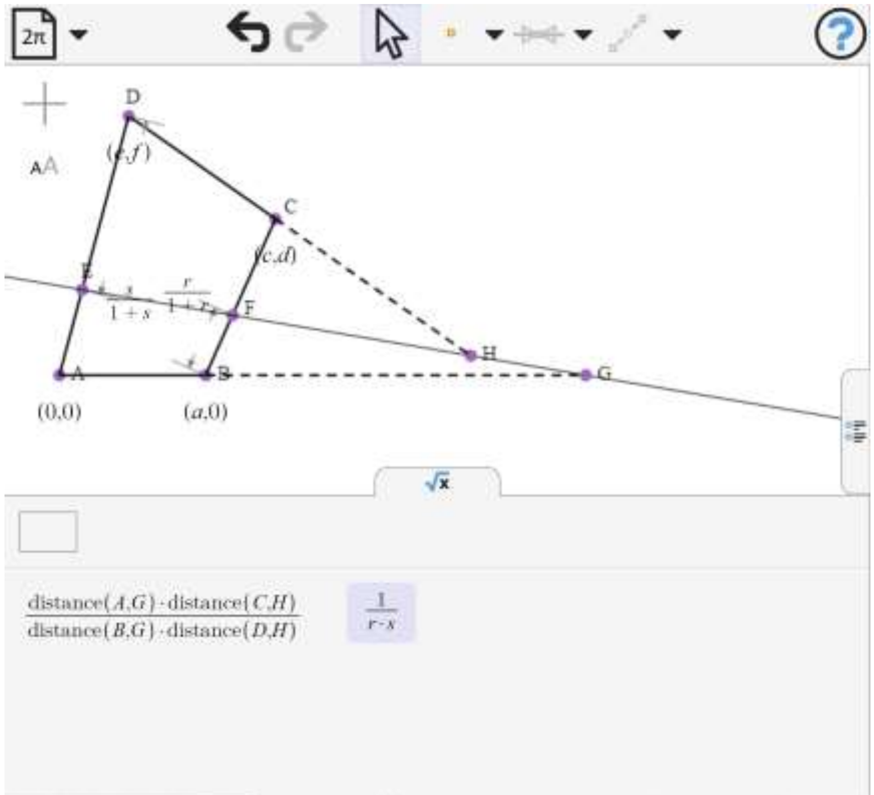


3. Menelaus Theorem for a Quadrilateral



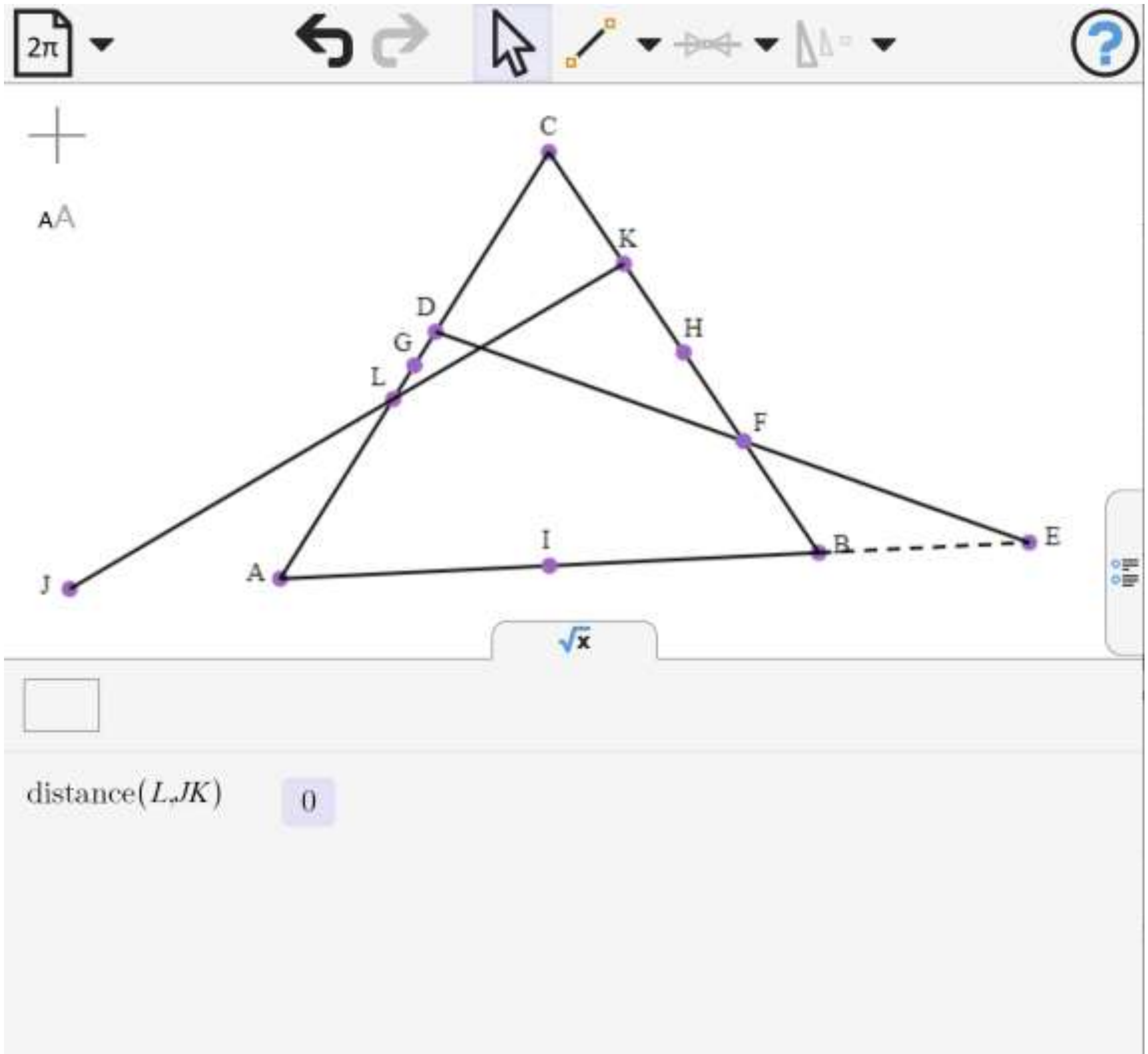
Here we have elected without loss of generality to make A the origin and the line y=b.

4. Menelaus Theorem for a Quadrilateral (done a little differently)



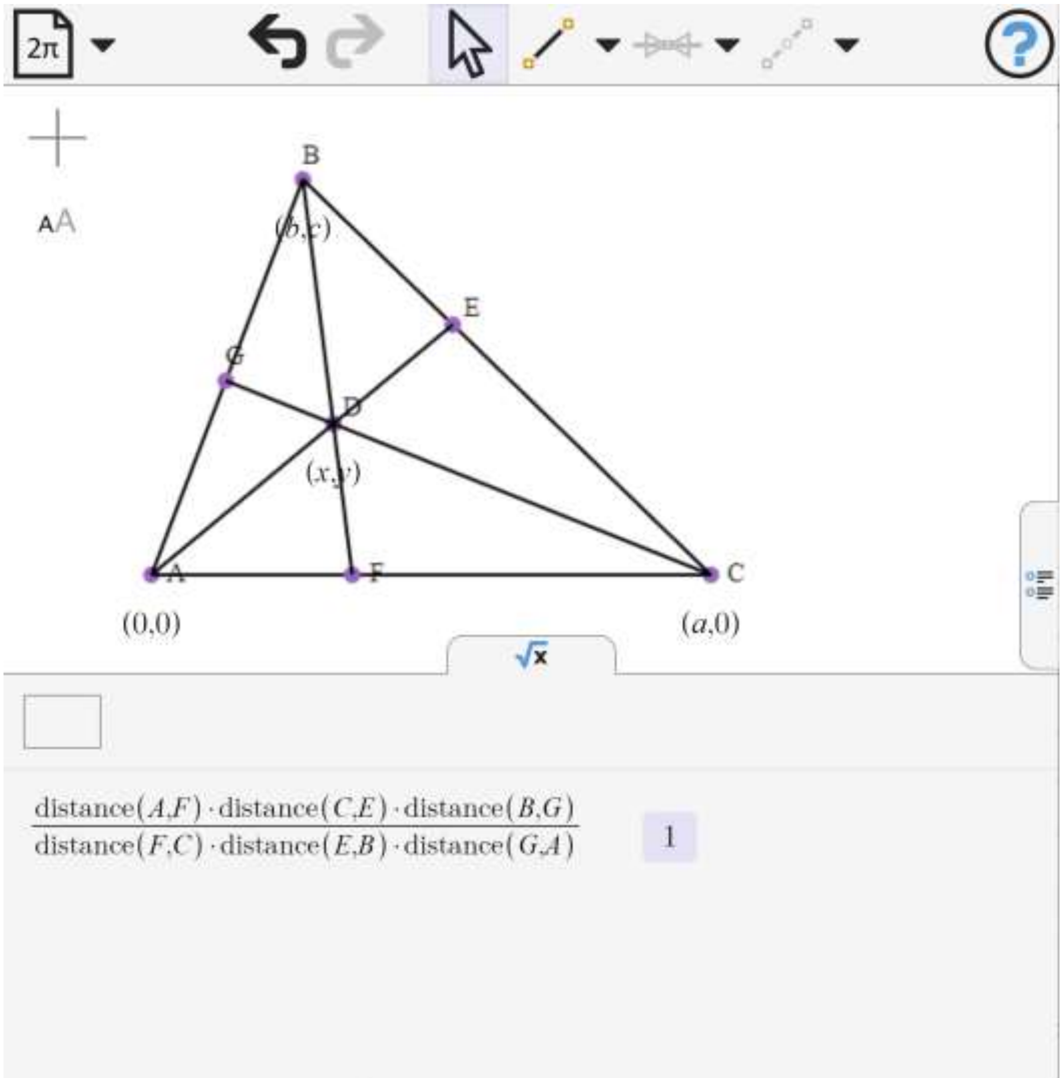
In this one, we specify the coordinates of the quad points and proportions for E and F, and measure the other proportions. A useful trick is to make one of the coordinates (0,0), and another (a,0) which cuts down on the complexity of the results.

5. The isotomic points of three collinear points are collinear



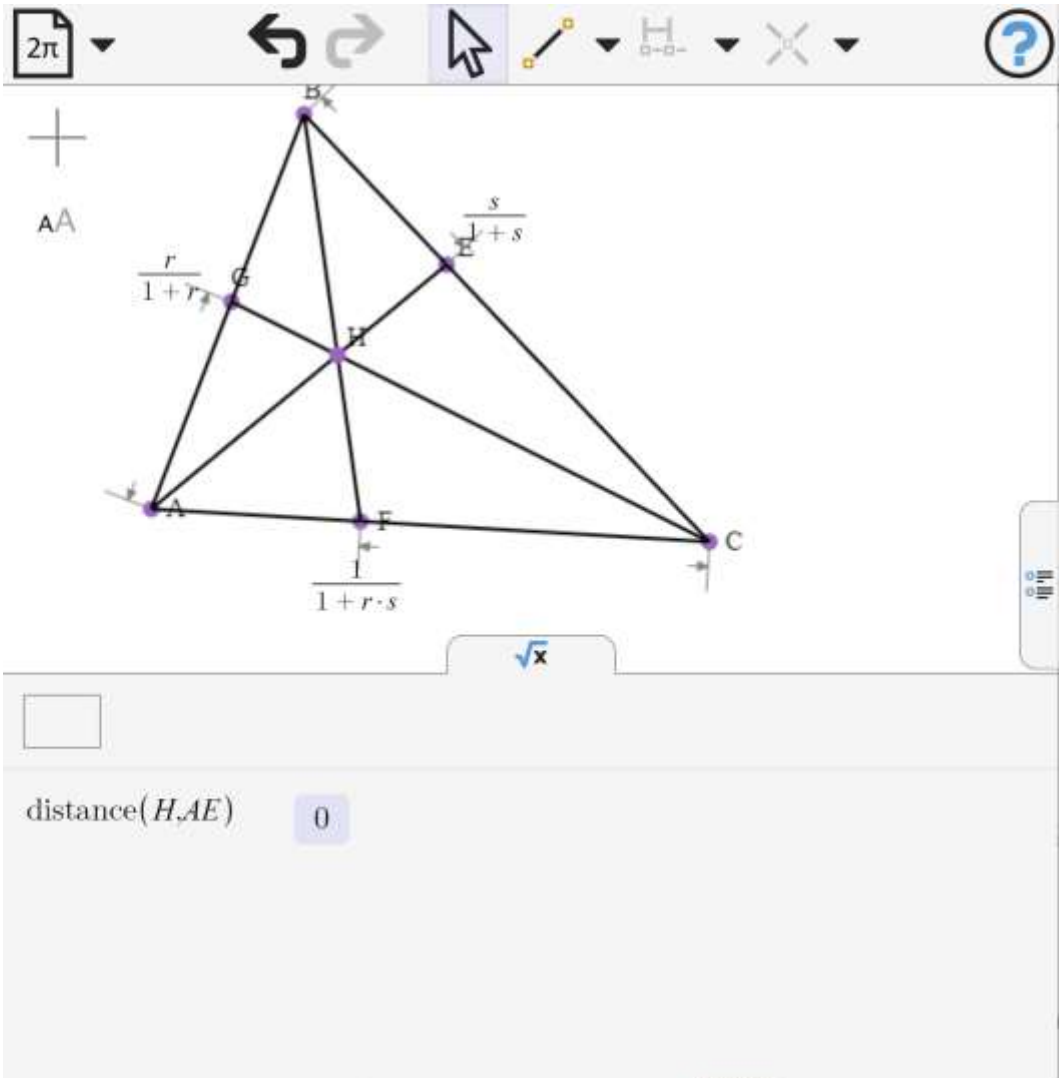
Isotomic points are created by reflecting points in segment midpoints. (reflection in a point is done in Geometry Expressions using dilation of scale -1).

6. Ceva’s Theorem



Here is a direct proof of Ceva’s theorem.

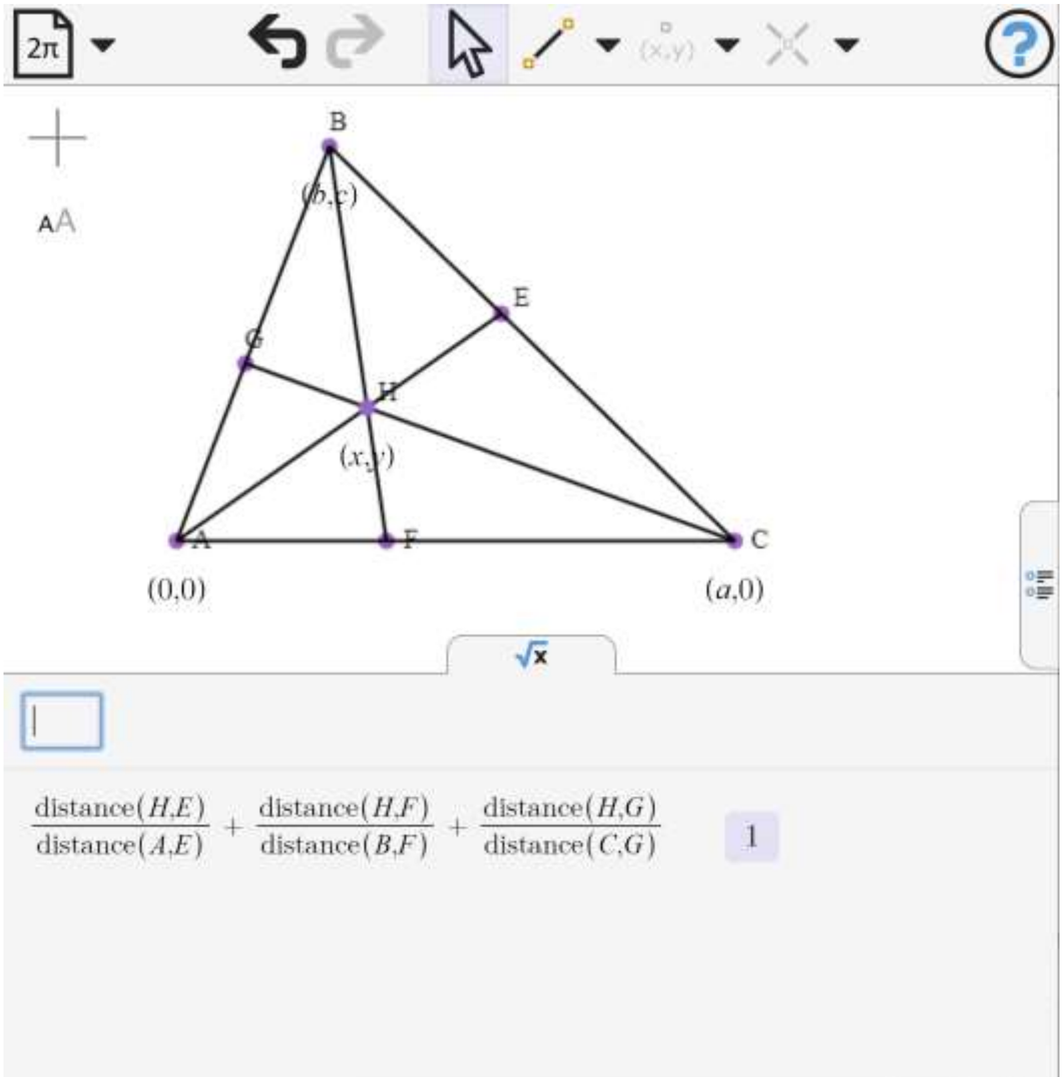
7. Converse of Ceva's Theorem



Here we have set H to be the intersection of lines AD and BE. We measure its distance to line CF

8. Ratios on the Cevian

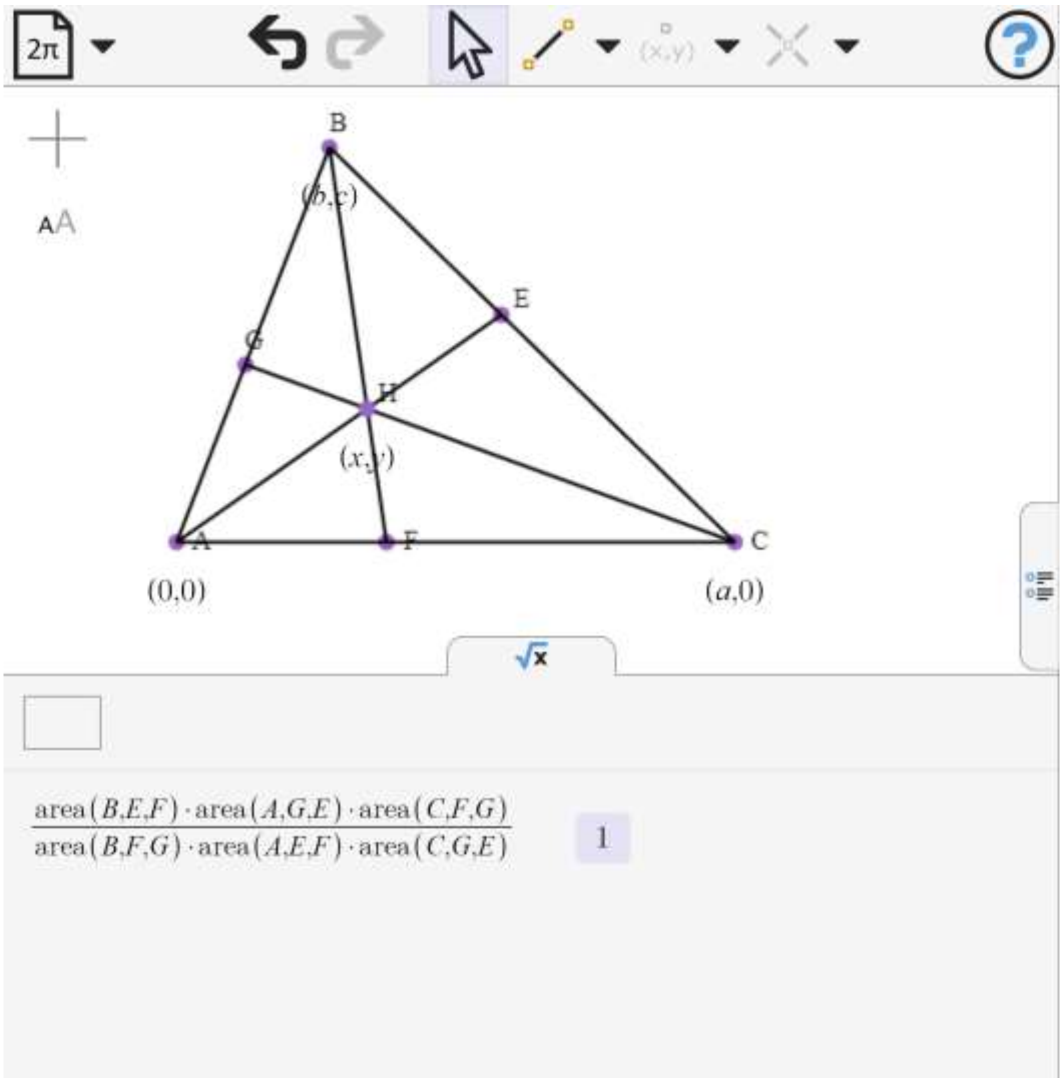
If LMN is the Cevian Triangle of point S in triangle ABC, we have $\frac{SL}{AL} + \frac{SM}{BM} + \frac{SN}{CN} = 1$



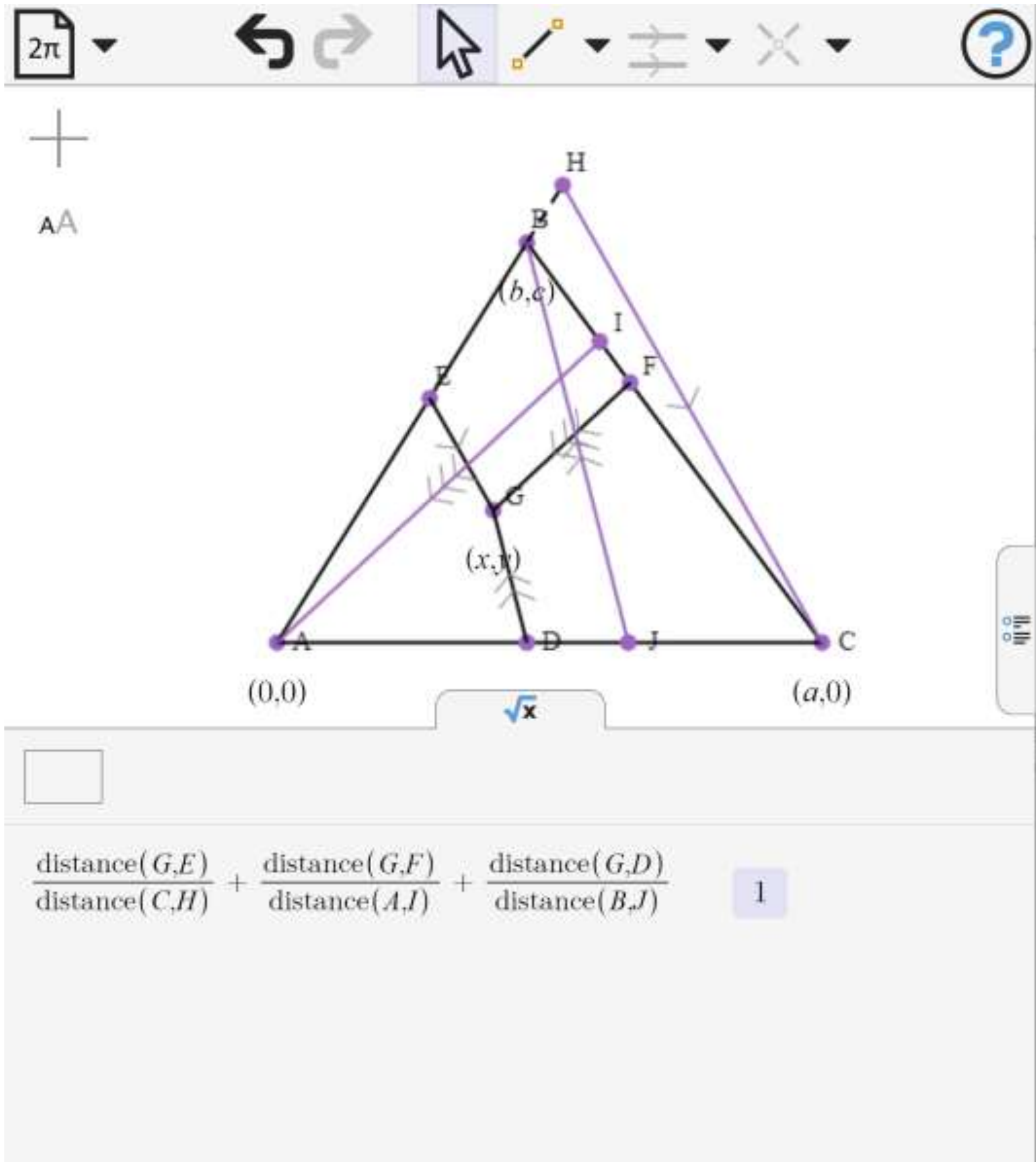
SN and CN are hidden for clarity.

9. Sub triangle areas

If EFF is the Cevian Triangle of point S in triangle ABC, we have the following area relationship



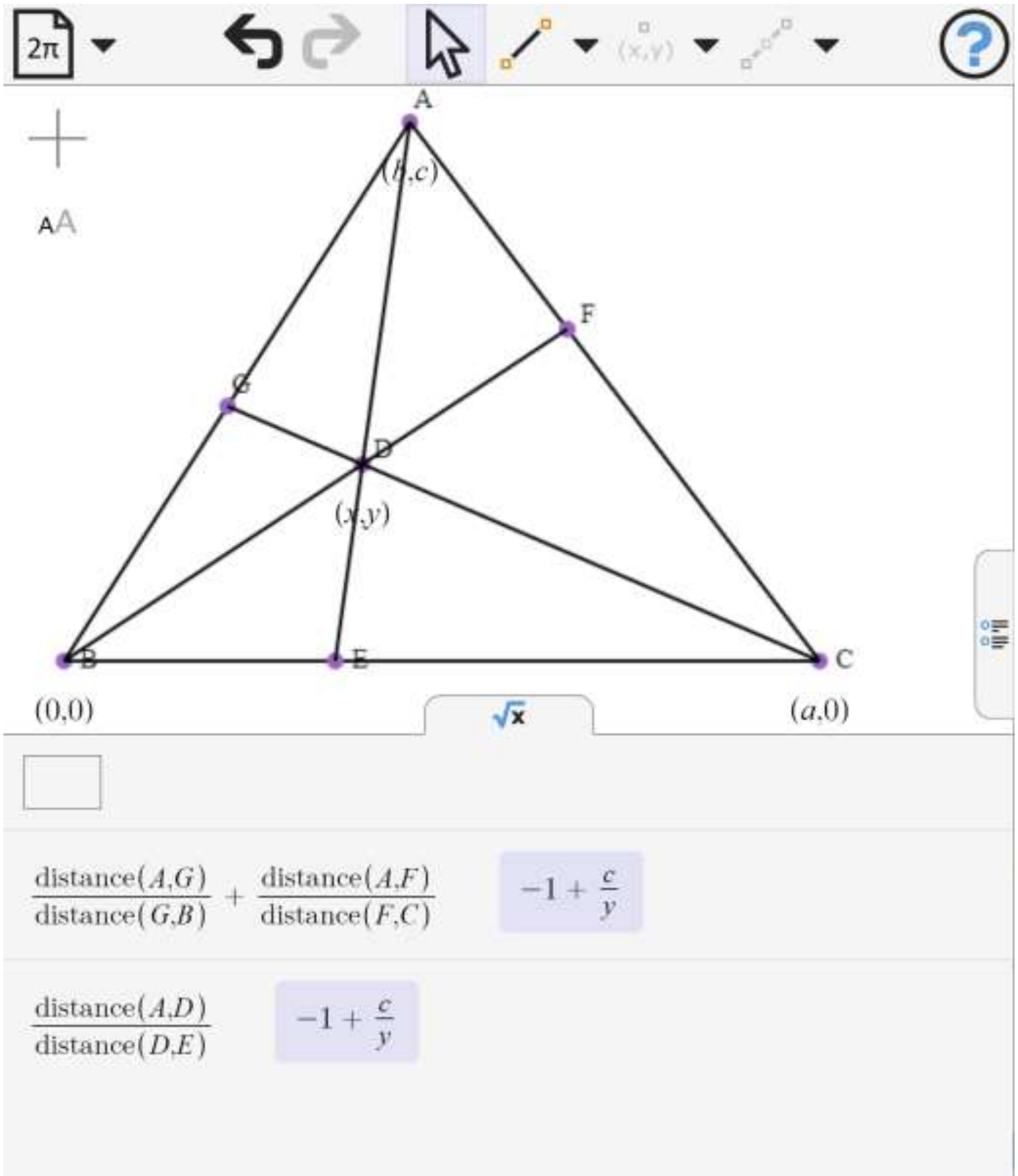
10. Ratios of parallels



D,E,F are the feet of the altitudes of ABC.G is an arbitrary point. H lies on AB and CH is parallel to GE. I lies on BC and AI is parallel to GF. J lies on AC and BJ is parallel to GD. The sum of the ratios of the parallel sides is 1.

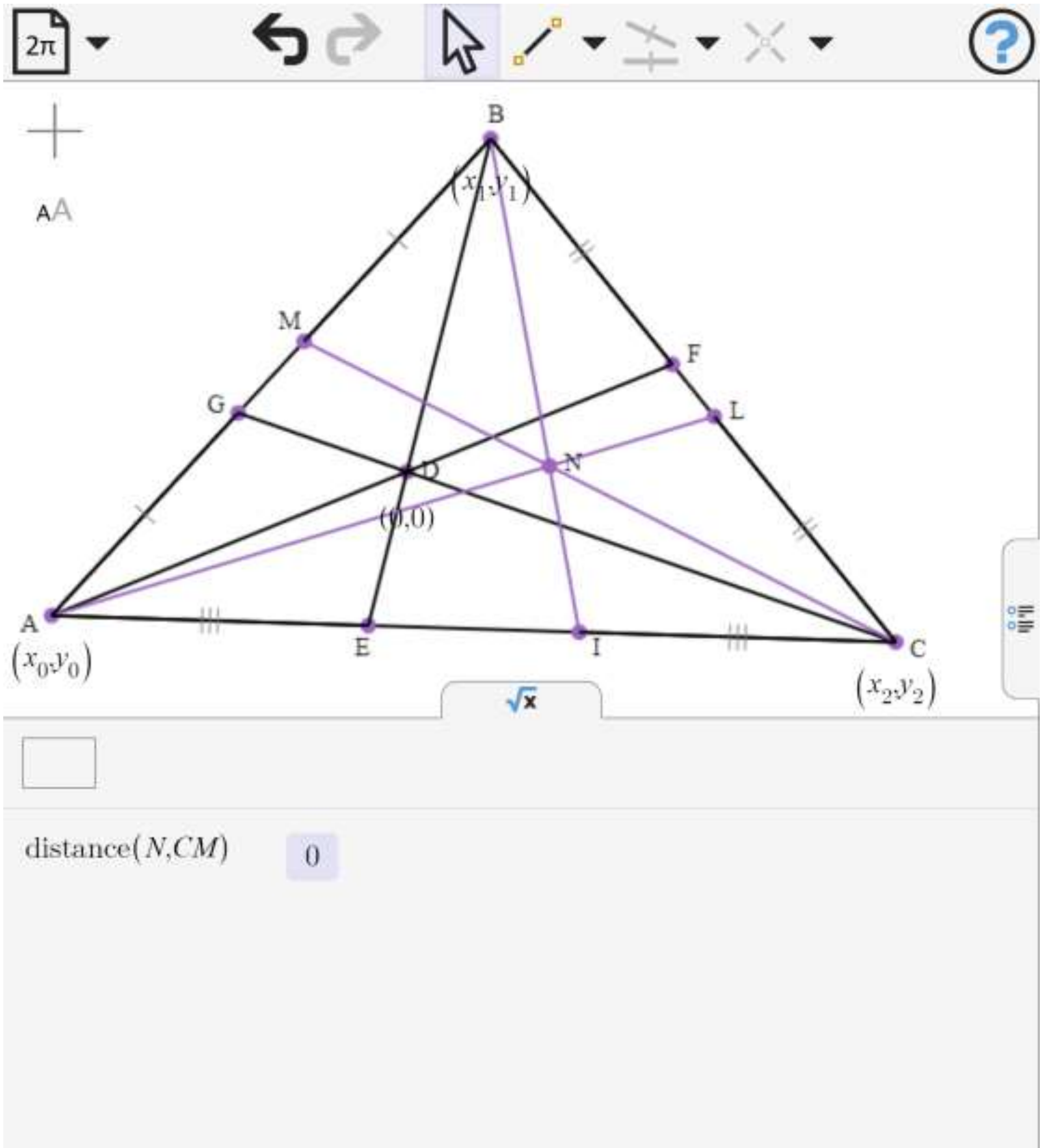
11. More ratios

If EFG is the Cevian triangle of the point D for the triangle ABC, we have: $\frac{AD}{DE} = \frac{AG}{GB} + \frac{AF}{FC}$



12. Isotomics of Cevian point

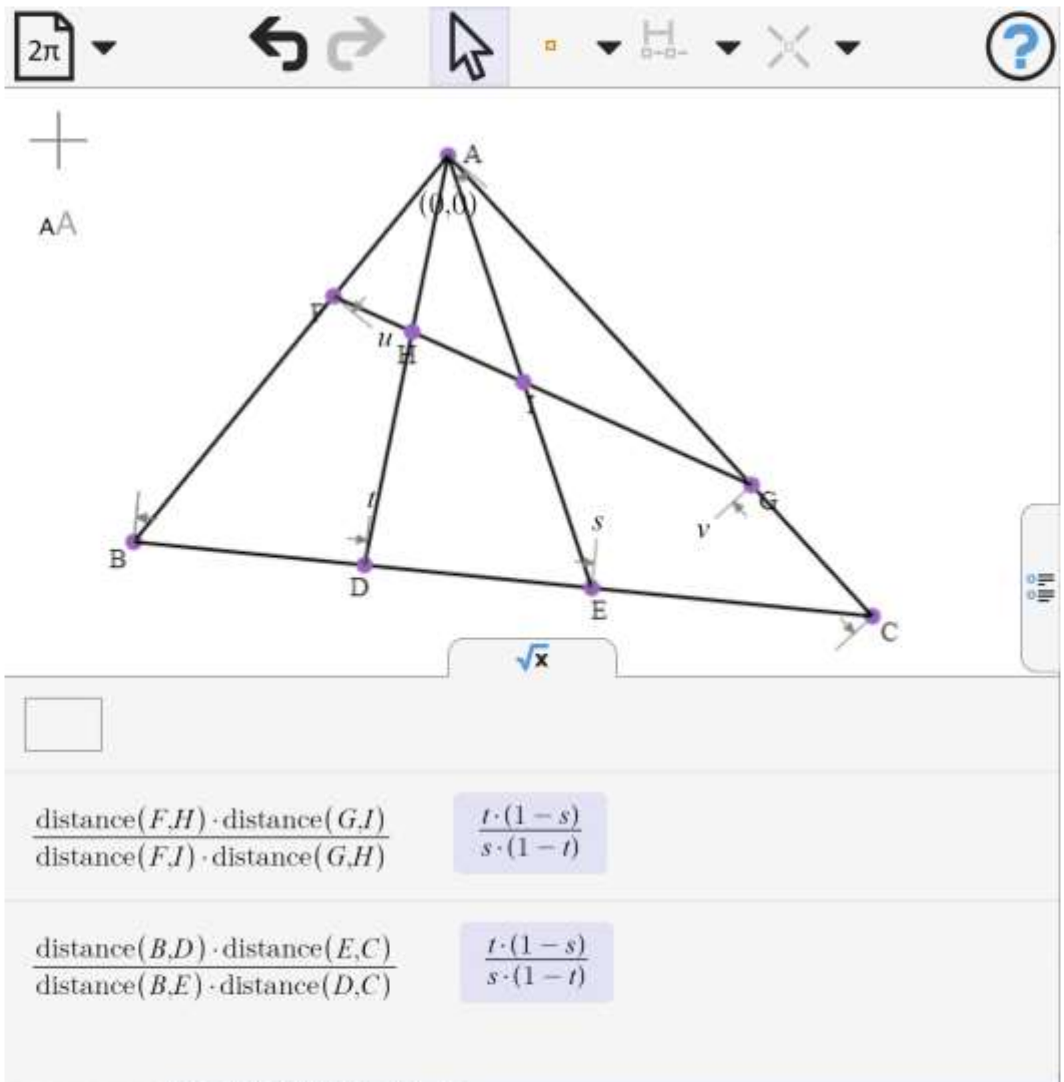
If the three lines joining three points marked on the sides of a triangle to the opposite vertices are concurrent, the same is true of the isotomics of the given points.



N is defined to be the intersection of AL and BI we measure its distance to CM

13. The cross ratio of four points on a line is unchanged by projection

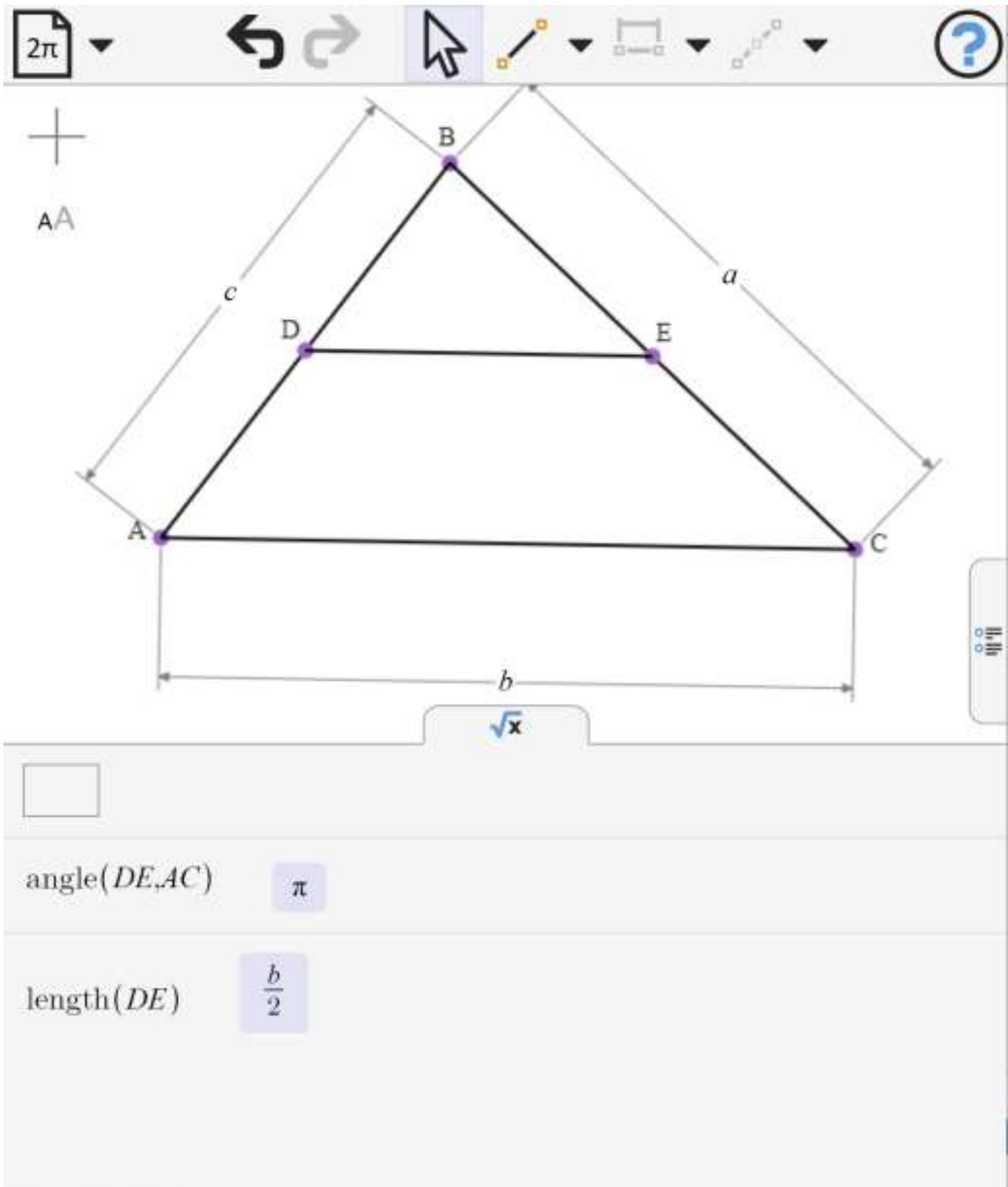
Let A,B,C,D be four collinear points. The cross ratio denoted (ABCD) = $\frac{CA/_{CB}}{DA/_{DB}}$



Medians and Centroids

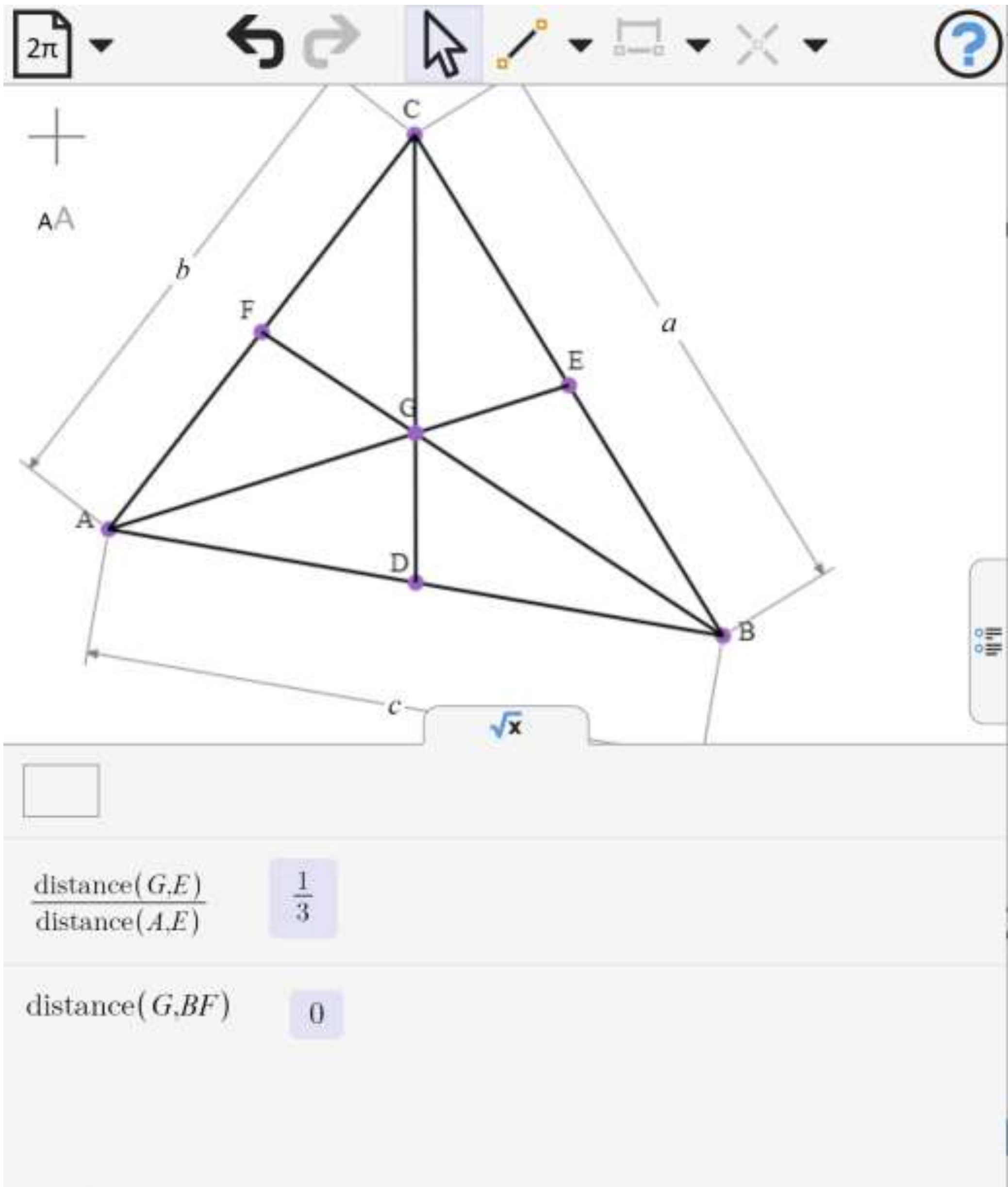
14. Line joining side midpoints

The line joining the midpoints of two sides of a triangle is parallel to the third side and is equal to one half the length.



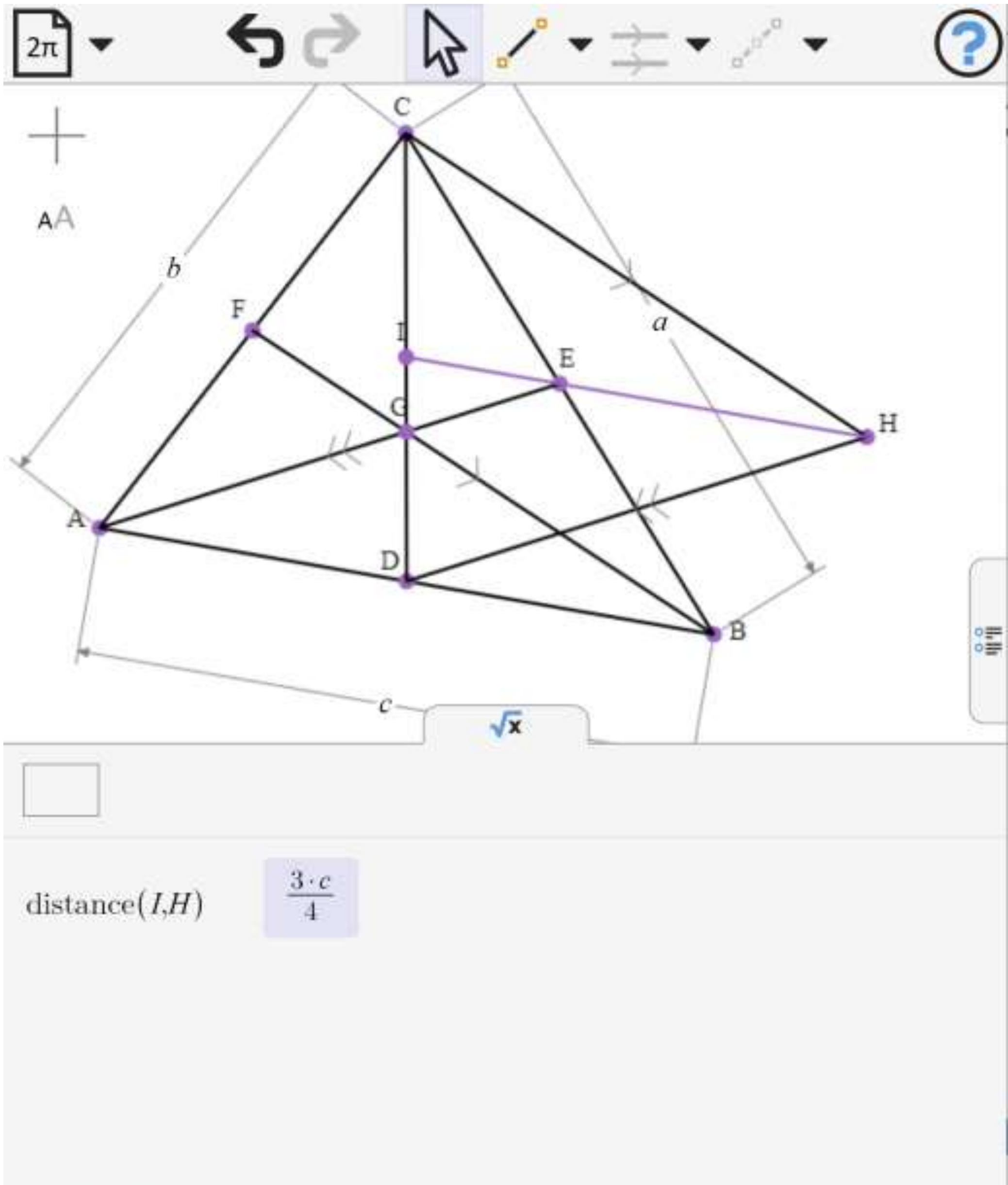
15. Centroid Theorem

The three medians of a triangle meet in a point and each median is trisected by this point.



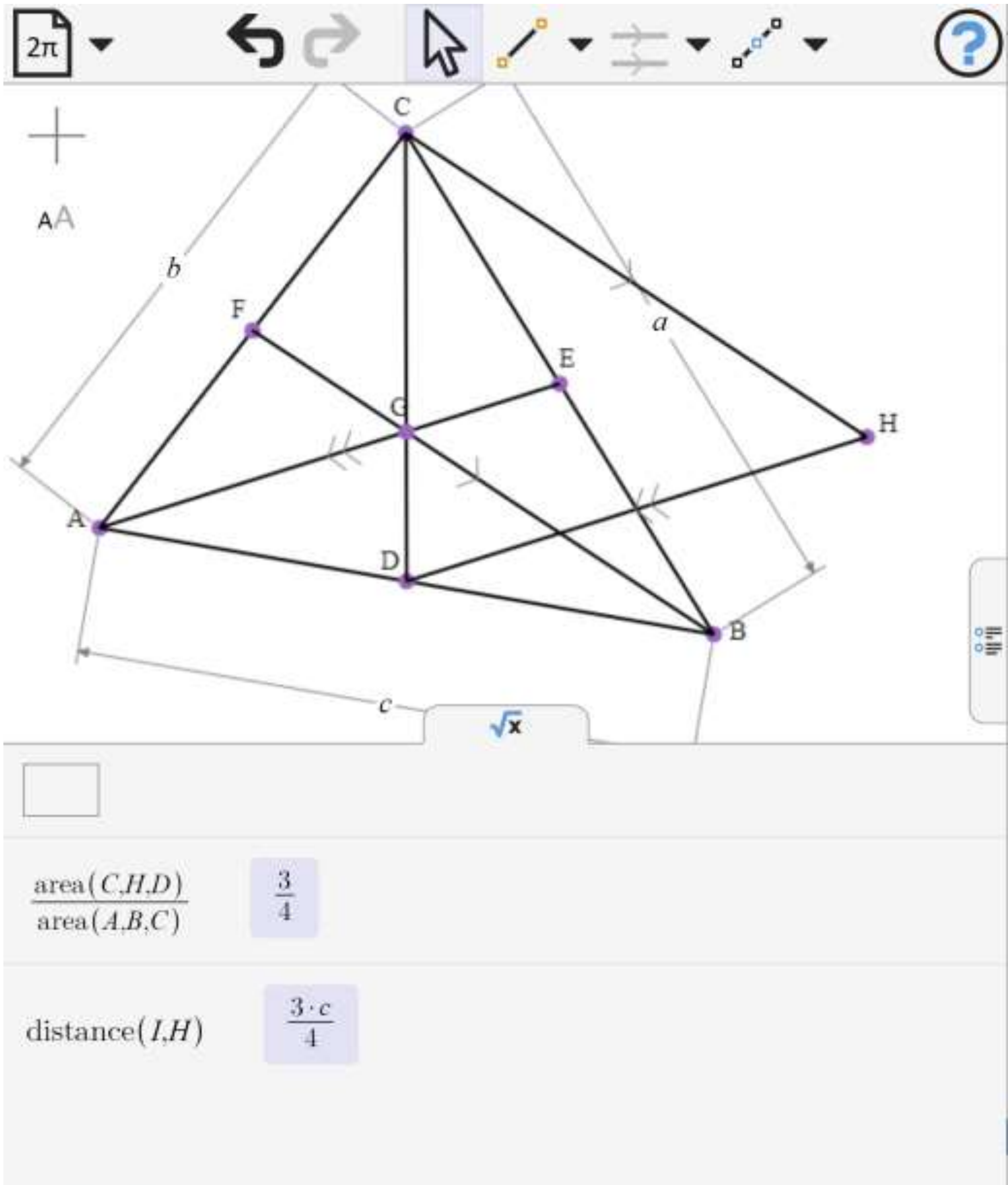
16. Median triangle

With the medians of a triangle a new triangle is constructed. The medians of the second triangle are equal to three quarters of the respective sides of the given triangle.



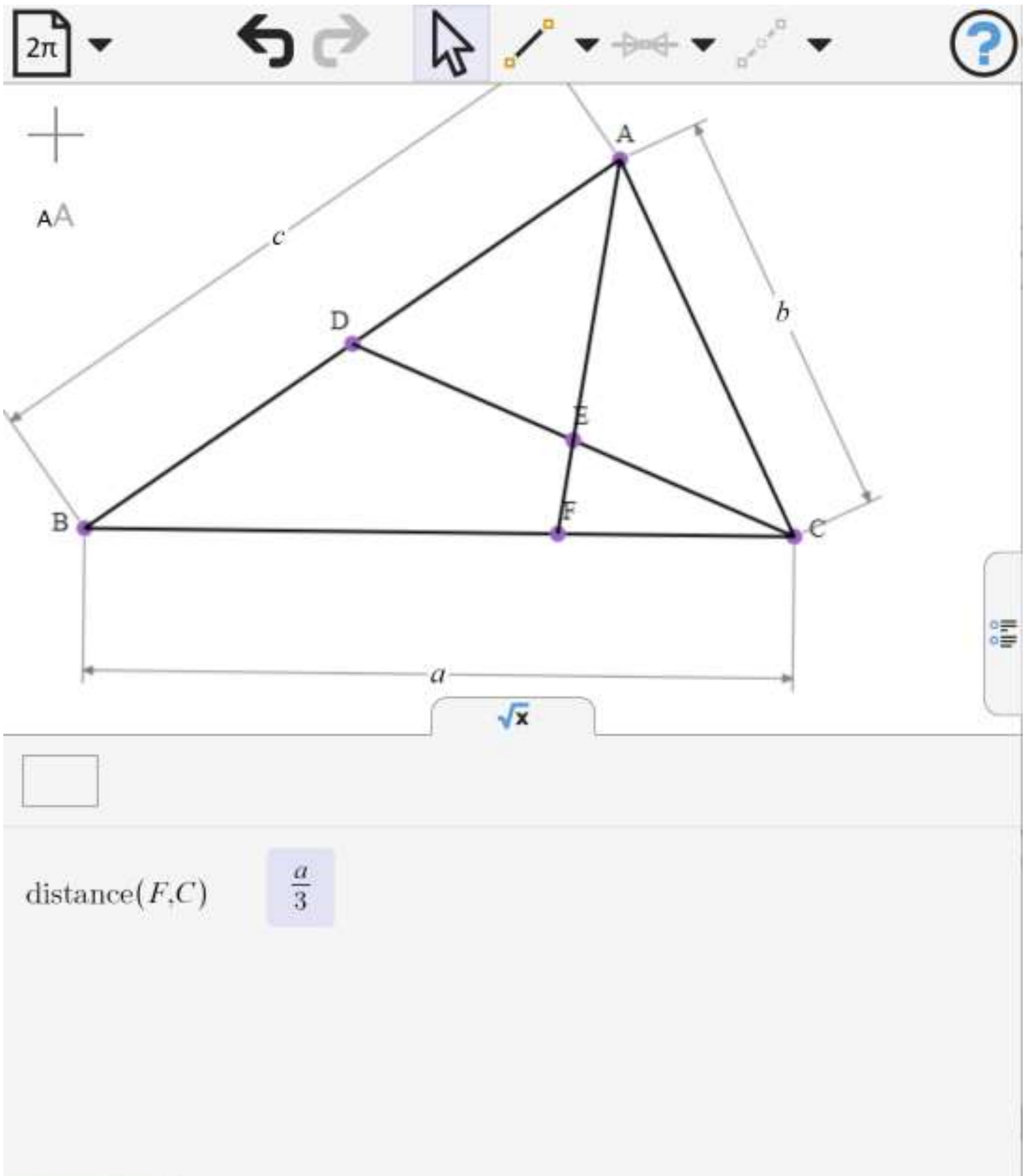
17. Area of the median triangle

The area of the triangle having for sides the medians of a triangle is equal to three quarters the area of the given triangle.



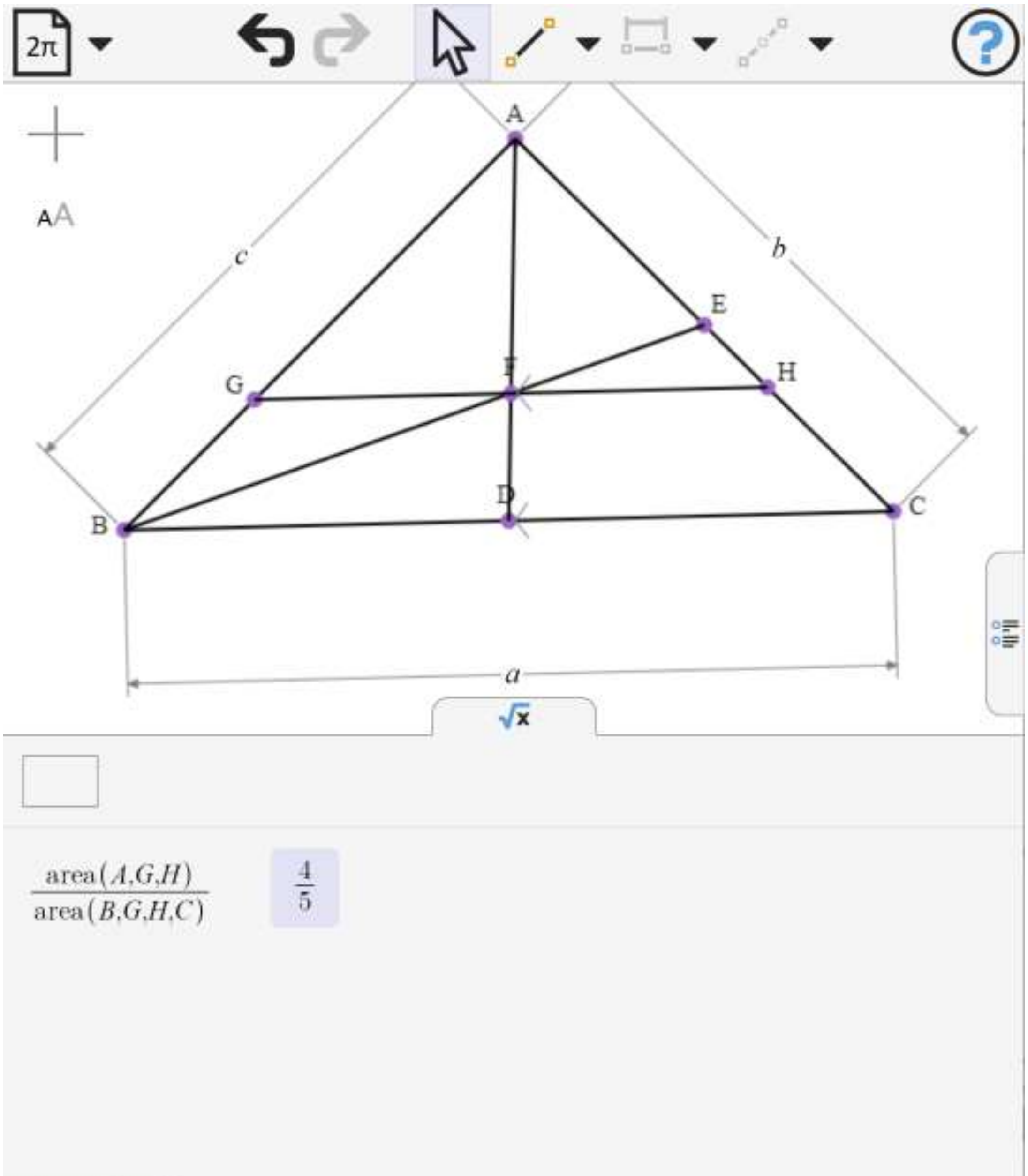
18. Median midpoint

Show that the line joining the midpoint of a median to a vertex of the triangle trisects the side opposite the vertex considered.



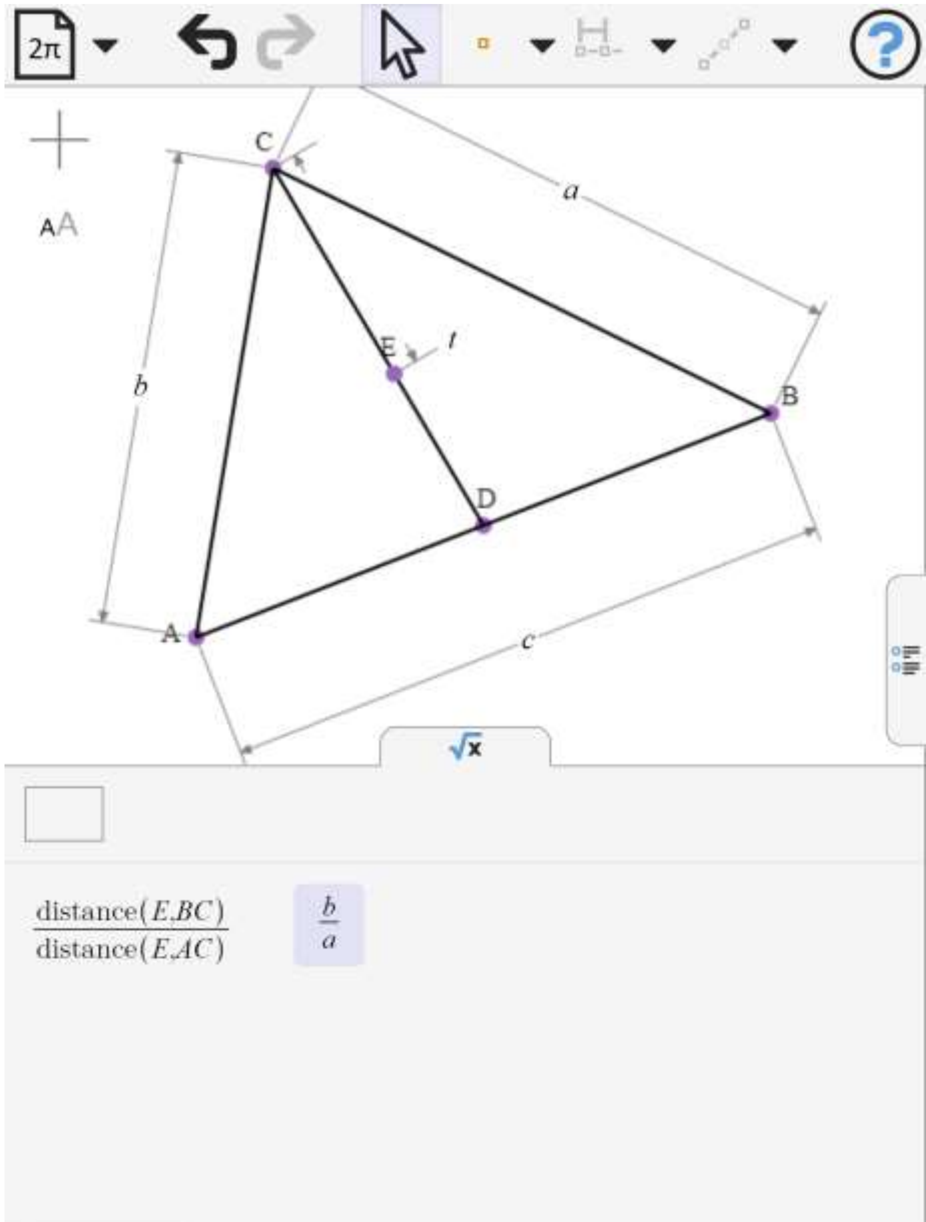
19. Parallel through the centroid

Show that a parallel to a side of a triangle through the centroid divides the area of the triangle into two parts, in the ratio of 4:5



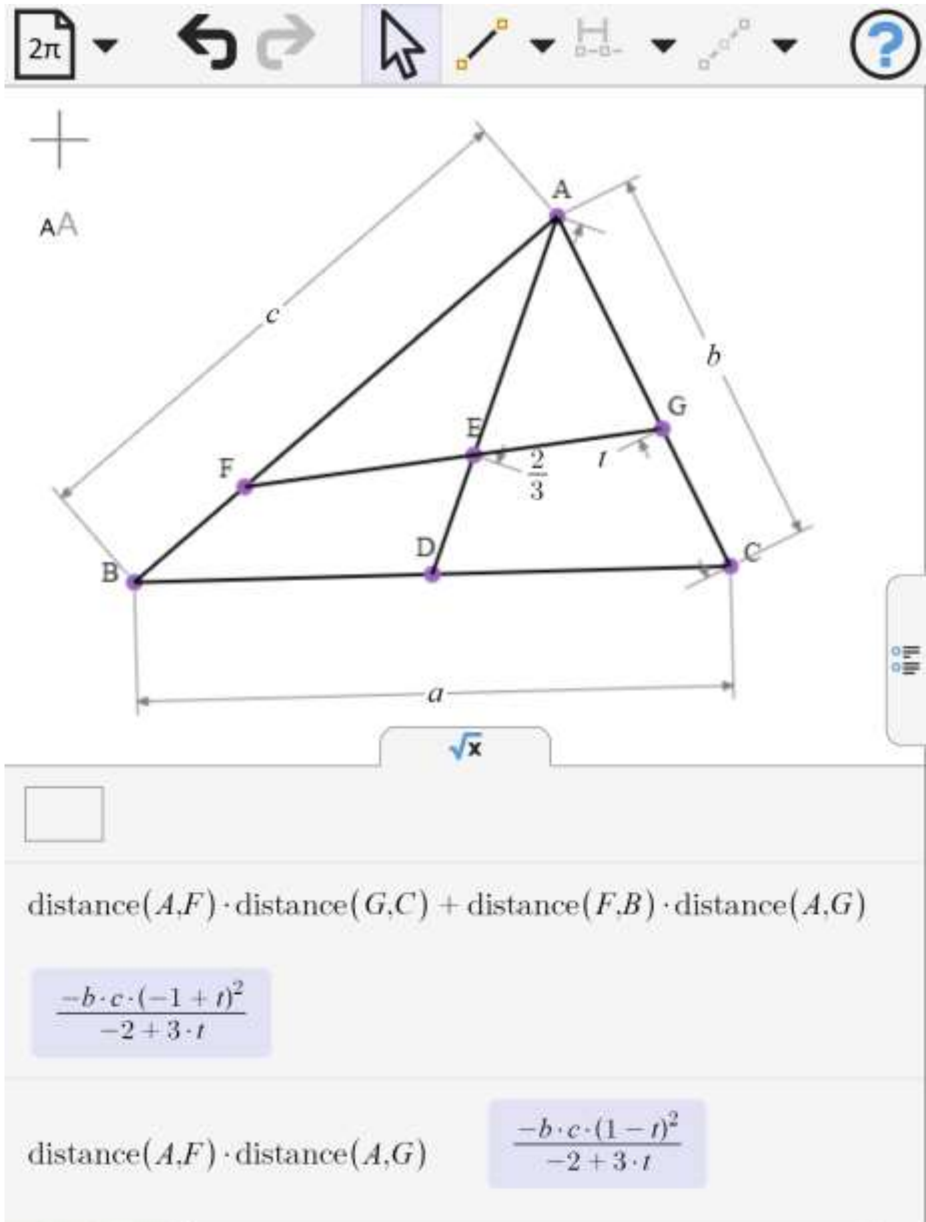
20. Distances from a median point to triangle sides

Show that the distances of a point on a median of a triangle from the sides including the median are inversely proportional to these sides.



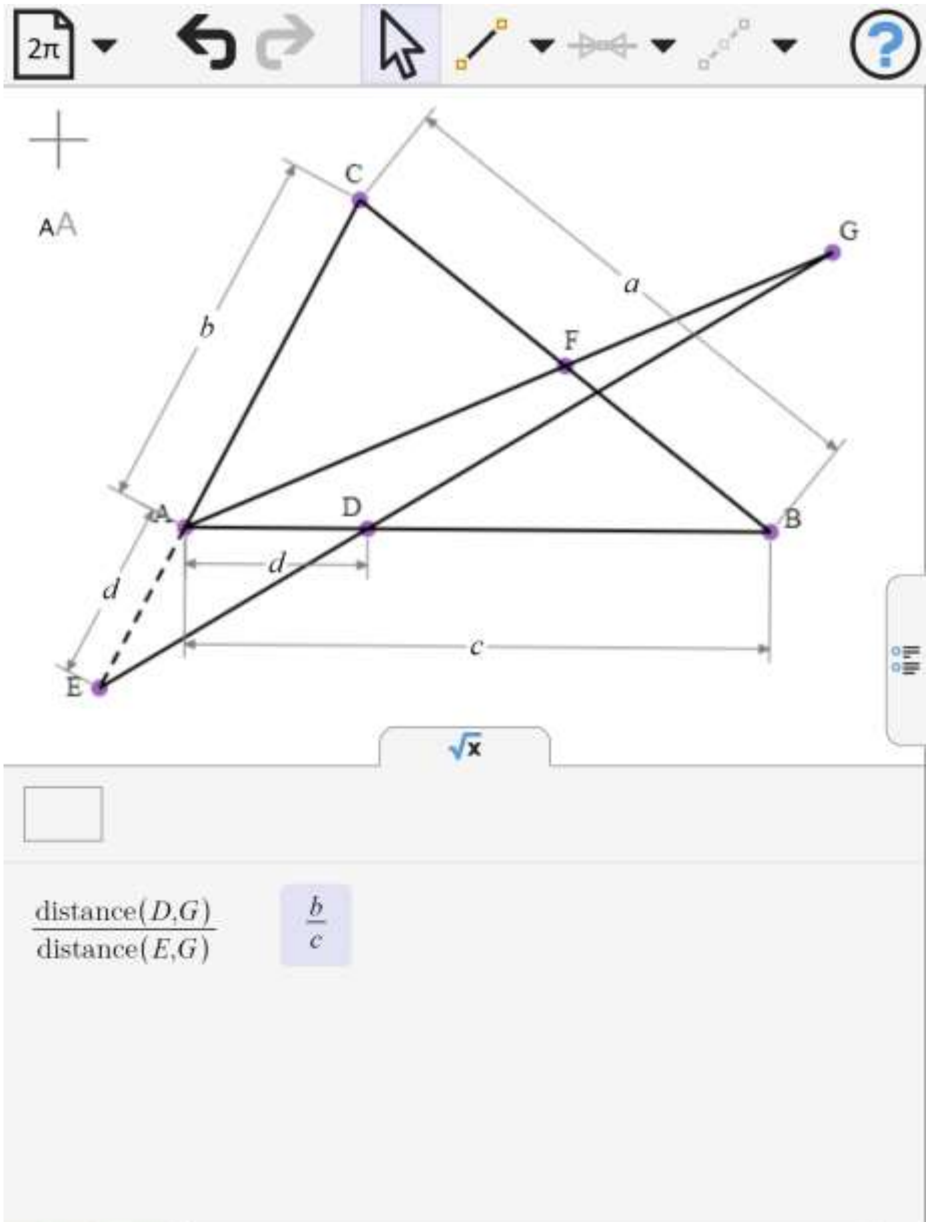
21. A line through the centroid

Show that, if a line through the centroid G of the triangle ABC meets AB in M and AC in N we have AN*MB+AM*NC=AM*AN



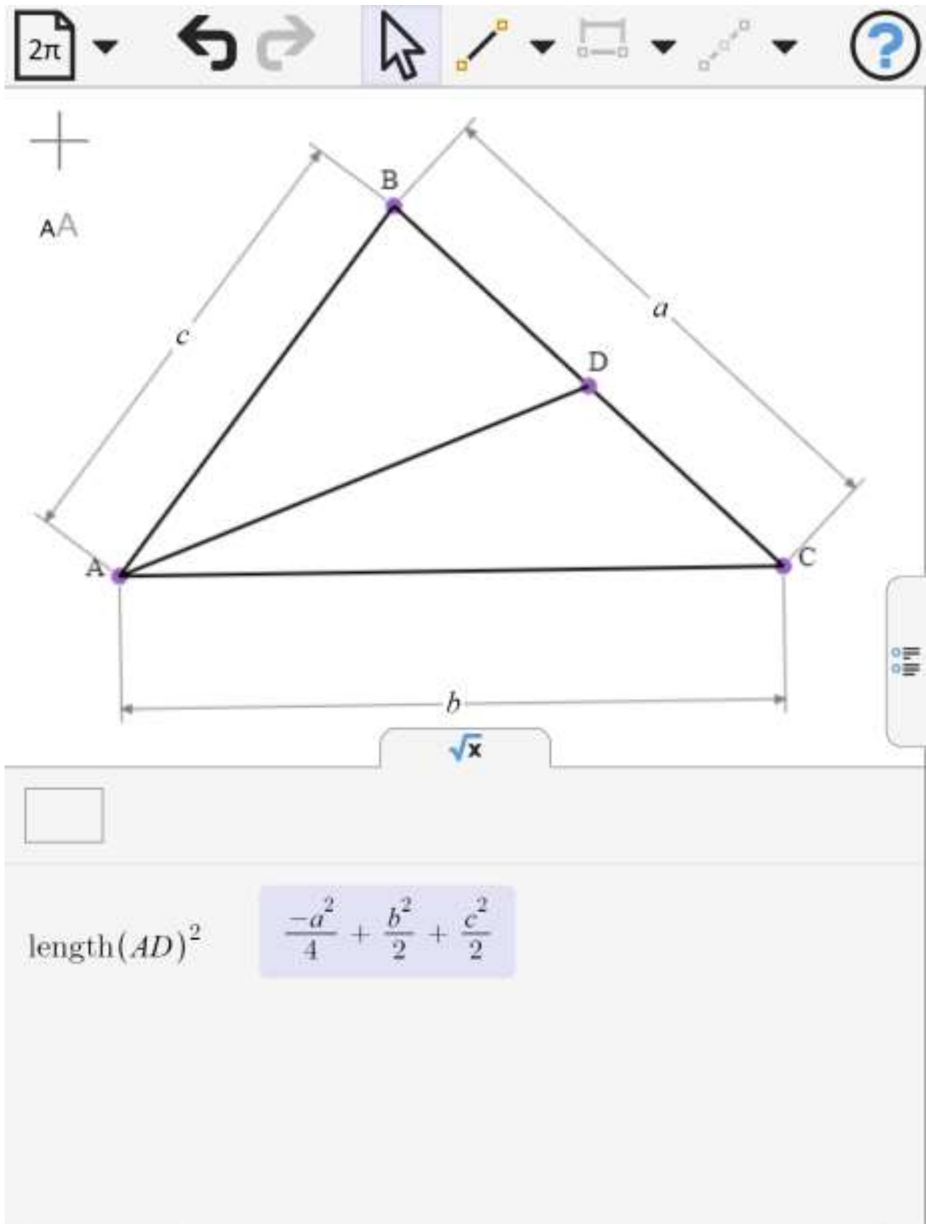
22. Median dividing a particular line

Two equal segments AD, AE are taken on the sides AB, AC of the triangle ABC. Show that the median issued from A divides DE in the ratio of the sides AC, AB



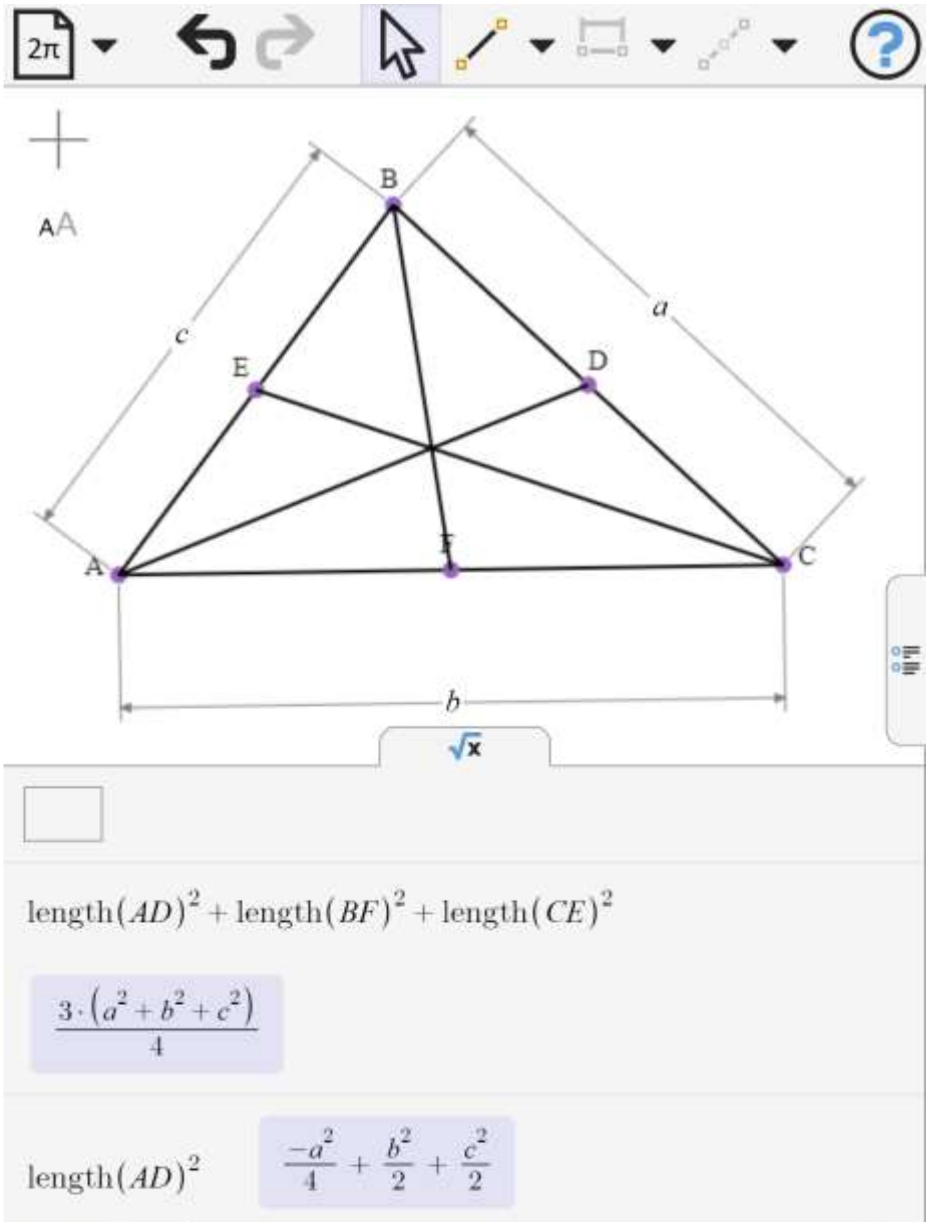
23. Median length

Compute the square of the lengths of the medians



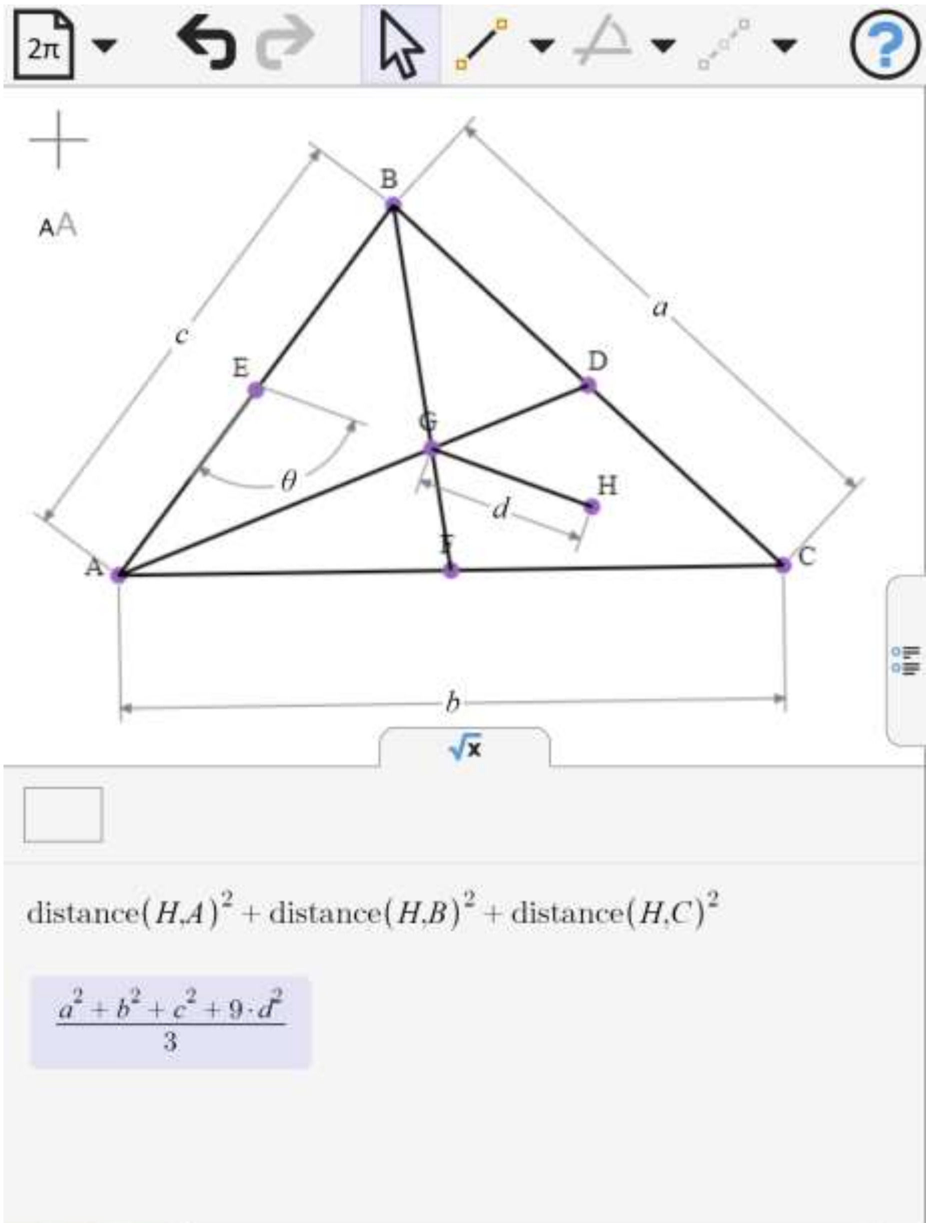
24. Sum of squares of the medians

The sum of the squares of the medians is equal to ¾ the sum of squares of the sides of the original triangle



25. Sum of squares of the distance of a point from the vertices in terms of its distance from the centroid

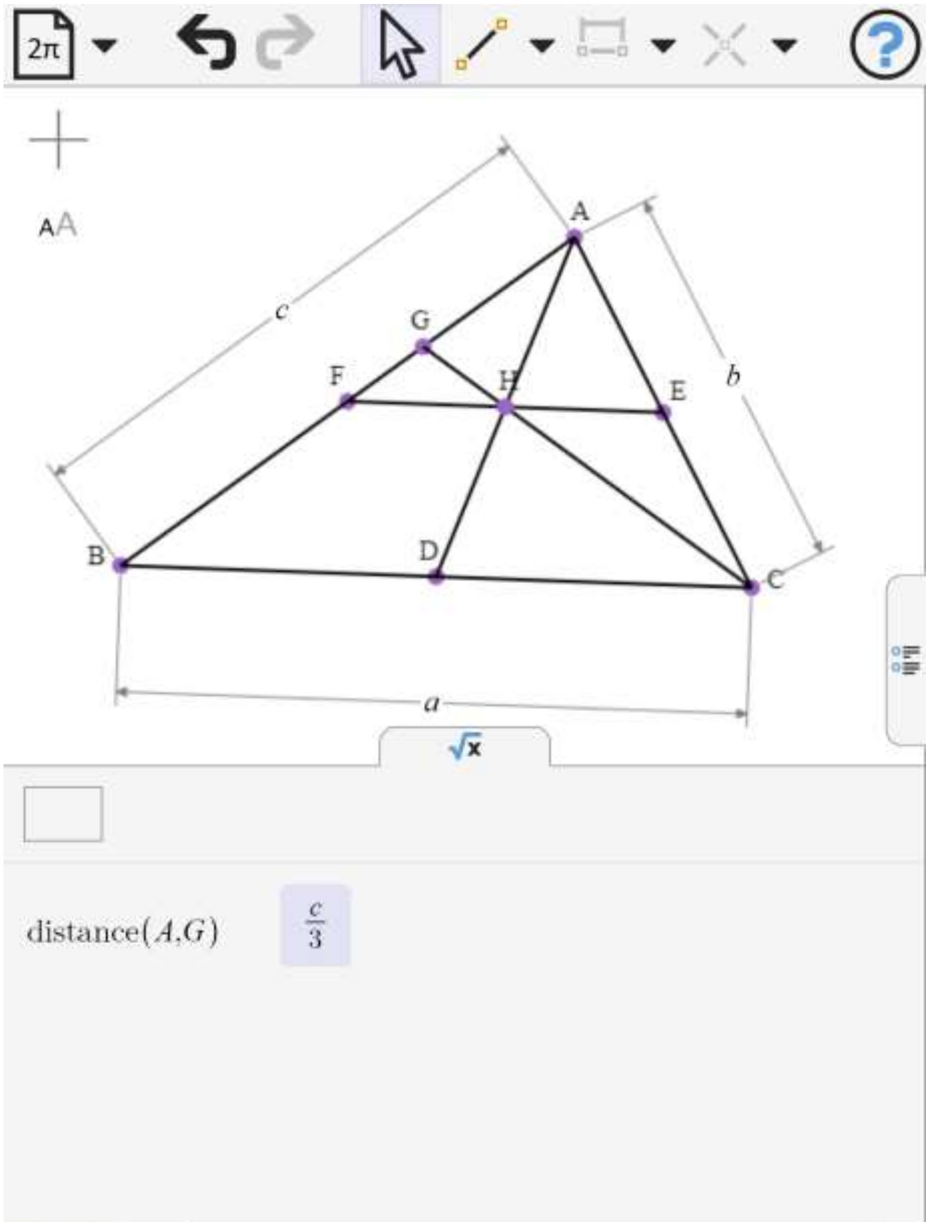
If two points are equidistant from the centroid of a triangle, the sums of the squares of their distances from the vertices of the triangle are equal.



The sum of squares is independent of θ, hence the result.

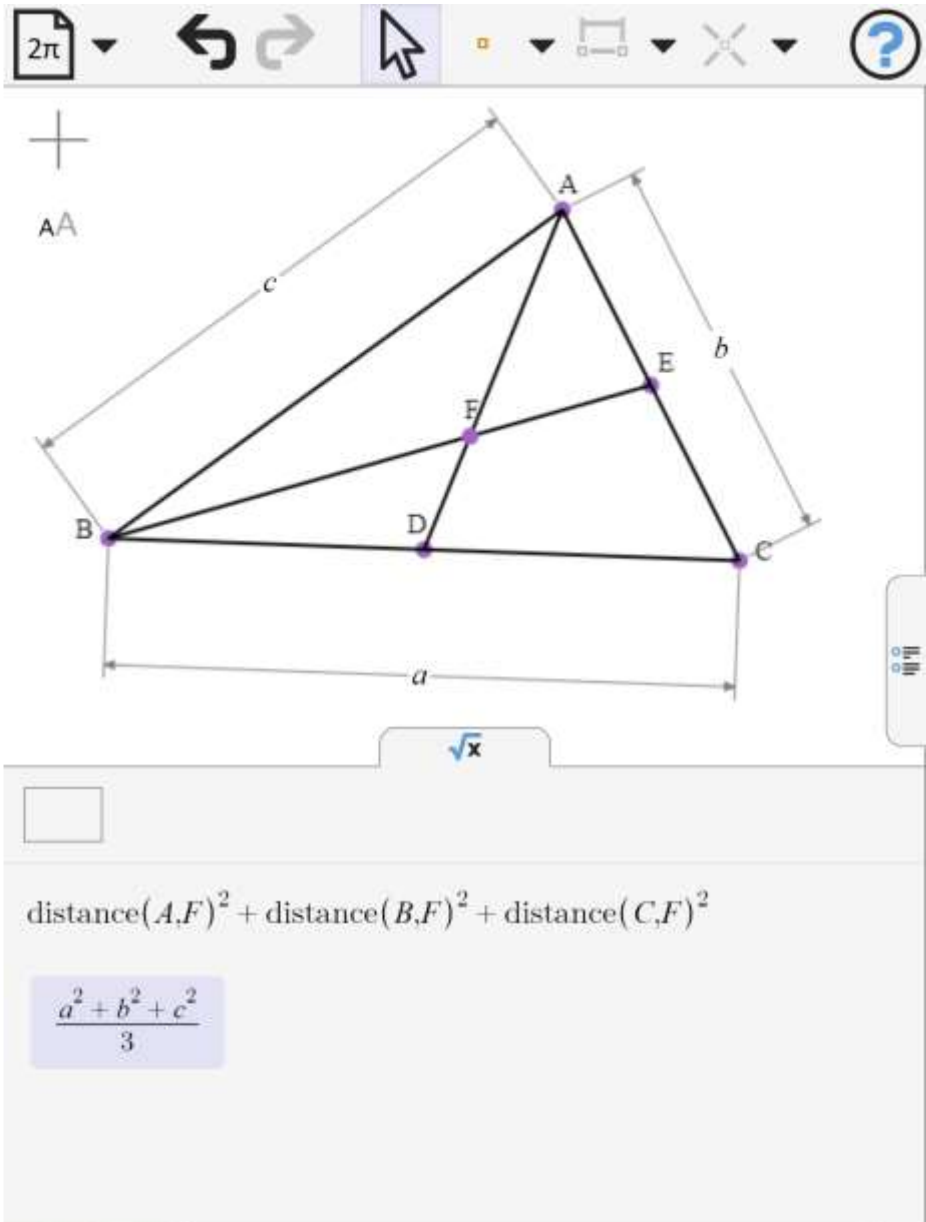
26. A median meets a side of the medial triangle

Let D,E,F be the midpoints of BC, CA, AB of triangle ABC. The median AD meets FE in H, and CH meets AB in G. Show that AB=3AG.



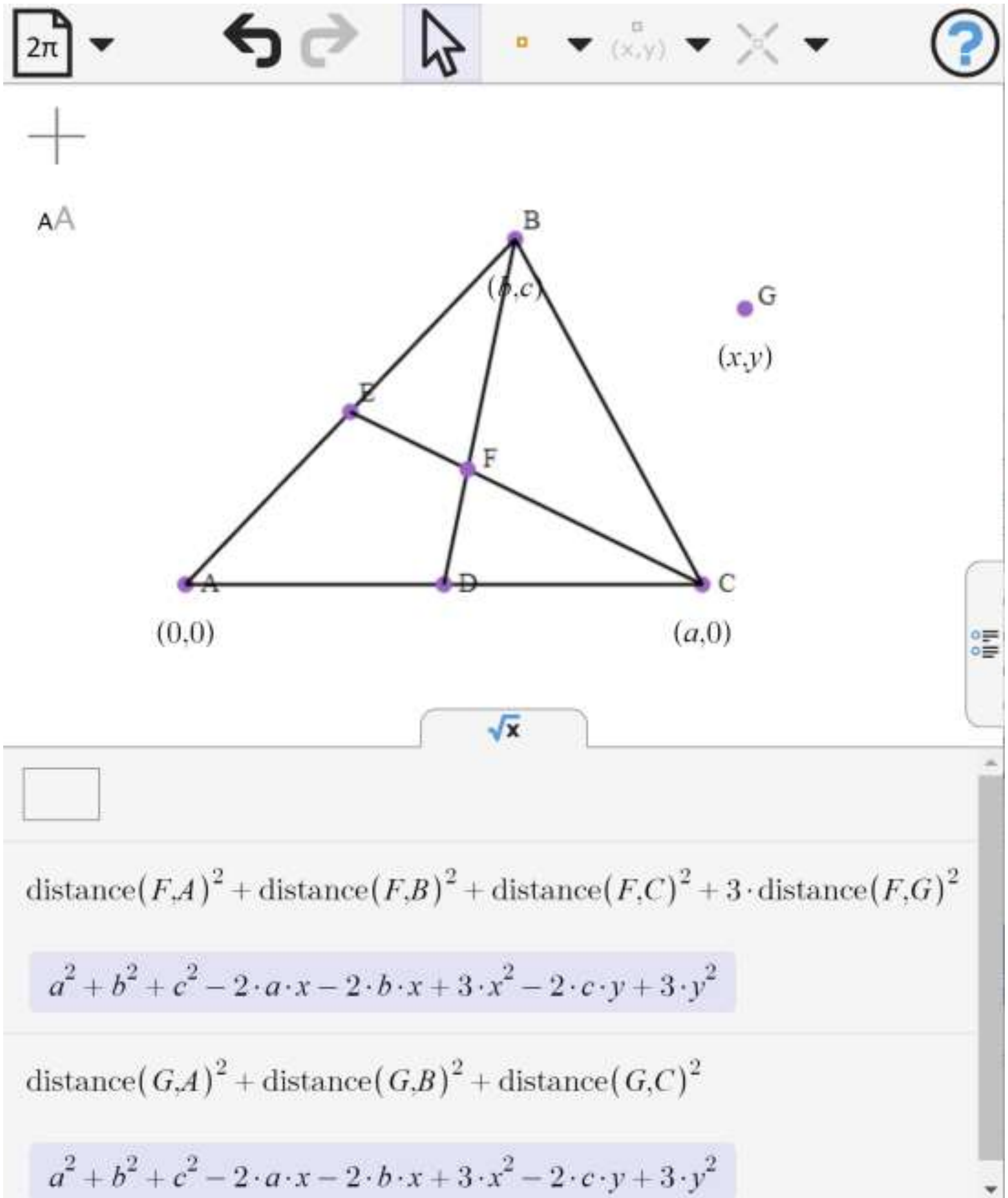
27. Sum of squares of the distances from the vertices to the centroid

The sum of squares of the distances of the centroid of a triangle from the vertices is equal to one third the sum of squares of the sides.



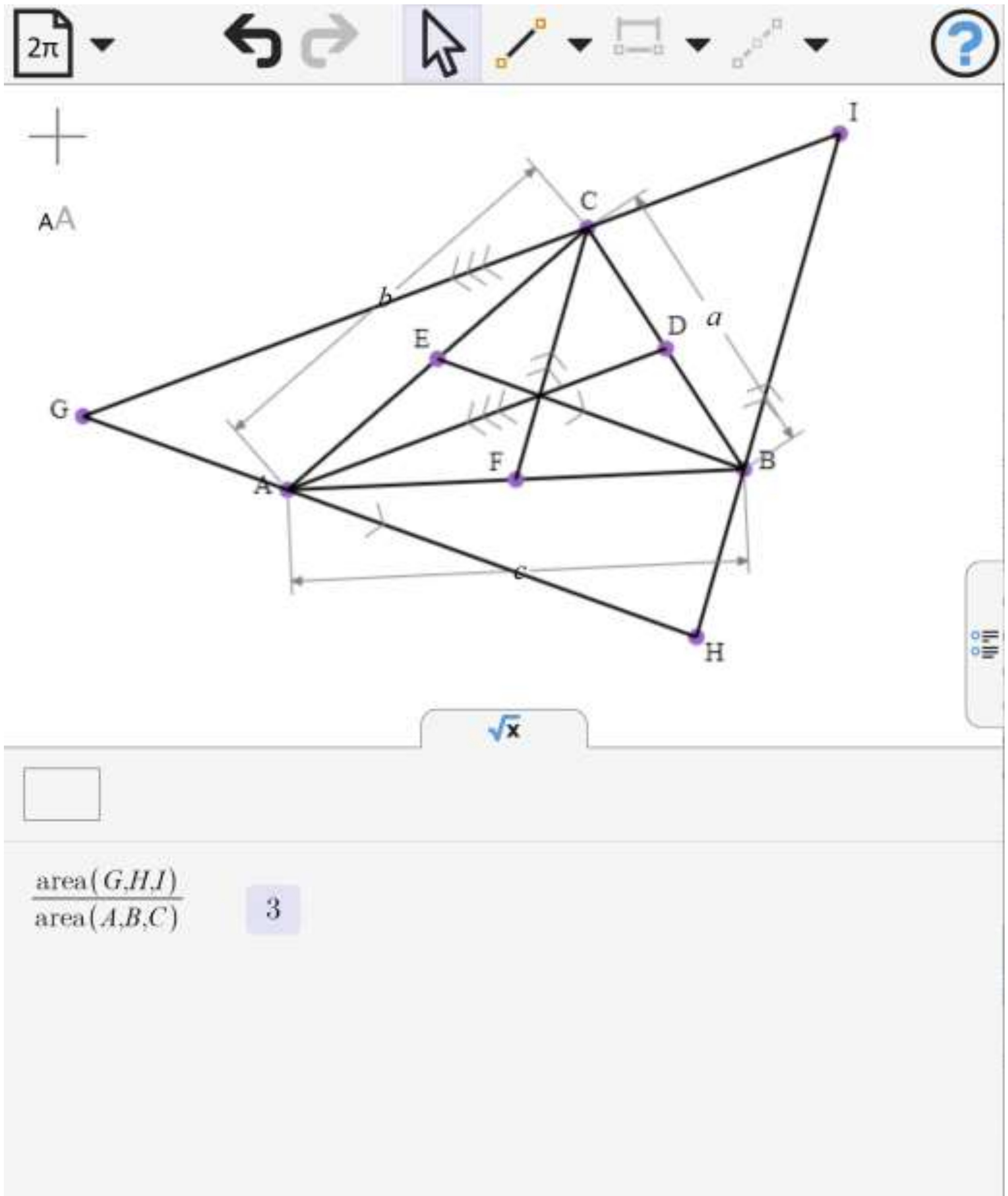
28. Sum of squares of the distances of a point from the vertices of a triangle

If G is any point in the plane of the triangle ABC and F is the centroid of ABC, we have $GA^2+GB^2+GC^2=FA^2+FB^2+FC^2+3FG^2$



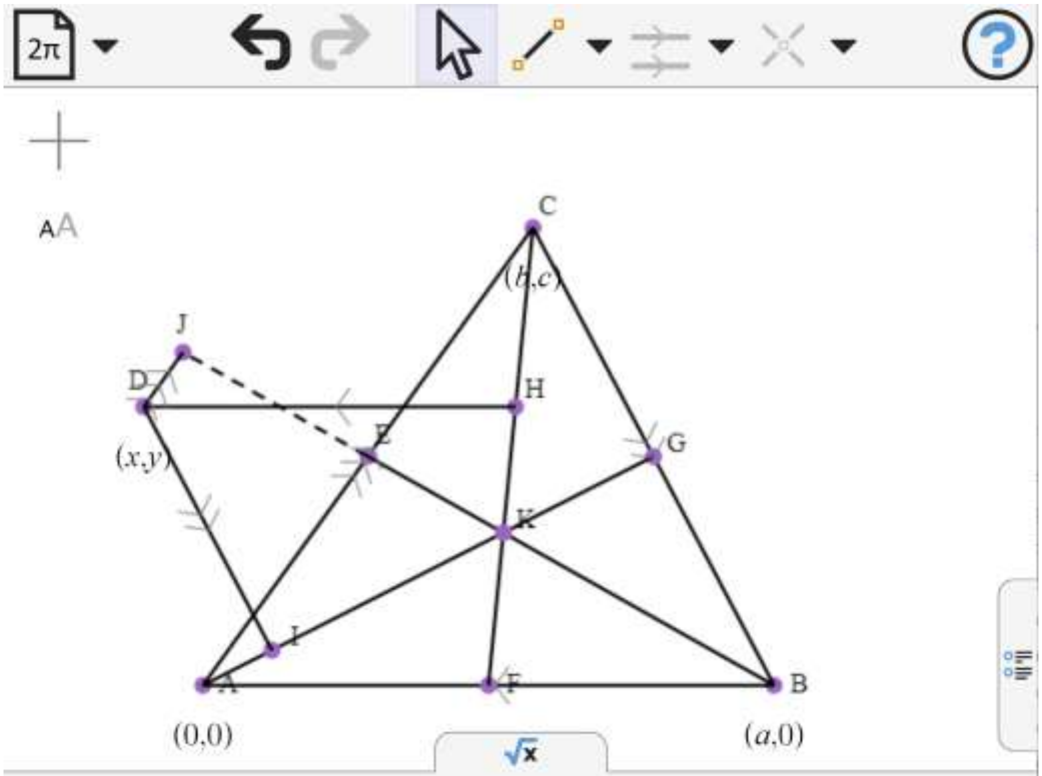
29. Exscribed triangle parallel to the medians

Show that the parallels through the vertices A, B, C of the triangle ABC to the medians of this triangle issued from the vertices B, C, A respectively form a triangle whose area is three times the area of the original triangle



30. Parallels to triangle sides through a point

The parallels to the sides of a triangle ABC through the same point D meet the respective medians in the points H, I, J. Prove that we have KJ/KA = KJ/KB-KH/KC

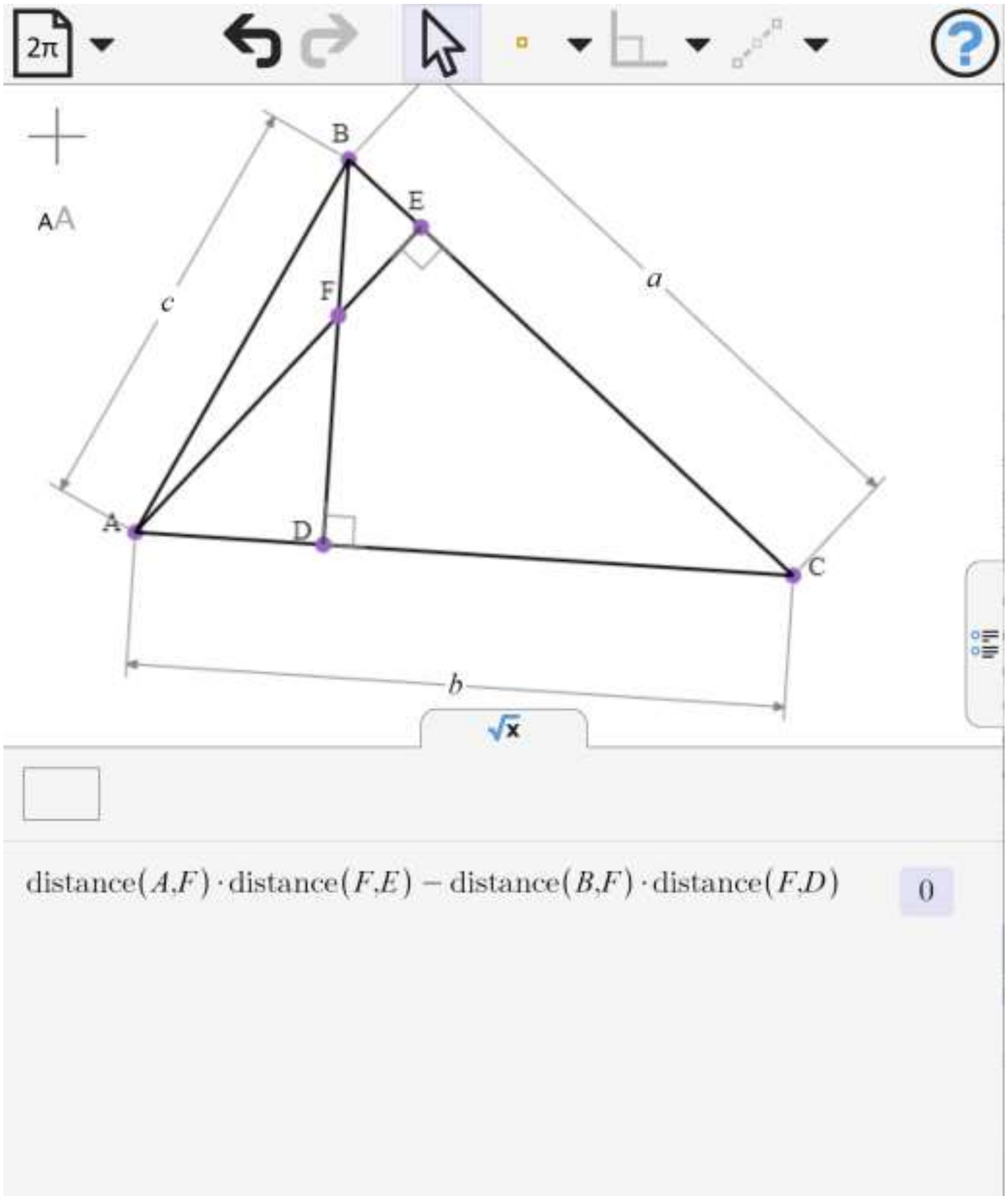


$$\frac{\text{distance}(K,J)}{\text{distance}(K,B)} - \frac{\text{distance}(K,H)}{\text{distance}(K,C)} \quad 1 - \frac{3 \cdot x}{2 \cdot a} - \frac{3 \cdot y}{2 \cdot c} + \frac{3 \cdot b \cdot y}{2 \cdot a \cdot c}$$

$$\frac{\text{distance}(K,I)}{\text{distance}(K,A)} \quad 1 - \frac{3 \cdot x}{2 \cdot a} - \frac{3 \cdot y}{2 \cdot c} + \frac{3 \cdot b \cdot y}{2 \cdot a \cdot c}$$

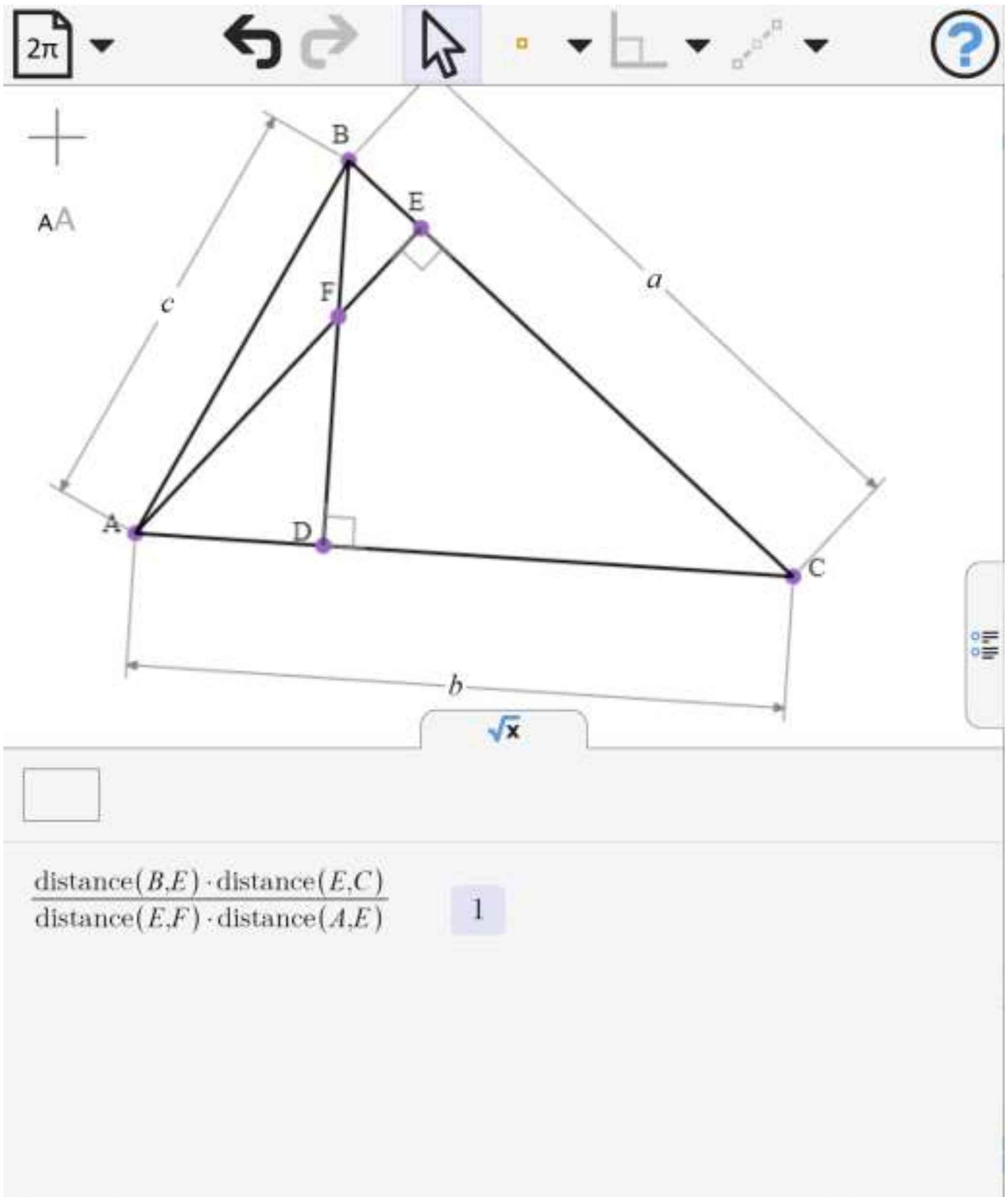
31. Altitude segments divided by orthocenter

In a given triangle, the three products of the segments into which the orthocenter divides the altitudes are equal.



32. Side segments divided by foot of altitude

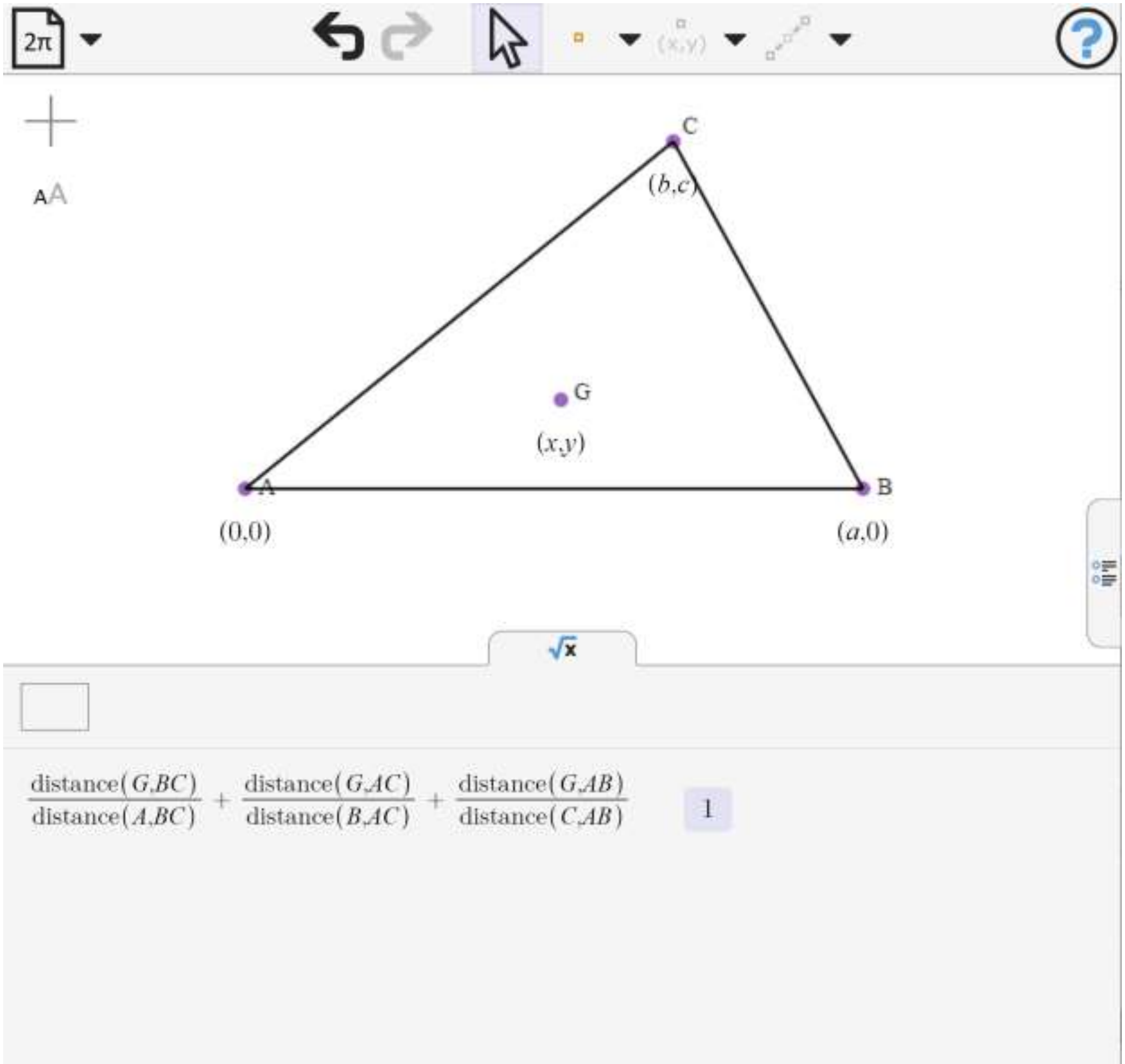
The product of the segments into which the side of a triangle is divided by the foot of the altitude is equal to this altitude multiplied by the distance of the side from the orthocenter.



33. Distances of a point from the sides of a triangle related to altitude lengths

If p,q, r are the distances of a point inside a triangle ABC from the sides of the triangle, show that

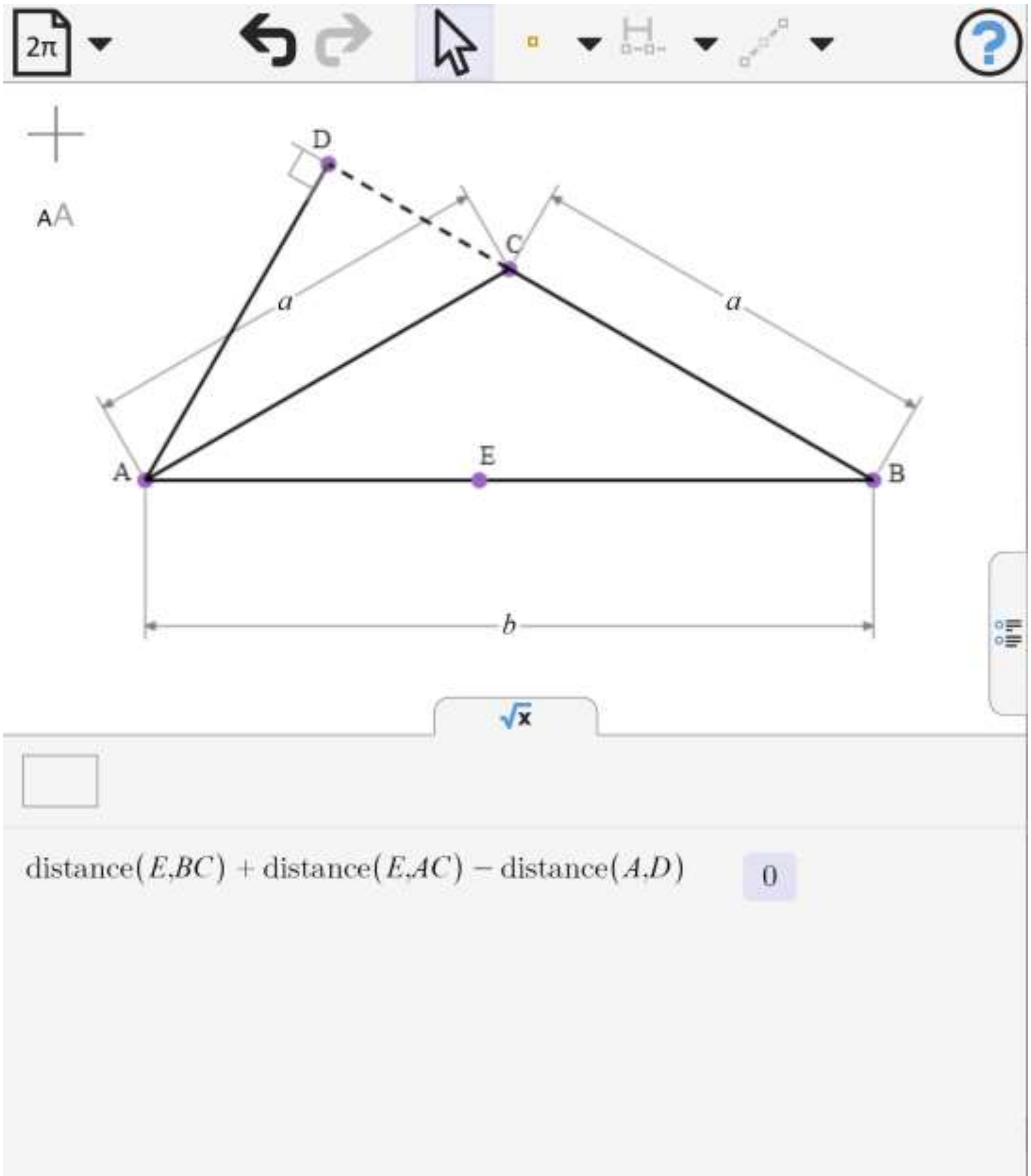
$\frac{p}{h_a}+\frac{q}{h_b}+\frac{r}{h_c}$=1.



Specifying the triangle by coordinates allows us to give the location of O by coordinate.

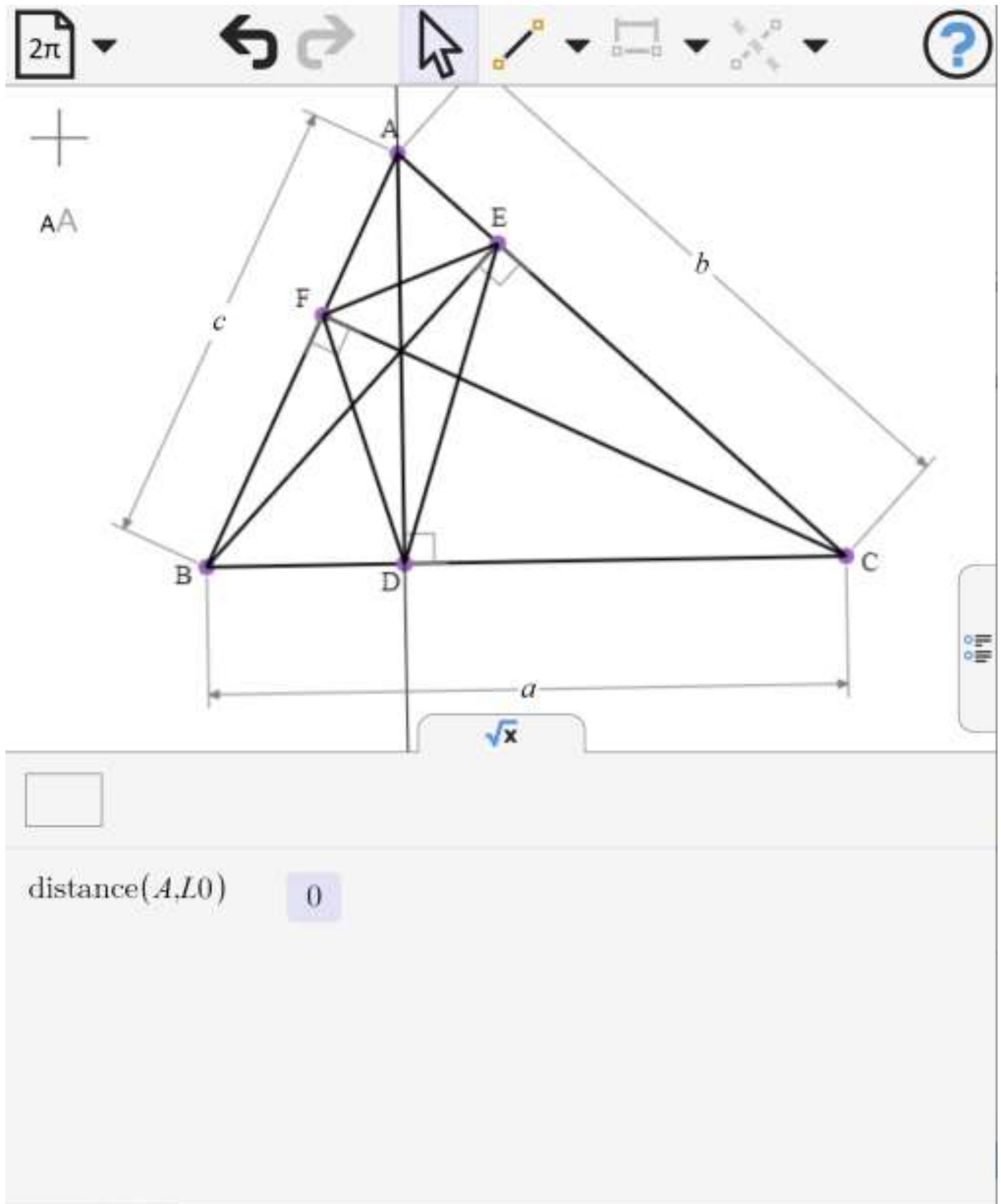
34. distances of a point on the base of an equilateral triangle to the other sides

Show that the sum of the distances of a point on the base of an isosceles triangle to its two sides is equal to the altitude on that side.



35. Altitudes are angle bisectors of the orthic triangle

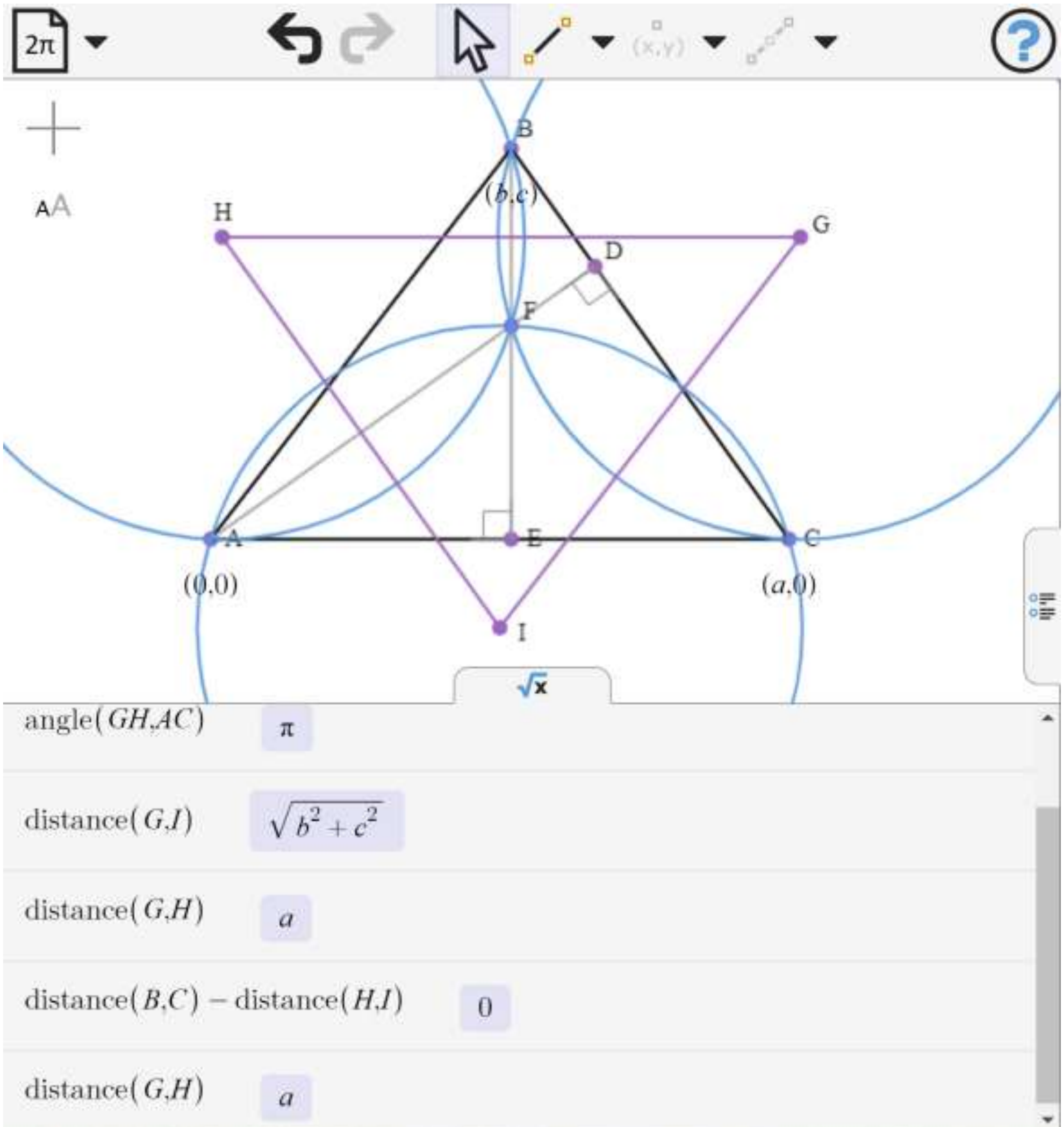
The altitudes of a triangle bisect the internal angles of its orthic triangle.



We can show that point A lies on the bisector of angle FDE, hence the altitude is the bisector.

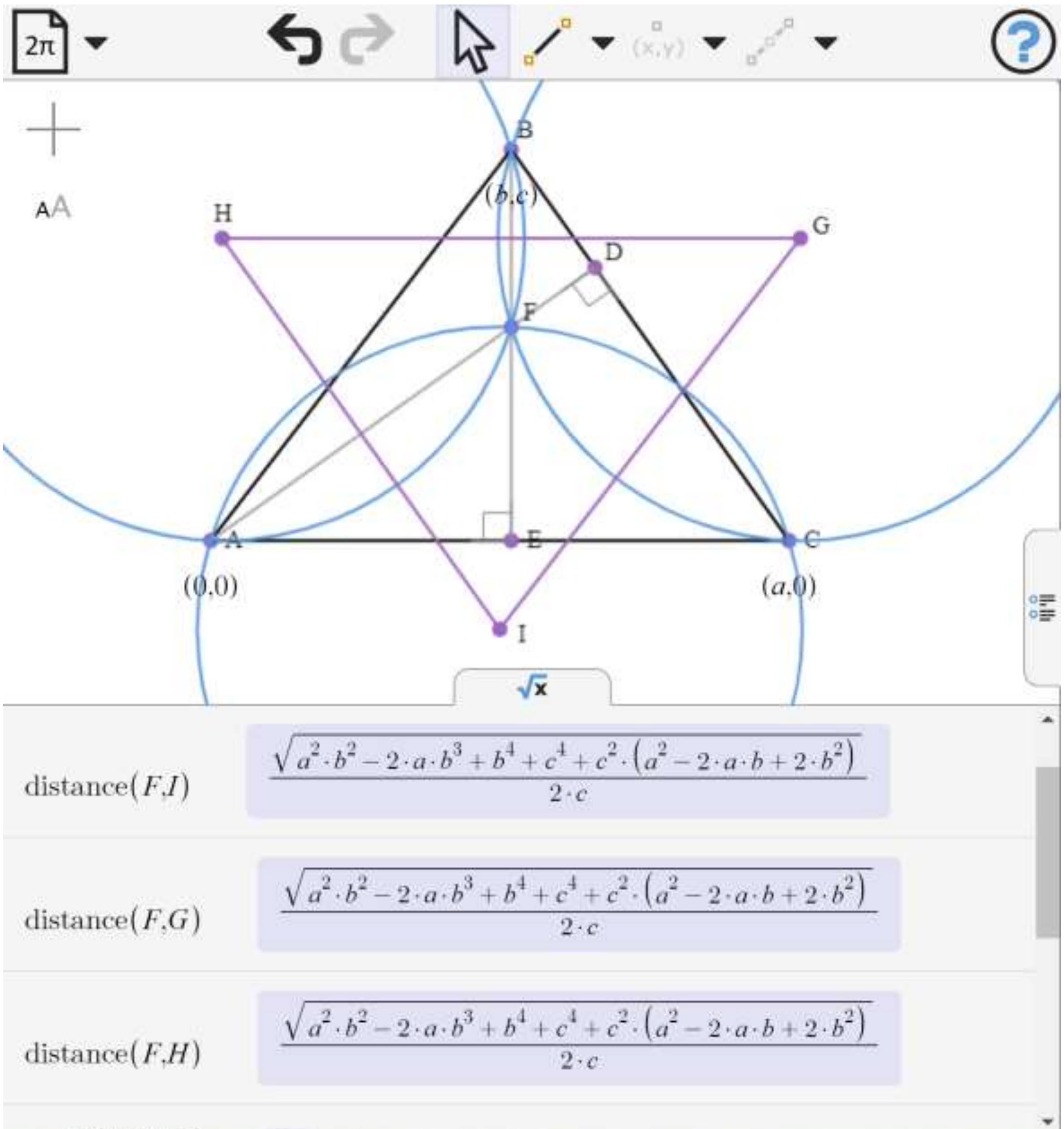
36. Circumcircles of two vertices and the orthocenter

Let F be the orthocenter of triangle ABC. Then the circumcenters of the four triangles ABF, ACF, FBC form a triangle congruent to ABC. The sides are parallel.



37. More on the above

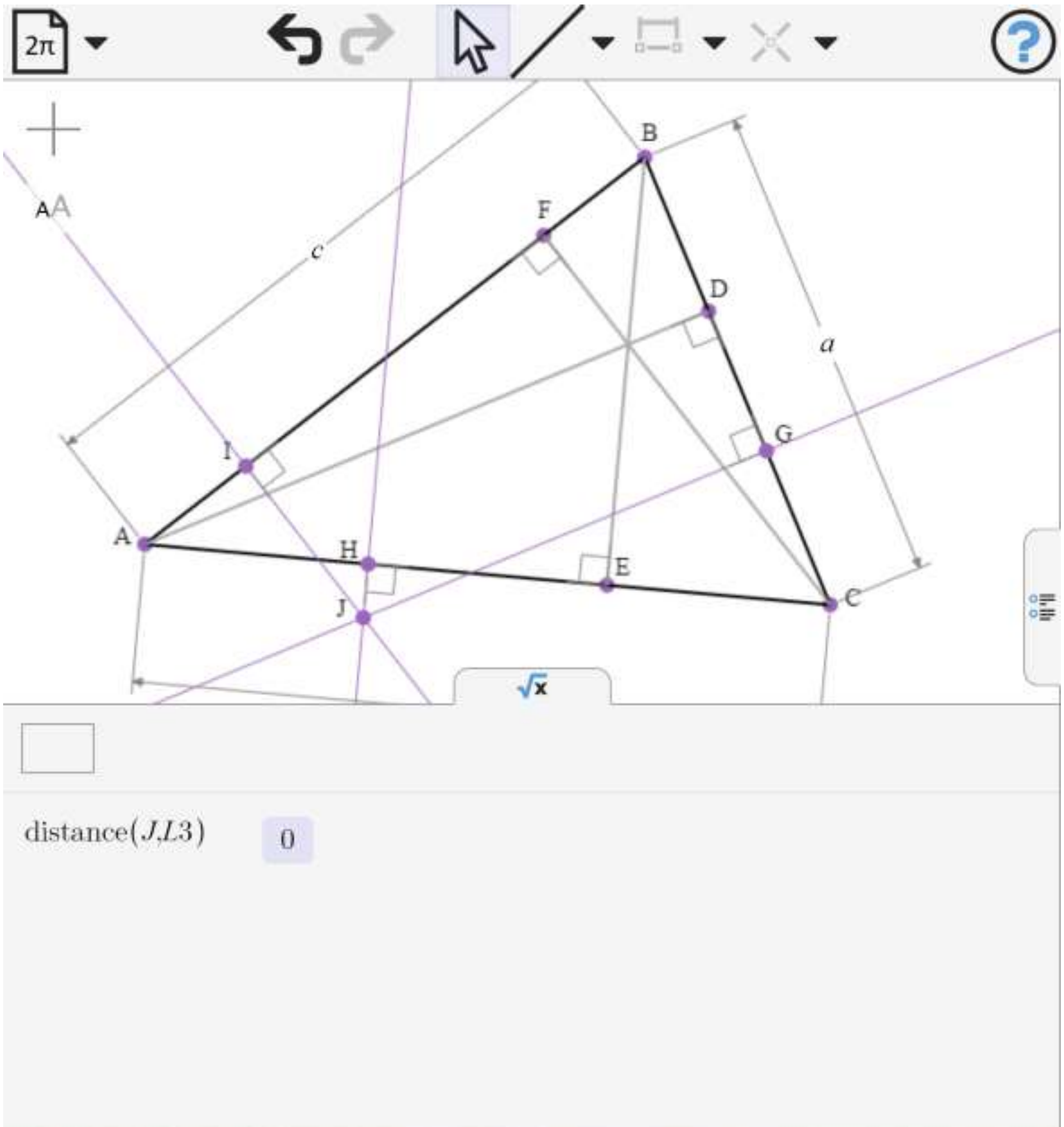
Continuing from Example 6.70, show that F is the circumcenter of GHI.



By inspection, the three lengths HI, HJ, HK are the same.

38. Perpendiculars at isotomic points to altitude feet

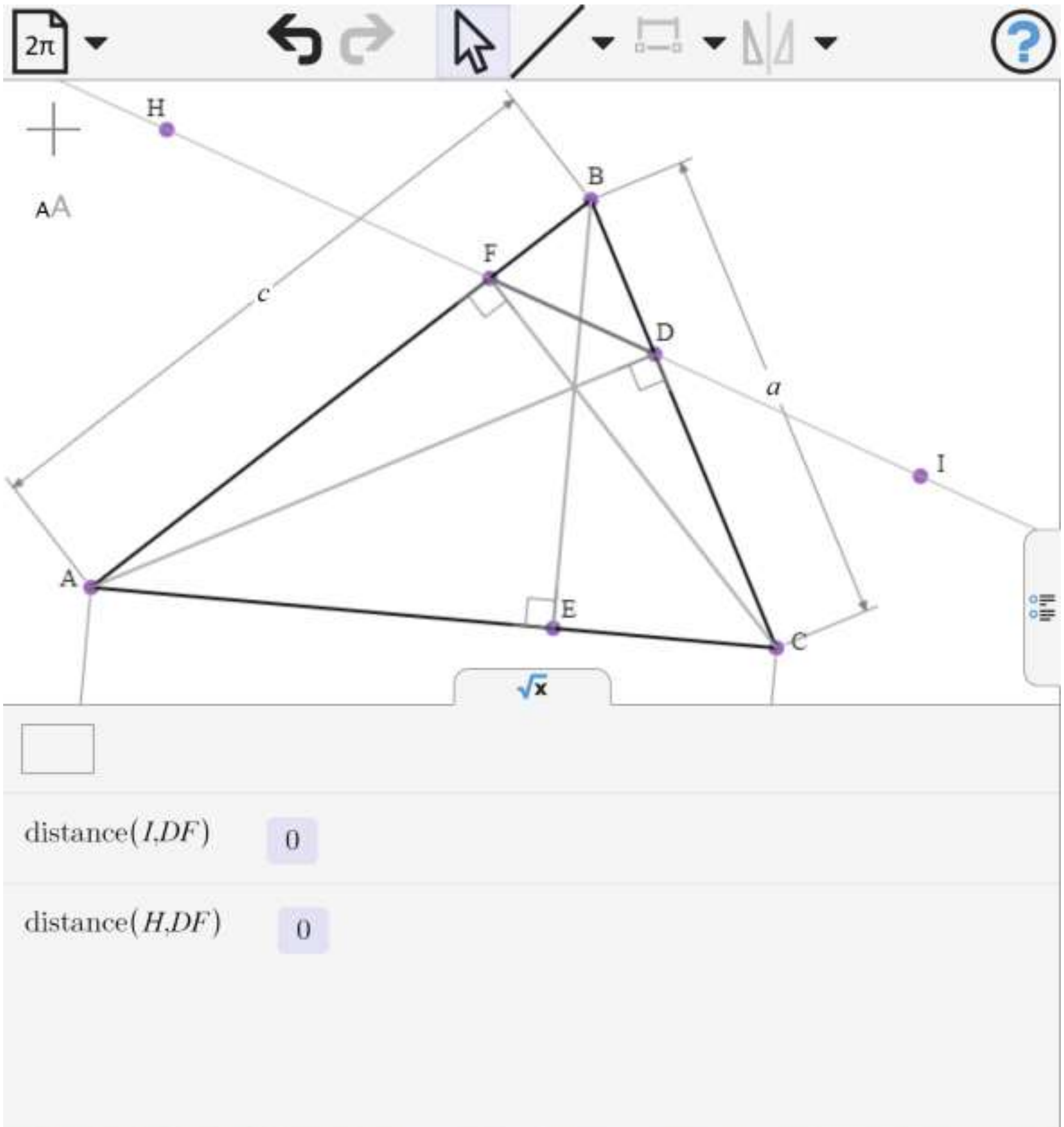
Show that the three perpendiculars to the sides of a triangle at the points isotomic to the foot of the respective altitudes are concurrent.



Note: Isotomic points are constructed by reflection in the perpendicular bisector. For clarity, these lines, though present, have been hidden in the diagram.

39. Reflections of altitude feet in triangle sides

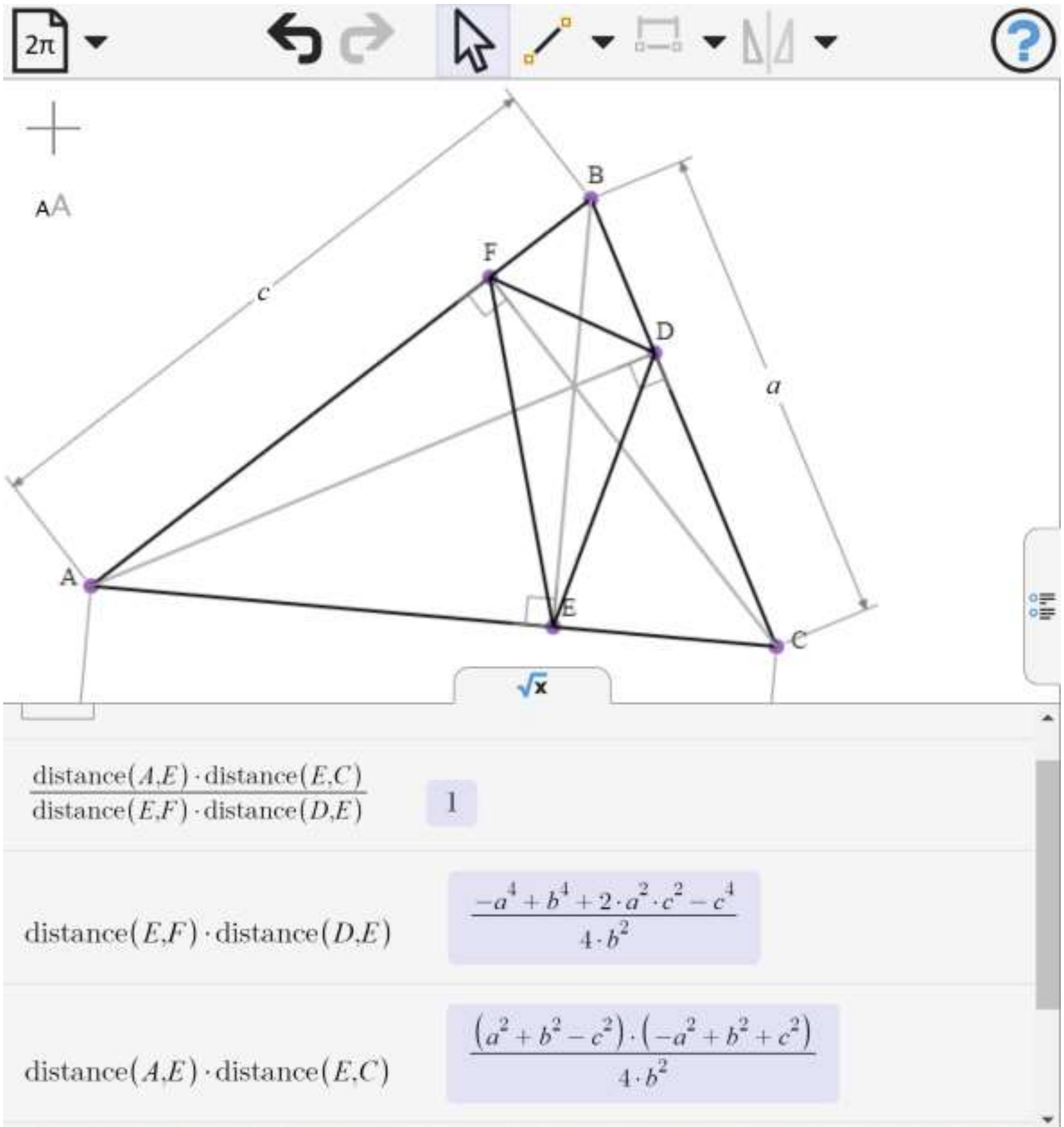
Show that the symmetries of the foot of the altitude to the base of the triangle with respect to the other two sides lie on the side of the orthic triangle relative to the base.



H,I are the images of E under reflection in AB and BC. We show their distance from the line DF is zero.

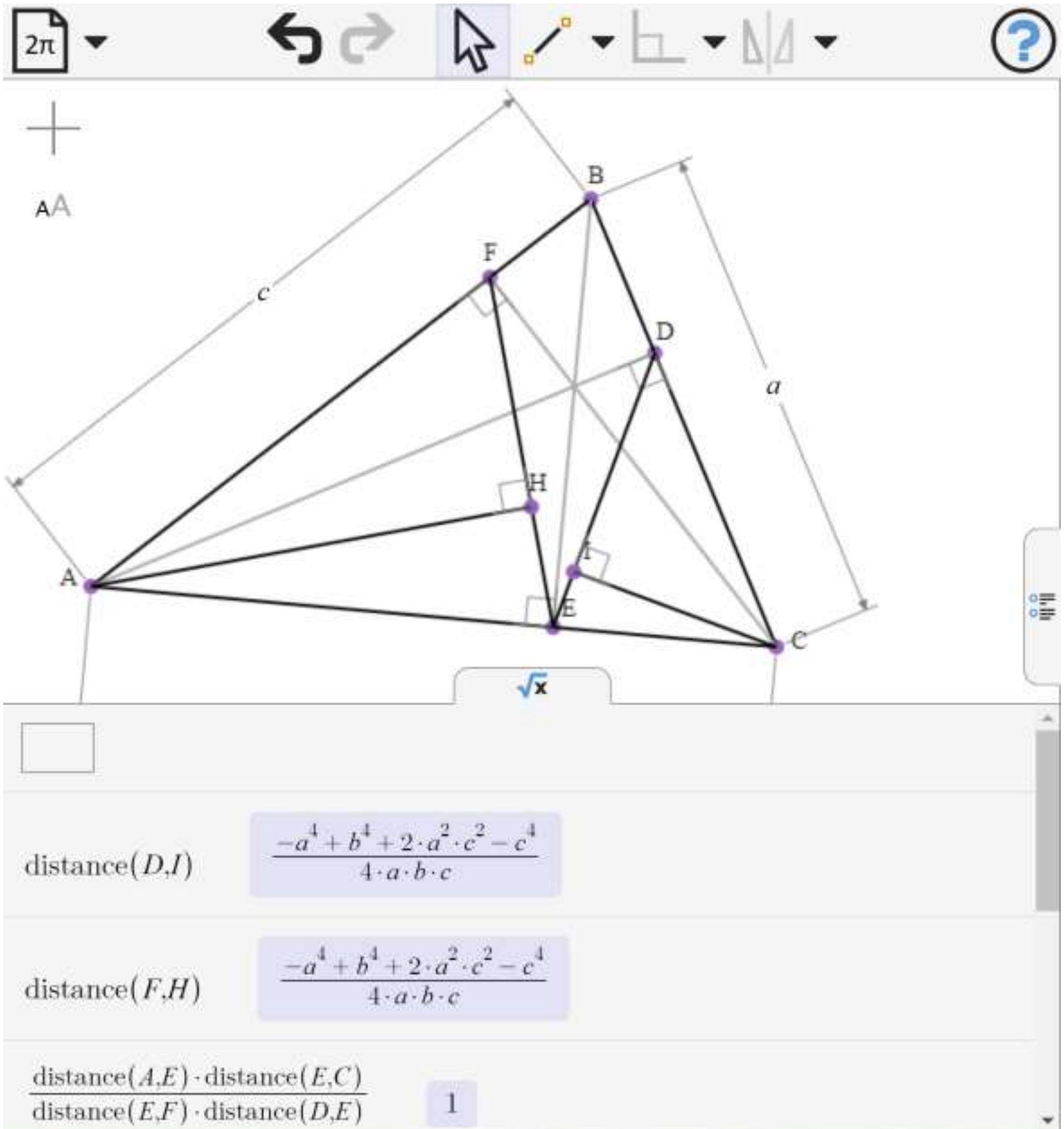
40. Products of segments involving the orthic triangle

Show that the product of the segments into which a side of a triangle is divided by the corresponding vertex of the orthic triangle is equal to the product of the sides of the orthic triangle passing through the vertex considered.



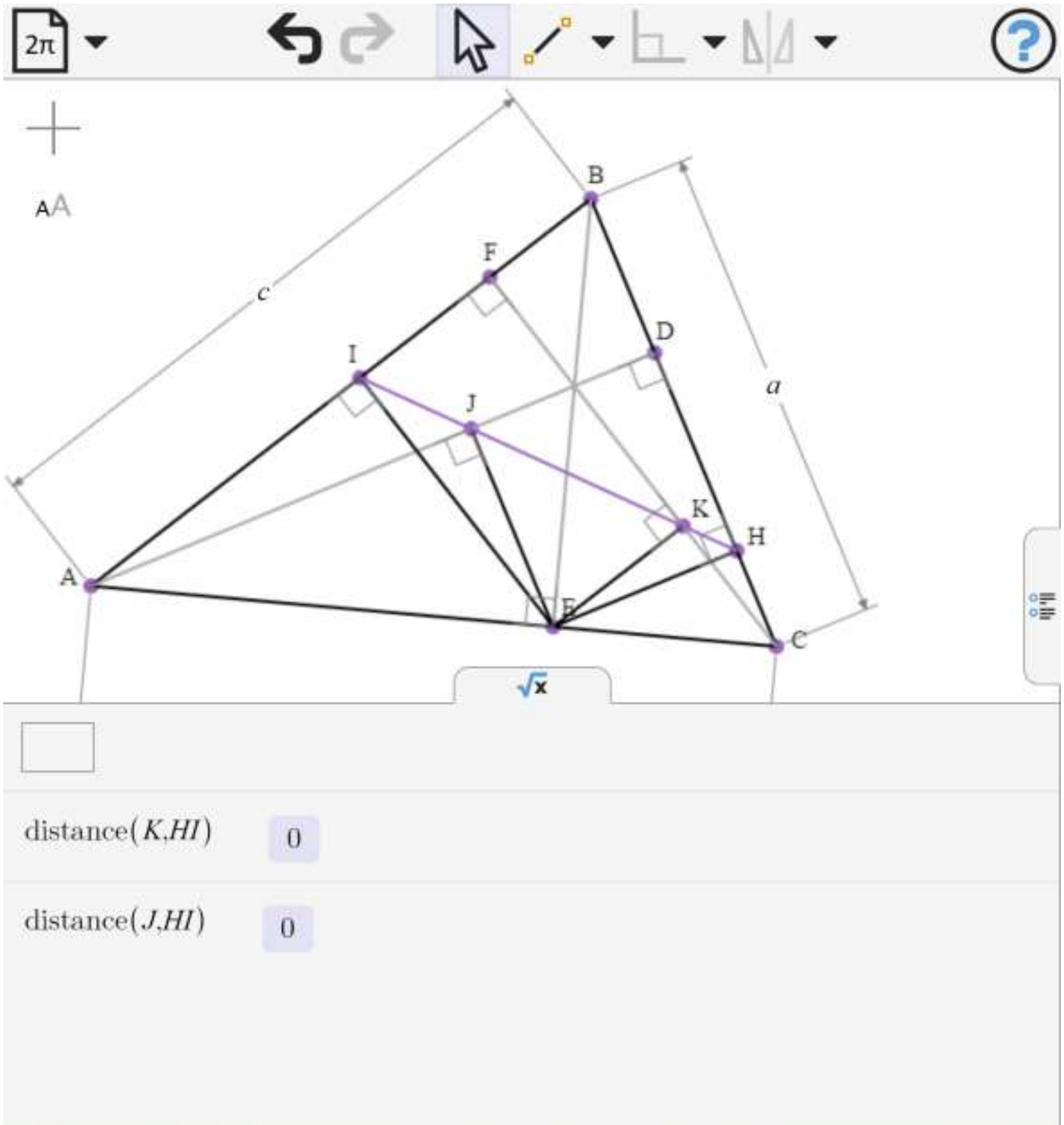
41. Perpendiculars to the sides of the orthic triangle

If P, Q are the feet of the perpendiculars from the vertices B, C of the triangle ABC on the sides DF, DE respectively, of the orthic triangle DEF, show that EQ=FP.



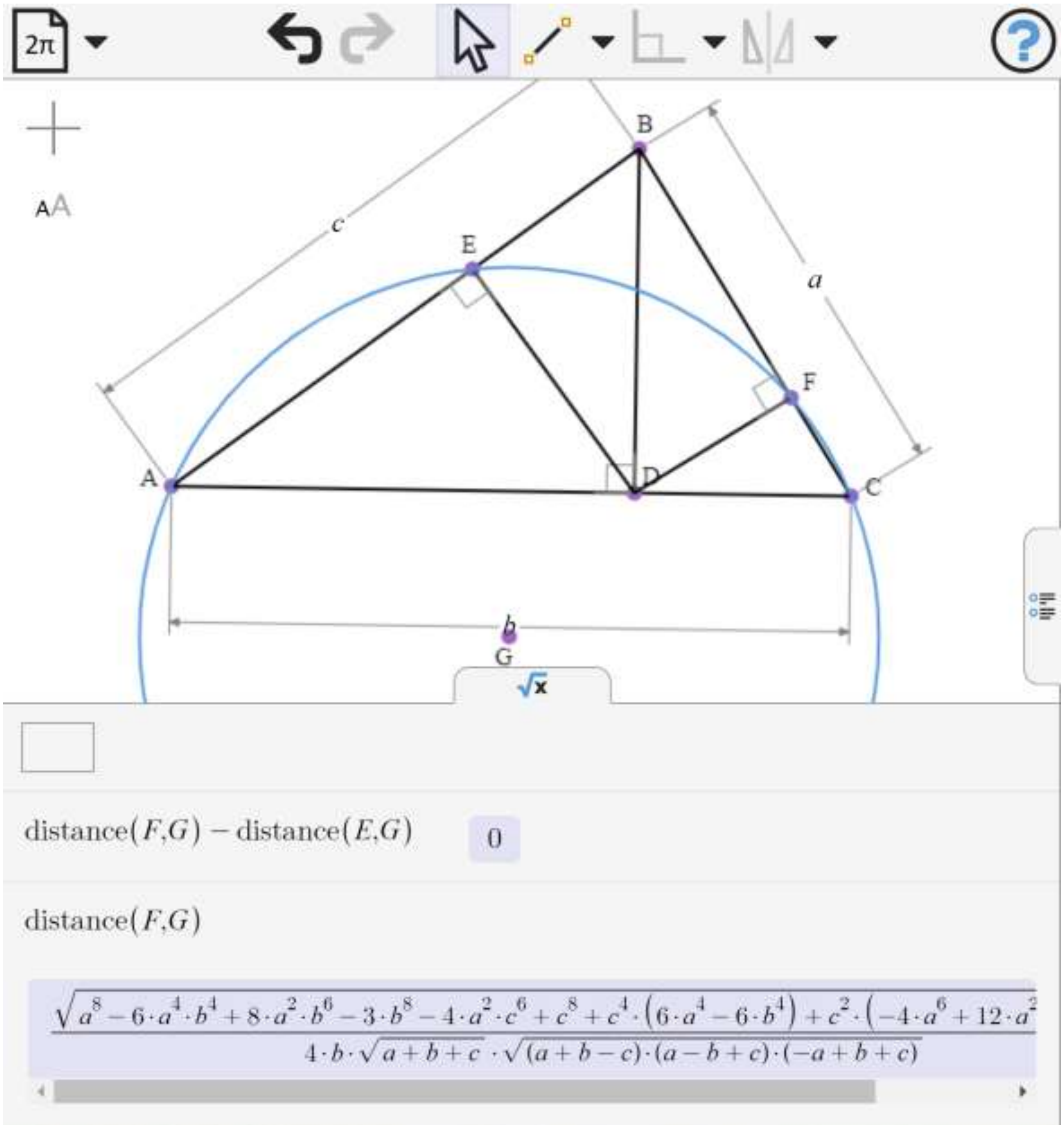
42. Projections of altitude feet onto sides and altitudes are collinear

The four projections of the foot of the altitude on a side of a triangle upon the other two sides and the other two altitudes are collinear.



43. Projections of altitude foot onto triangle sides cyclic with ends of base

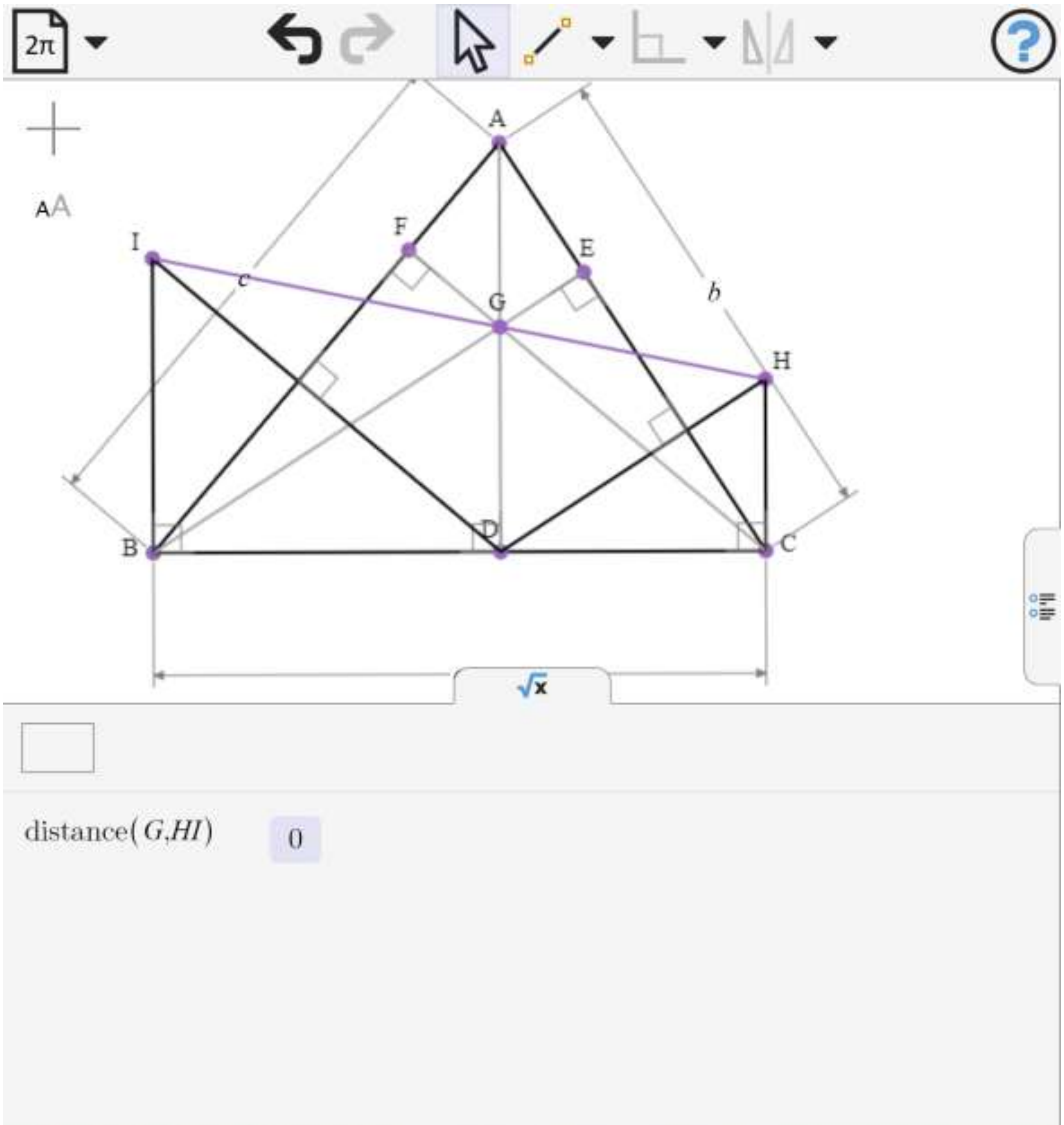
DE, DF are perpendiculars from the foot D of the altitude BD of the triangle ABC on the sides AB, BC. Prove that the points A, C, E, F are cyclic.



We put a circle through A,E,C then show that F is the same distance from the center of this circle as E.

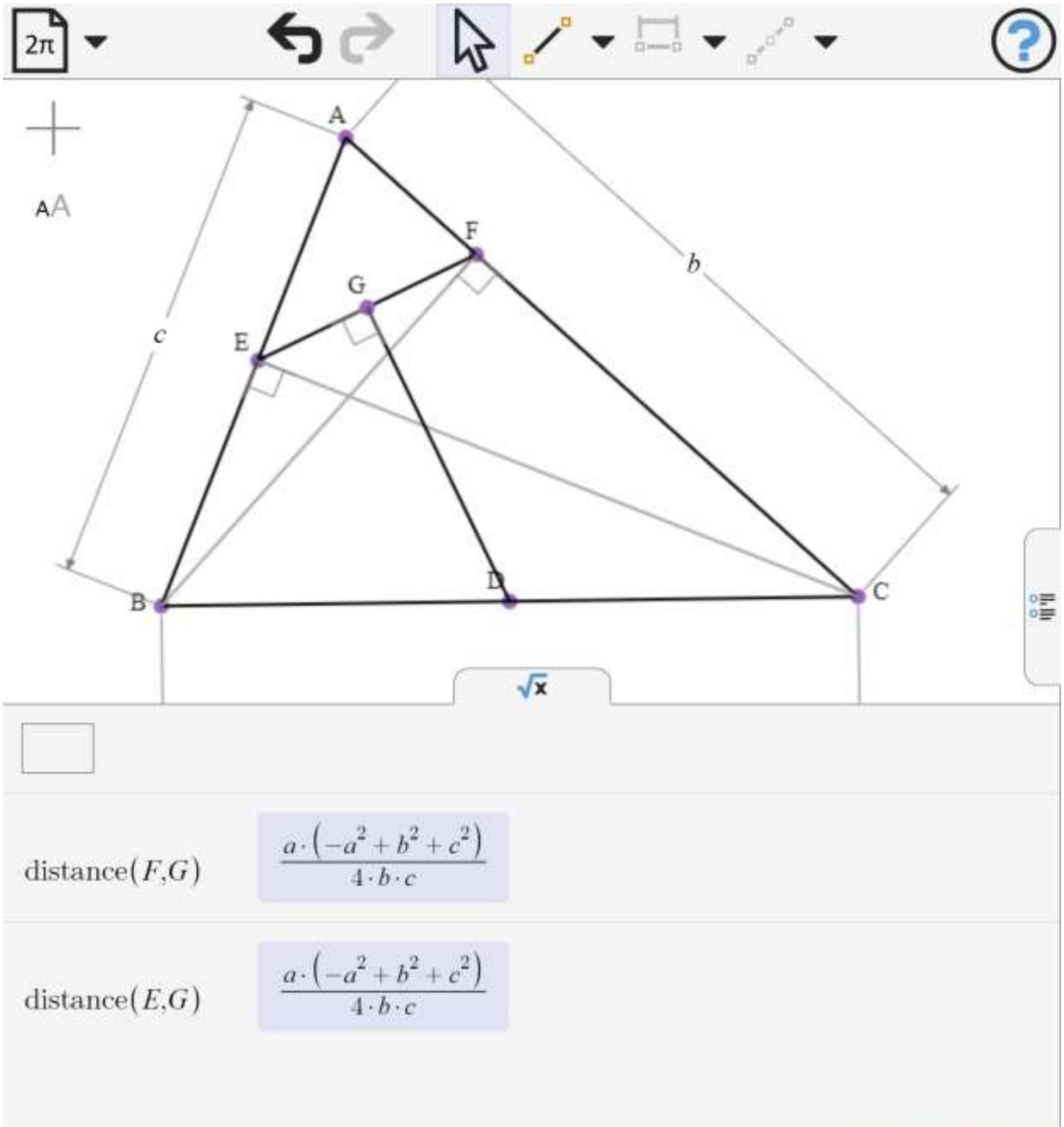
44. Perpendiculars to the other sides from an altitude base meet perpendiculars to the base from its end points

The perpendiculars DP, DQ dropped from the foot D of the altitude AD f the triangle ABC upon the sides AB, AC meet the perpendiculars BP, CQ erected to BC at B, C in the points P, Q respectively. Prove that the line PQ passes through the orthocenter H of ABC



45. Perpendicular from side midpoint bisects orthic triangle side

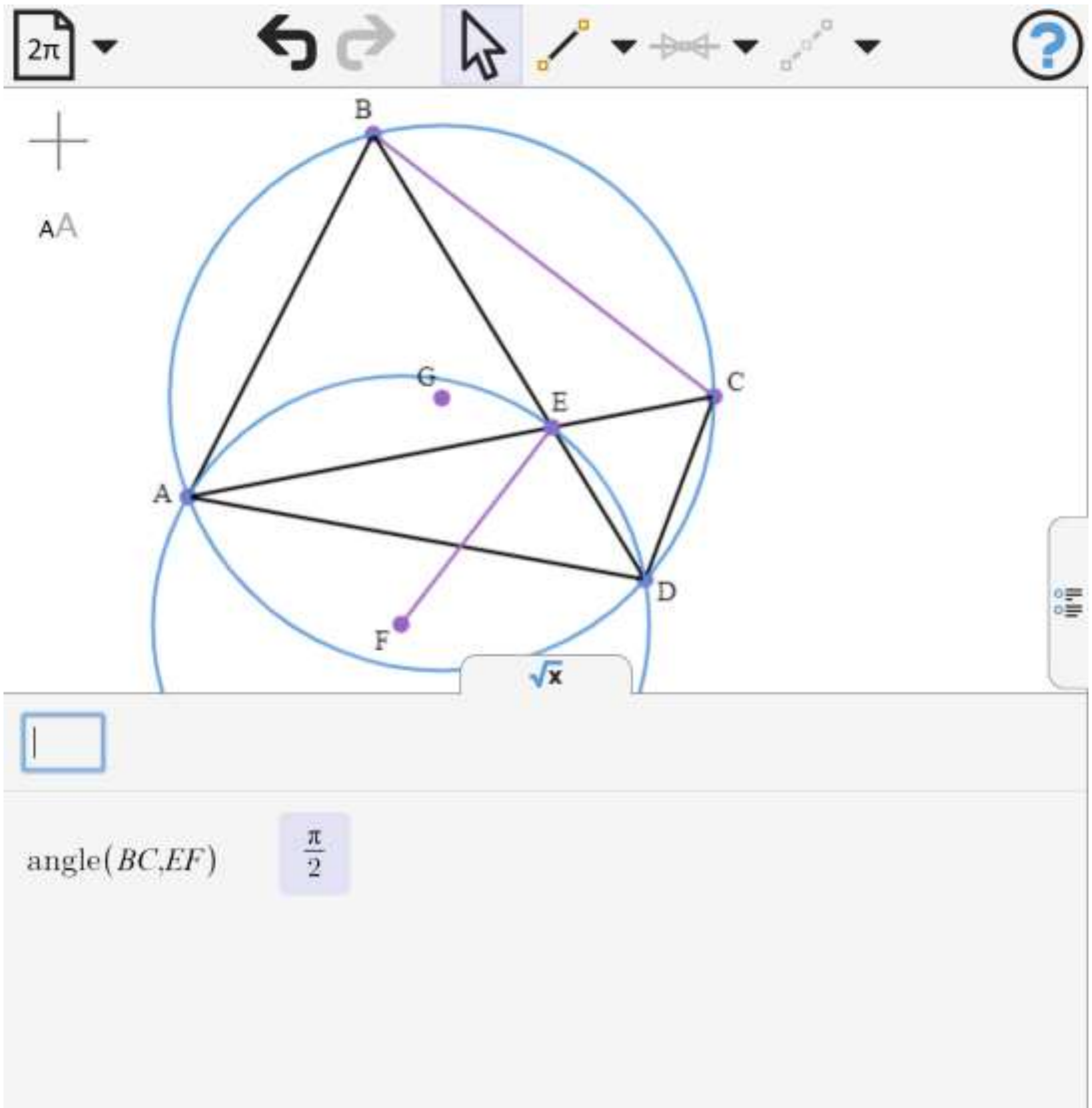
In triangle ABC, let D be the midpoint of the side BC, E and F the feet of the altitudes on AB, AC respectively. DG is perpendicular to EF at G. Show that G is the midpoint of DE.



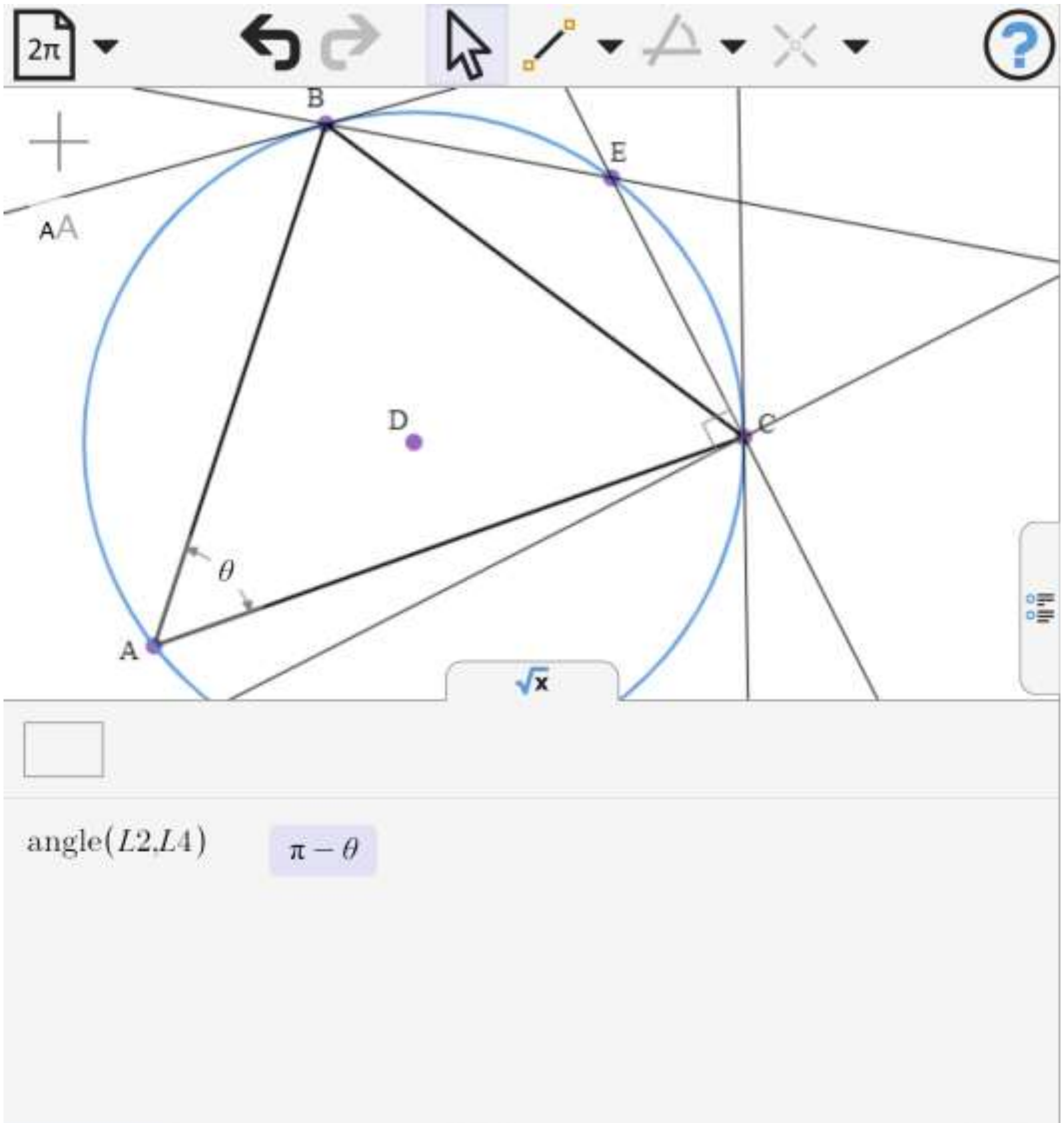
Angle Examples

46. Problem 6.294 from Zhang et al

Let ABCD be a cyclic quadrilateral and E the intersection point of its diagonals. Let F be the center of the circumcircle of AED. Then EF is perpendicular to BC

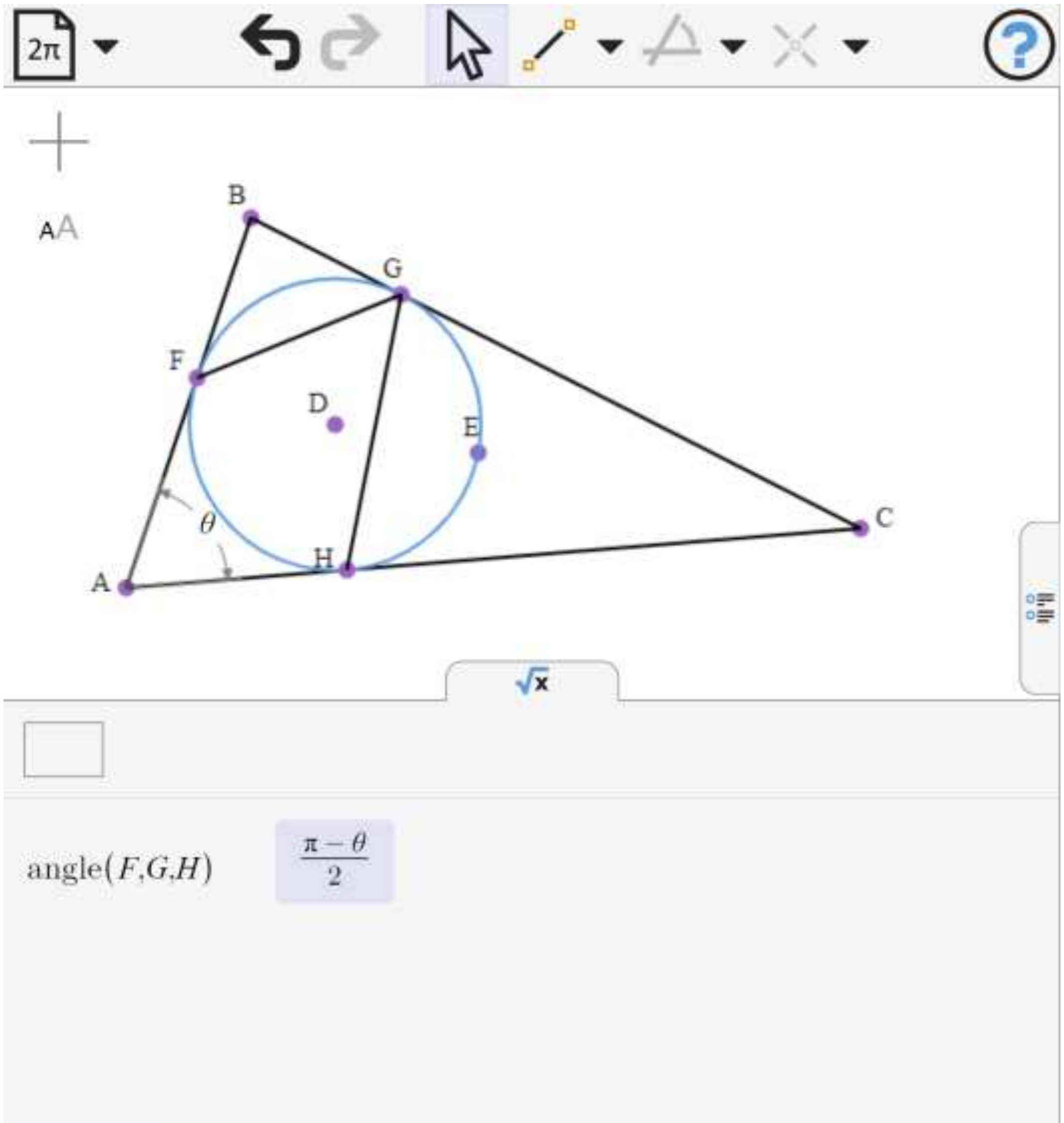


47. Angle bisectors of chord and tangents meet on the circle



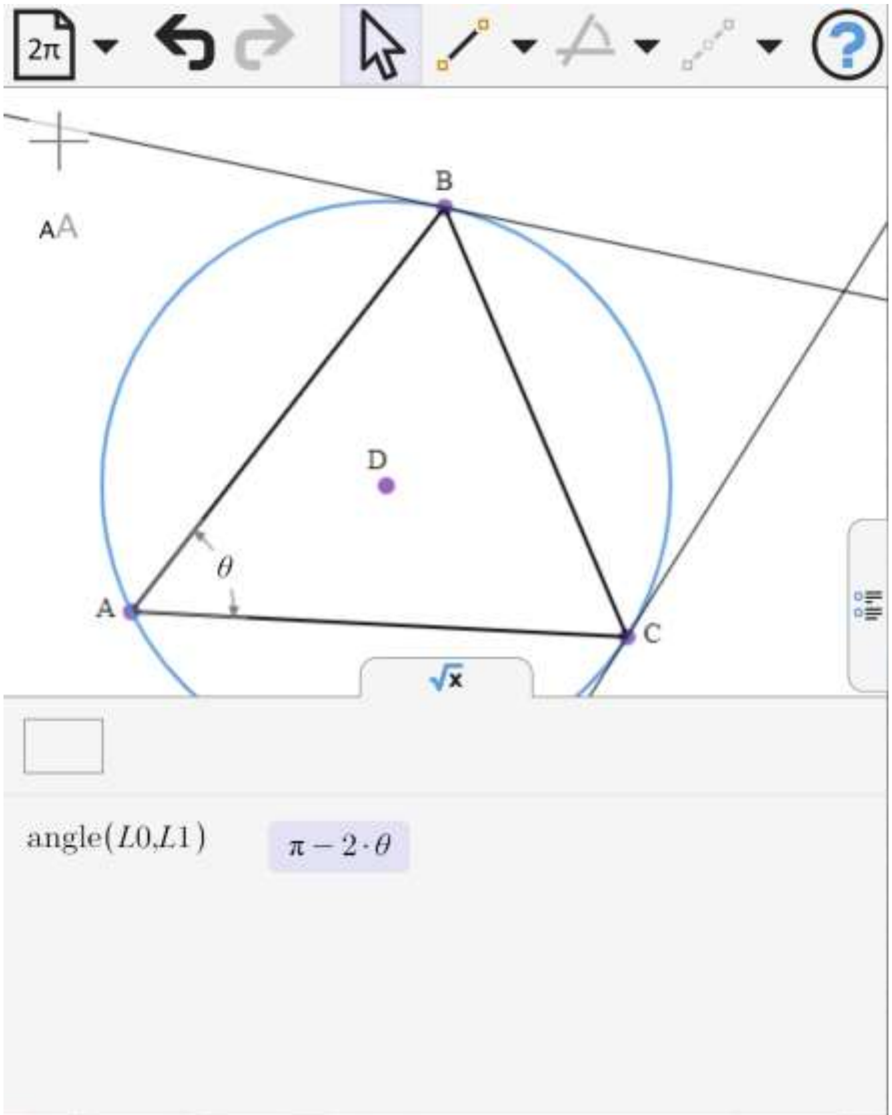
If chord BC subtends an angle θ on the circumference of a circle, the angle bisectors between the chord and the circle tangents at B and C are at angle $\pi - \theta$. We can infer that they meet on the circumference of the circle

48. Points of contact with the incircle



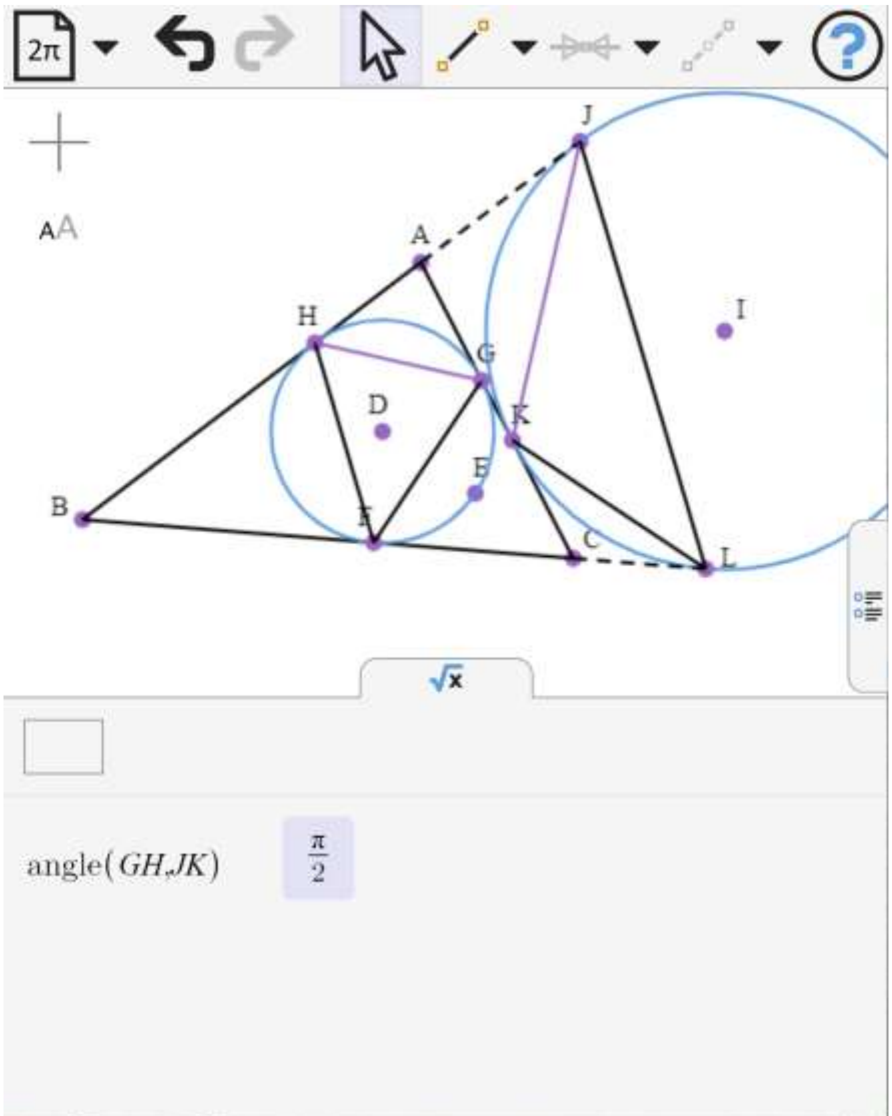
Given triangle ABC, let FGH be the points of contact between the incircle and sides AB, BC, CD respectively. If angle BAC is θ, then angle FGH is $\frac{\pi-\theta}{2}$

49. Tangent triangle



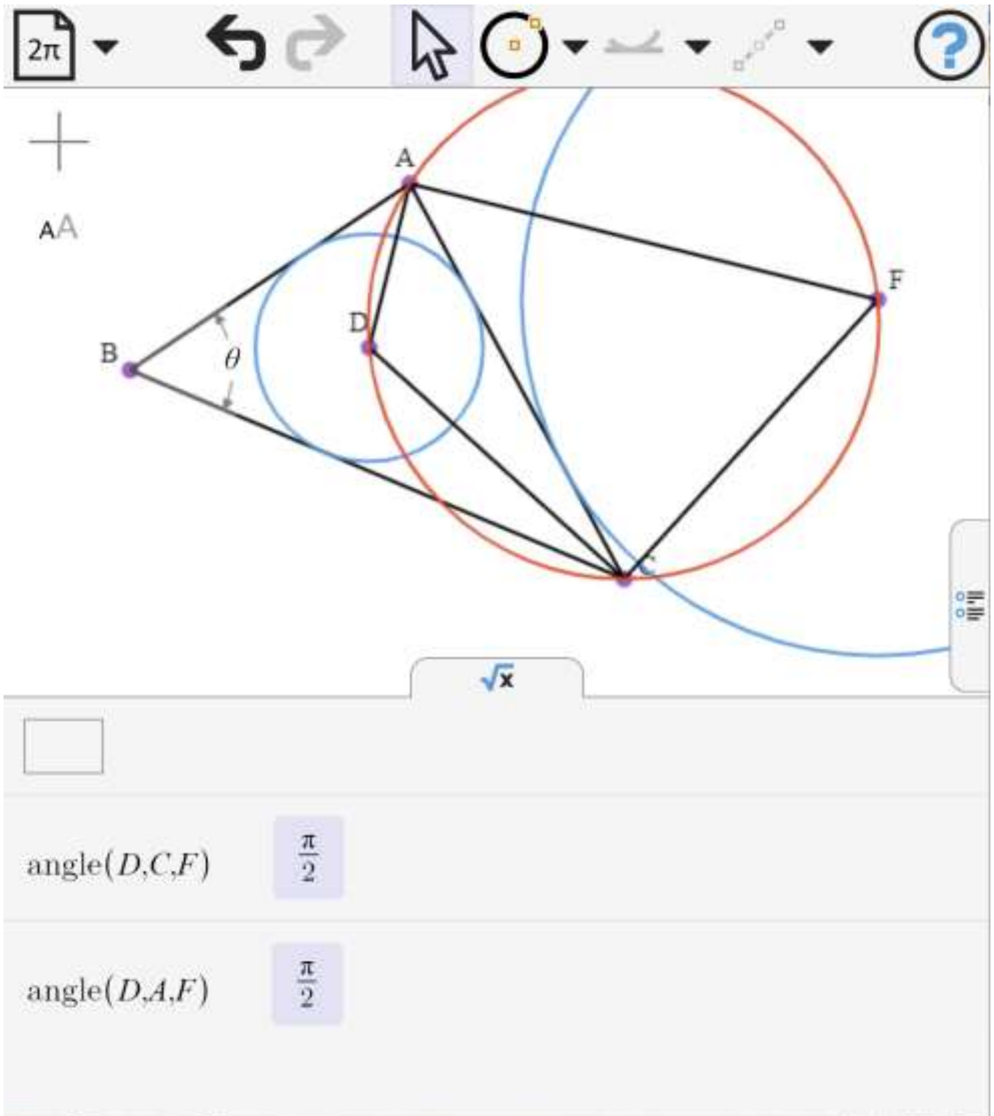
Conversely, the angle between the tangents to the circumcircle at B and C is $\pi - 2\theta$

50. Incircle and excircle



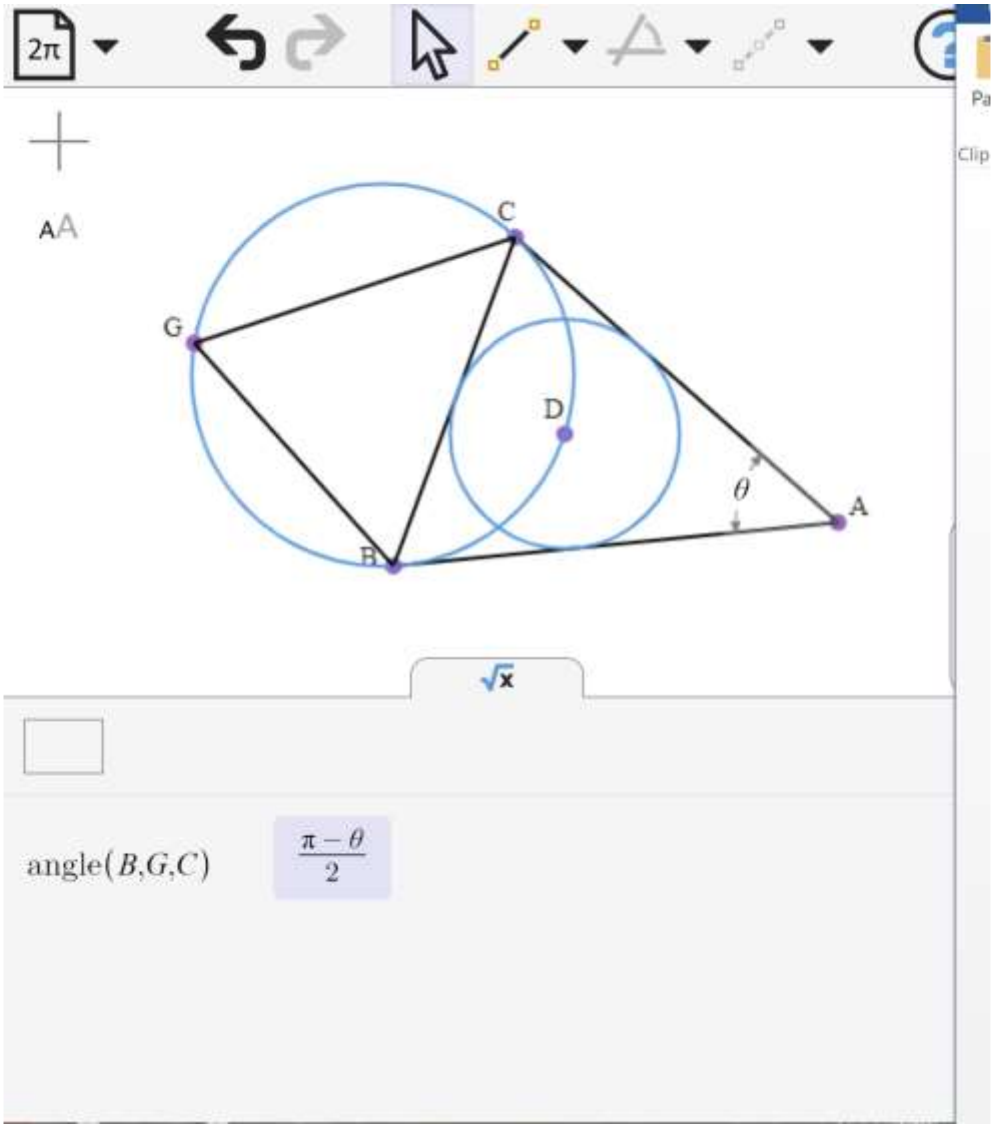
Let F,G,H be the points of contact between the incircle and sides BC, AC, AB of triangle ABC. Let ***C*** be the excircle external to AC. Let JKL be the points of contact between ***C*** and AB, AC, BC. Then GH is perpendicular to JK.

51. A cyclic quadrilateral in the incircle excircle diagram



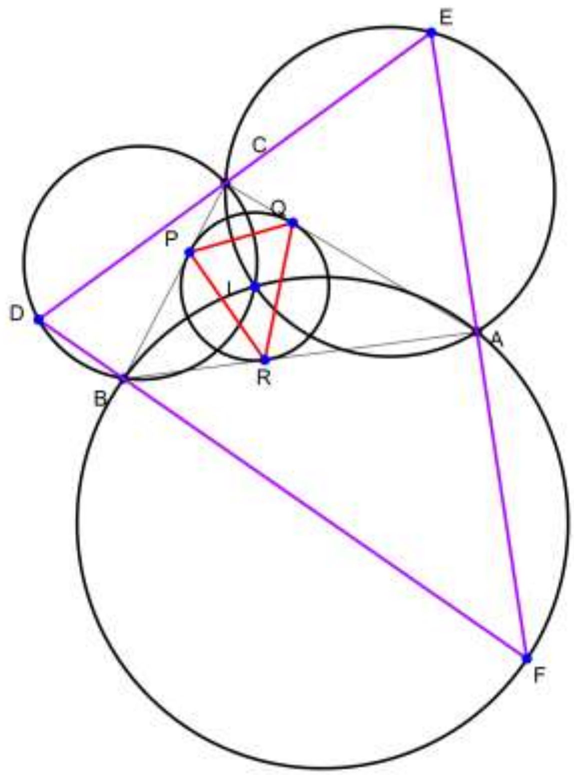
Let D be the incenter of ABC and F the center of the excircle external to AC. The quadrilateral AFCD is cyclic, with DF as a diameter.

52. Circumcircle of Incenter and two vertices



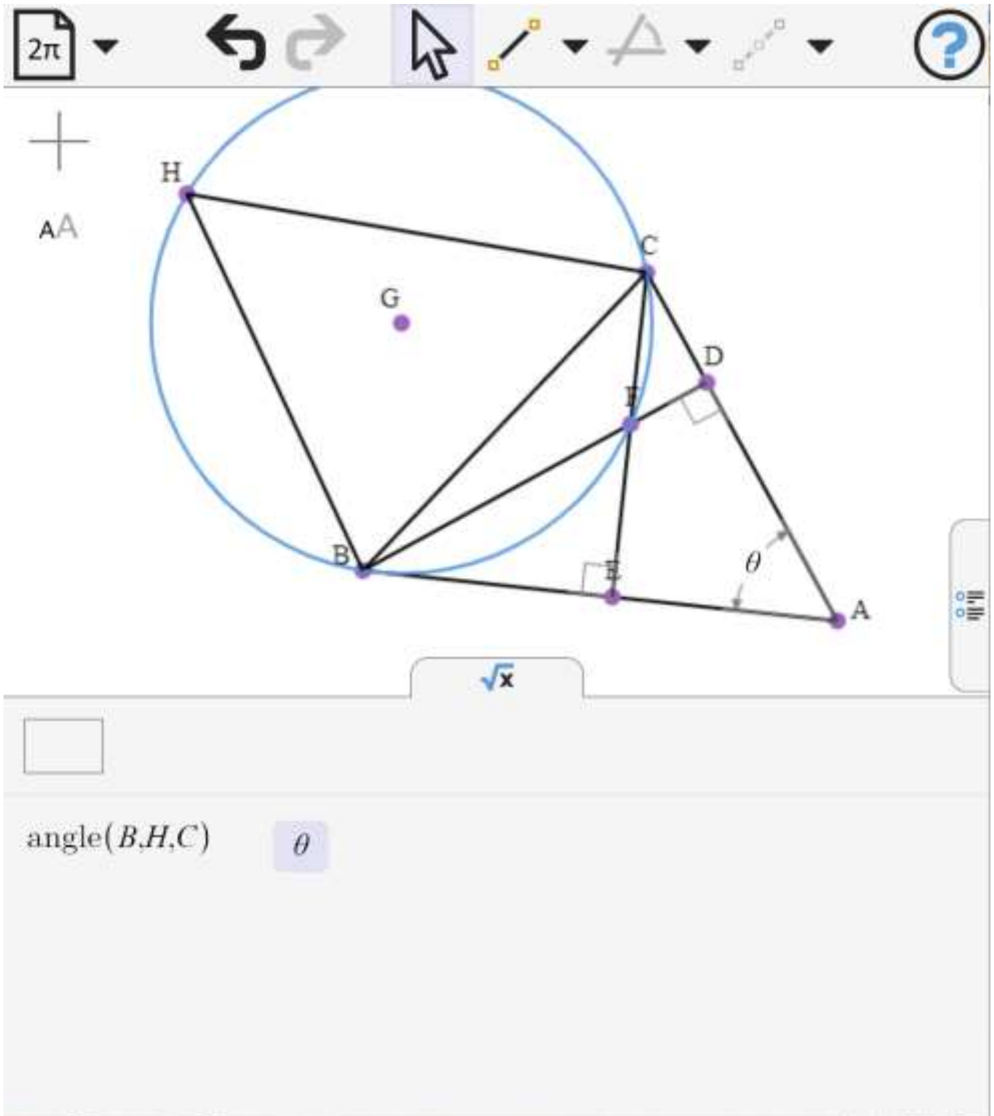
Let I be the incenter of triangle ABC, and let D be a point on the circumcircle of IAB. If angle BAC is is θ, then angle BDC is $\frac{\pi-\theta}{2}$

53. Three such circumcircles



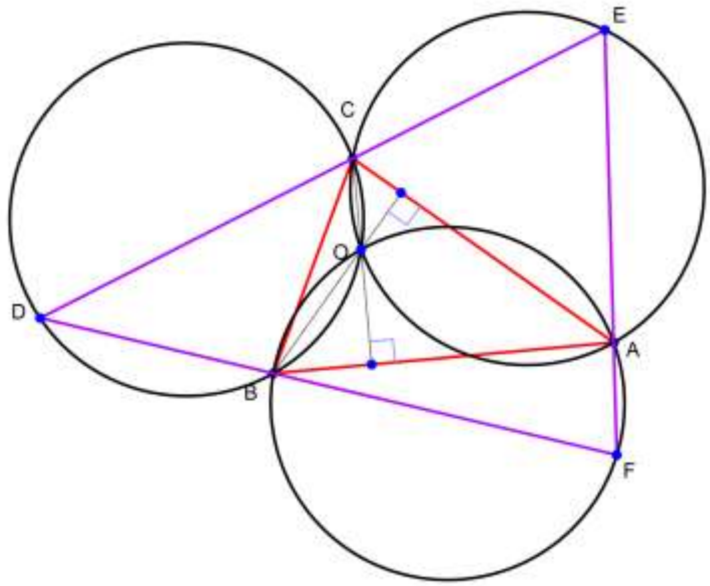
Let D lie externally on the circumcircle of BIC, let E be the intersection of DC extended and the circumcircle of CIA and let F be the intersection of EA extended and the circumcircle of AIB. Let PQR be the points of contact between the incircle and the sides BC, CA, AB. Then F, B and D are collinear and the triangle DEF is similar to the triangle PQR. This follows from the above results.

54. Circumcircle of two vertices and the orthocenter



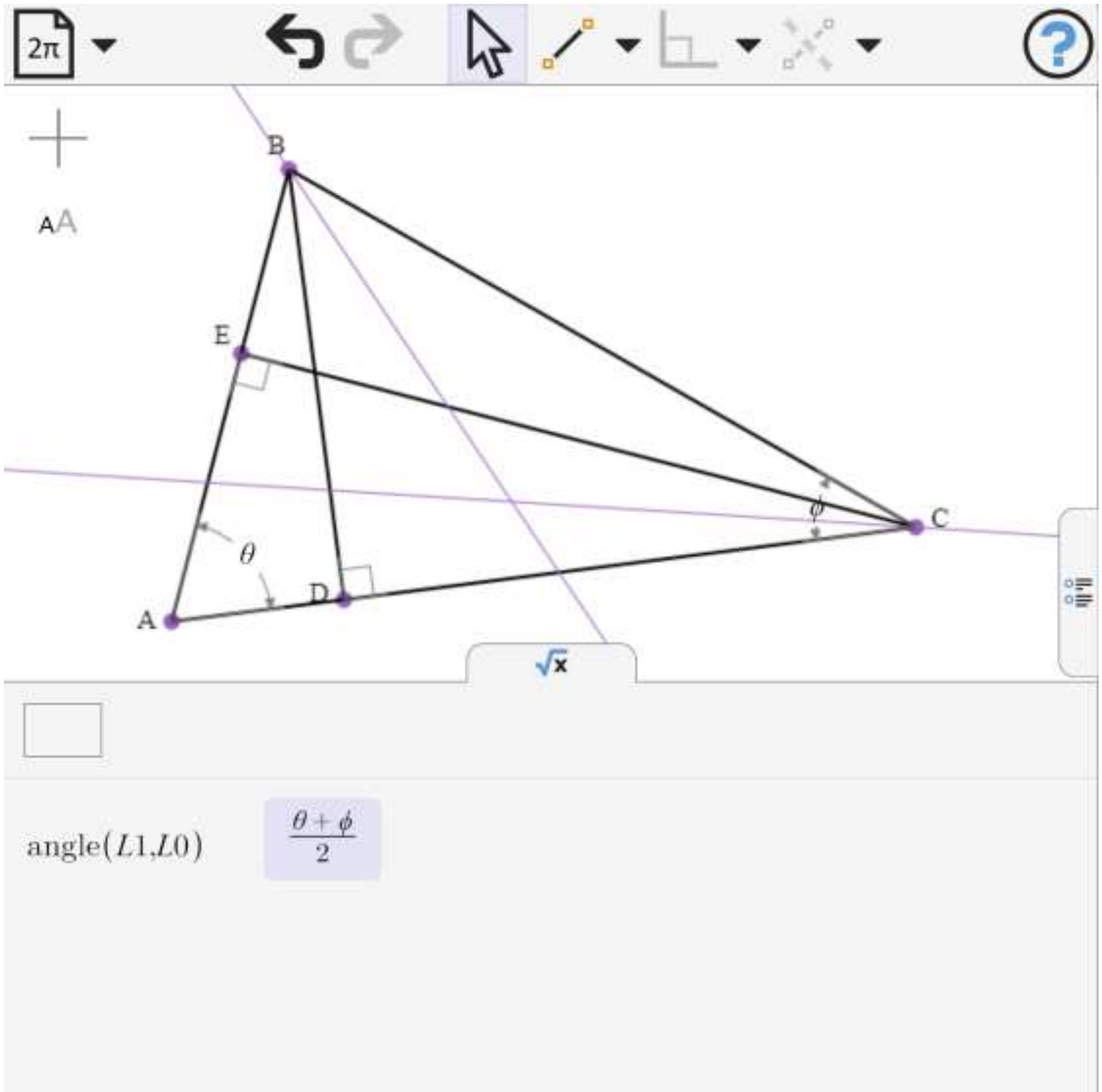
Let F be the orthocenter of triangle ABC, and let H be a point on the circumcircle of FAB. Angle BAC is equal to angle BHC.

55. Three such circumcircles

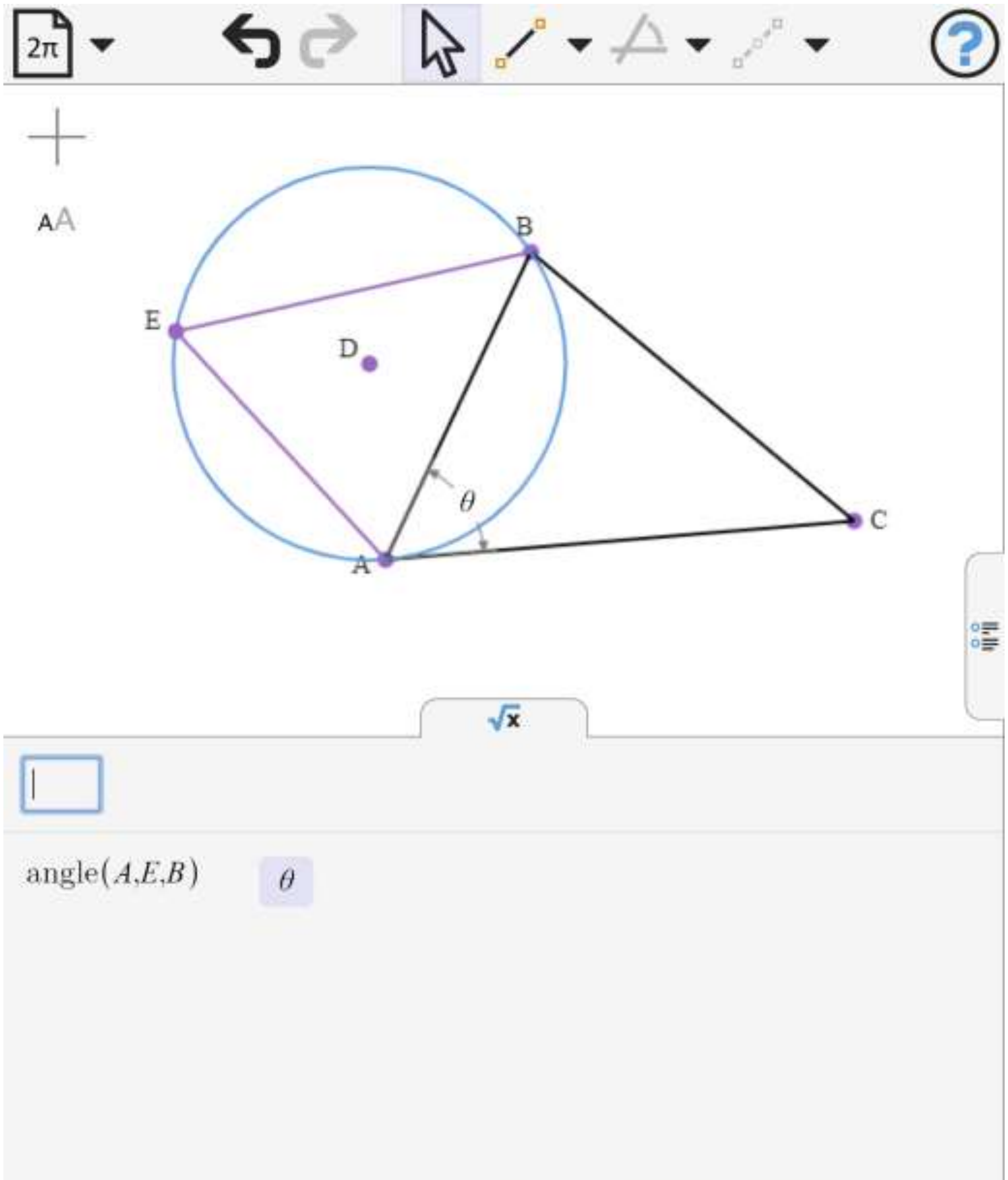


Let D lie externally on the circumcircle of BOC, let E be the intersection of DC extended and the circumcircle of COA and let F be the intersection of EA extended and the circumcircle of AOB. Then F, B and D are collinear and the triangle DEF is similar to the triangle ABC. This follows from the above result.

56. Altitude side angle bisectors

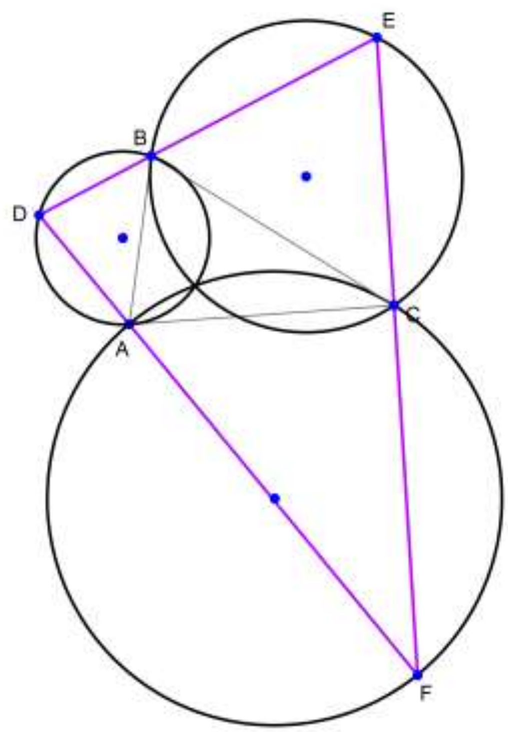


57. Mix-Linear circle



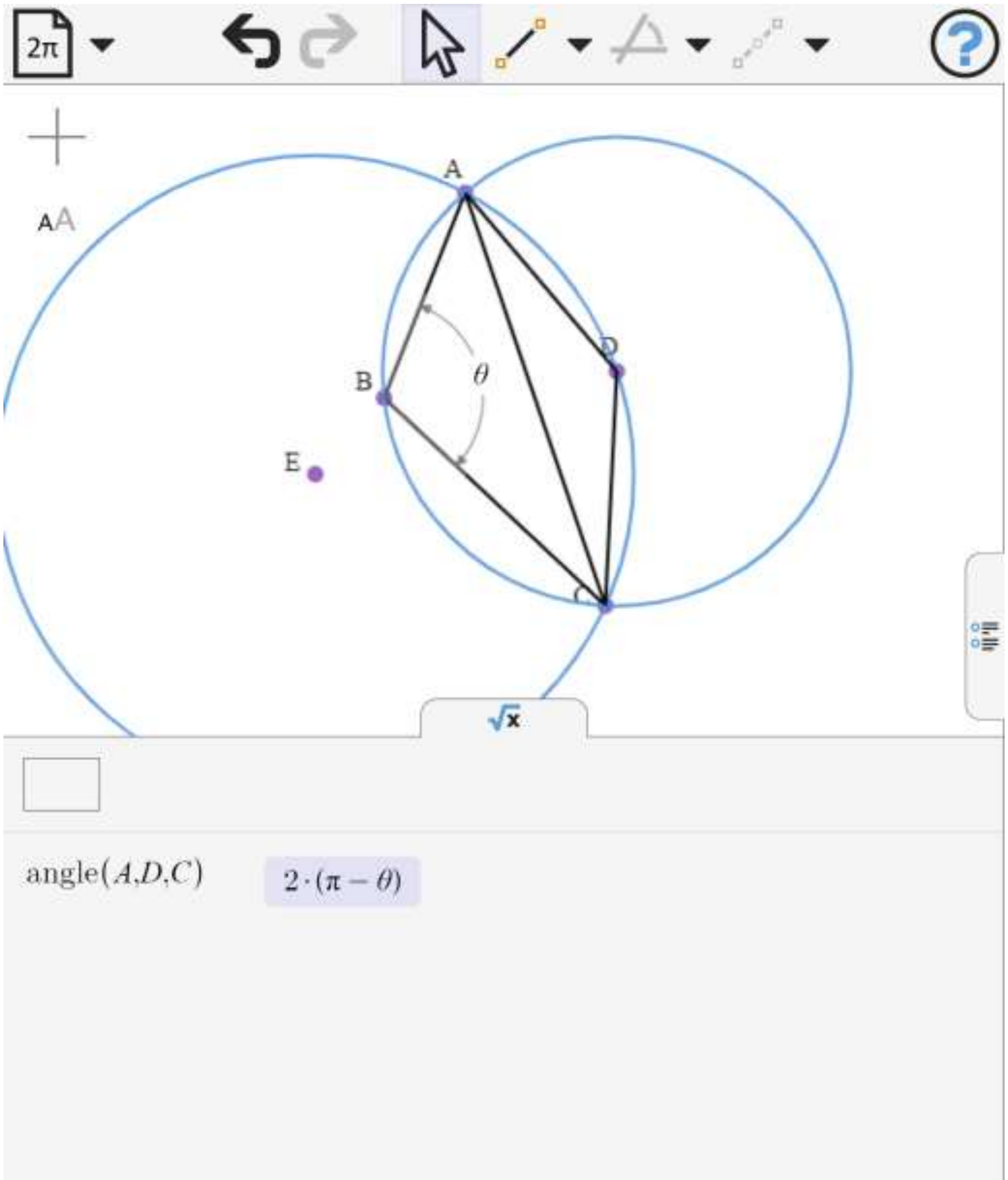
Let ***C*** be the circle through vertices A and B tangent to side AC of triangle ABC. Let D be an external point on this circle. Then ADB = BAC.

58. Three such circles



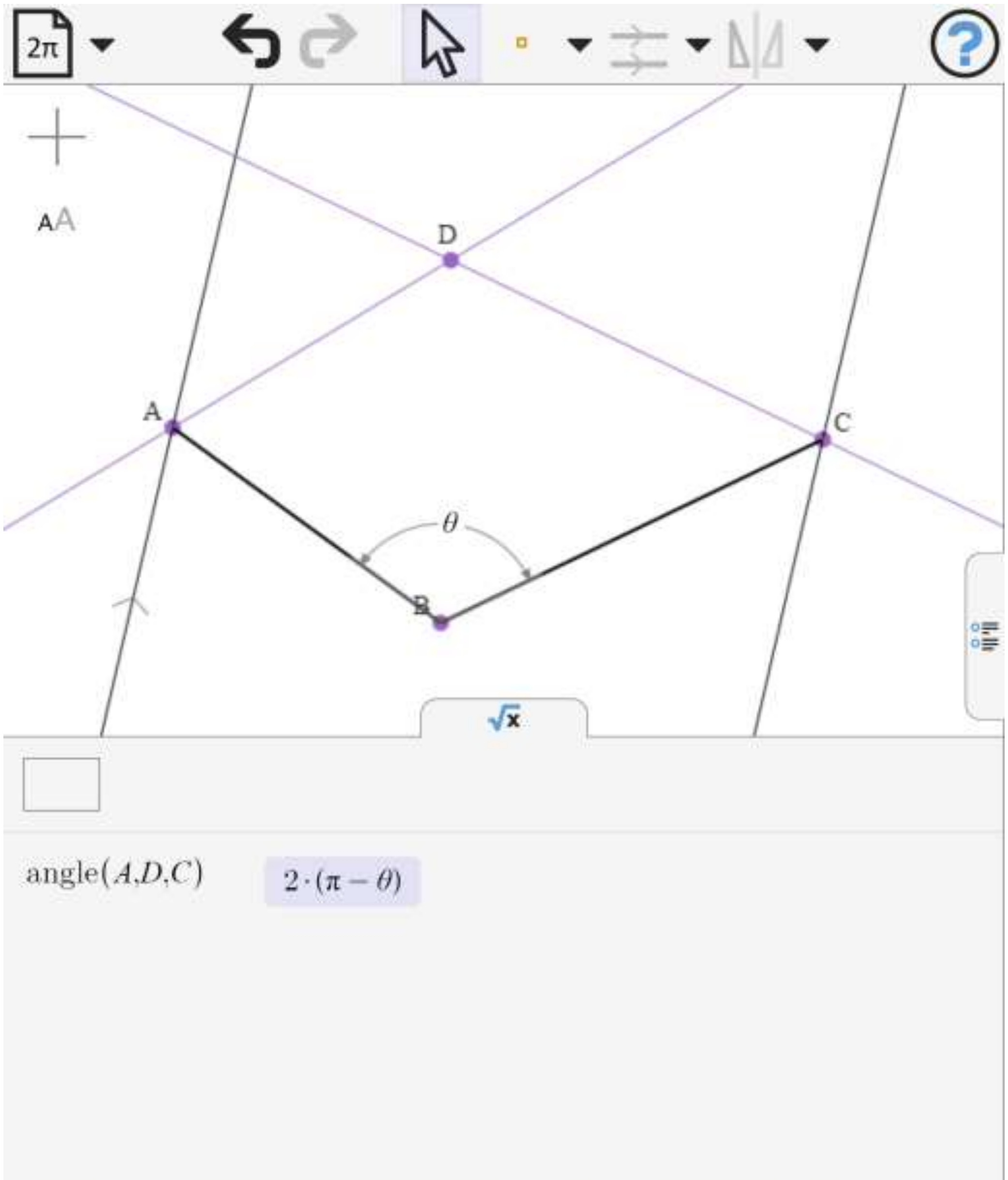
Let ***C*** be the circle through vertices A and B tangent to side AC. Let ***D*** be the circle through vertices A and B tangent to side AC. Let **E** be the circle through vertices A and B tangent to side AC. Let D lie externally on circle ***C***, let E be the intersection of DB extended and circle ***D*** and let F be the intersection of EC extended and circle ***E***. Then F, A and D are collinear and the triangle DEF is similar to the triangle ABC. This follows from the above result.

59. Angle subtended in the circumcircle of two vertices and the circumcenter



If angle ABC is θ, let D be the circumccenter of ABC. Then angle ADC=2(π-θ).

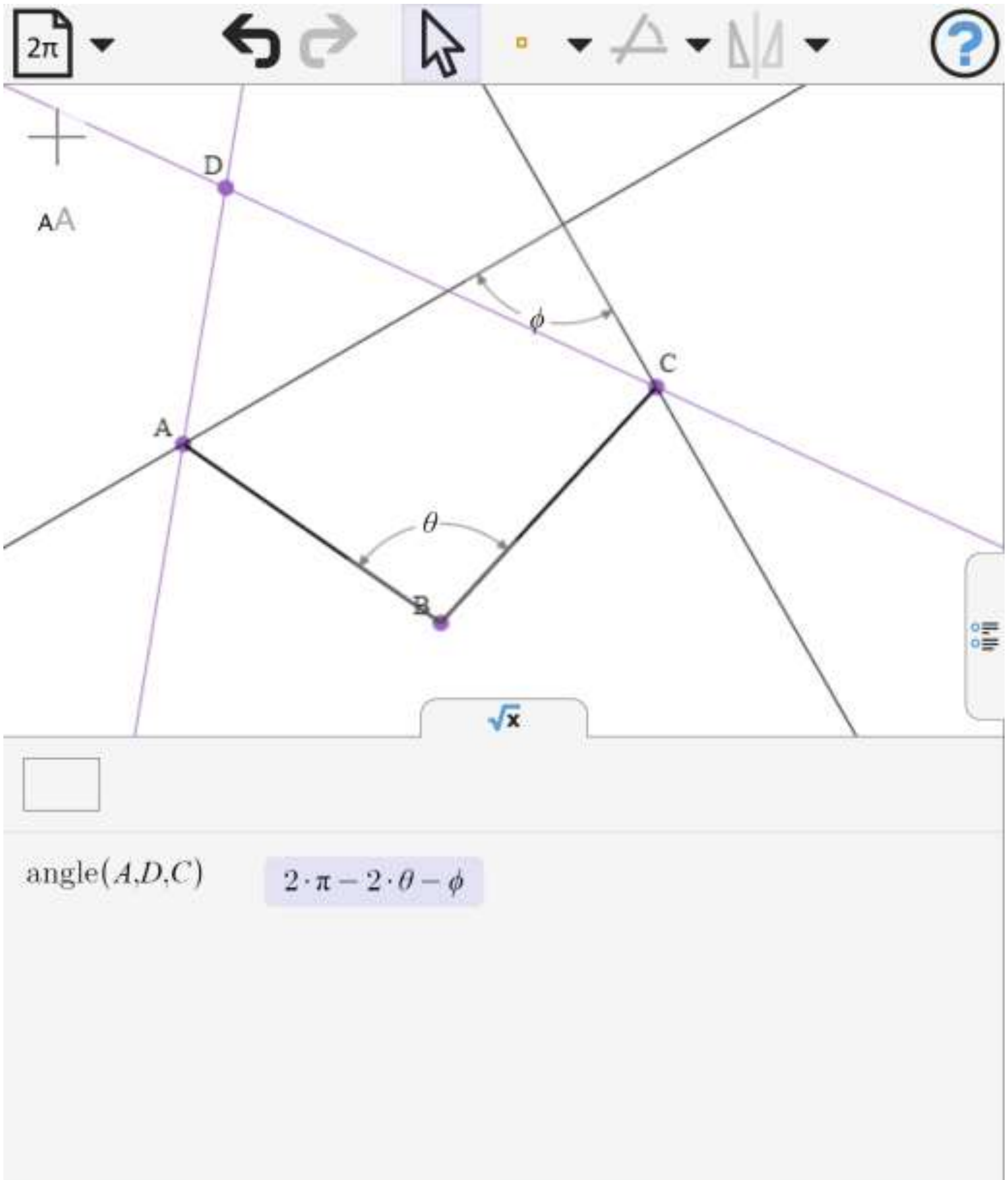
60. Reflection of parallel rays



Given triangle ABC with angle ABC=θ, the images of two parallel rays under reflection in BA and BC meet at angle 2(π-θ)

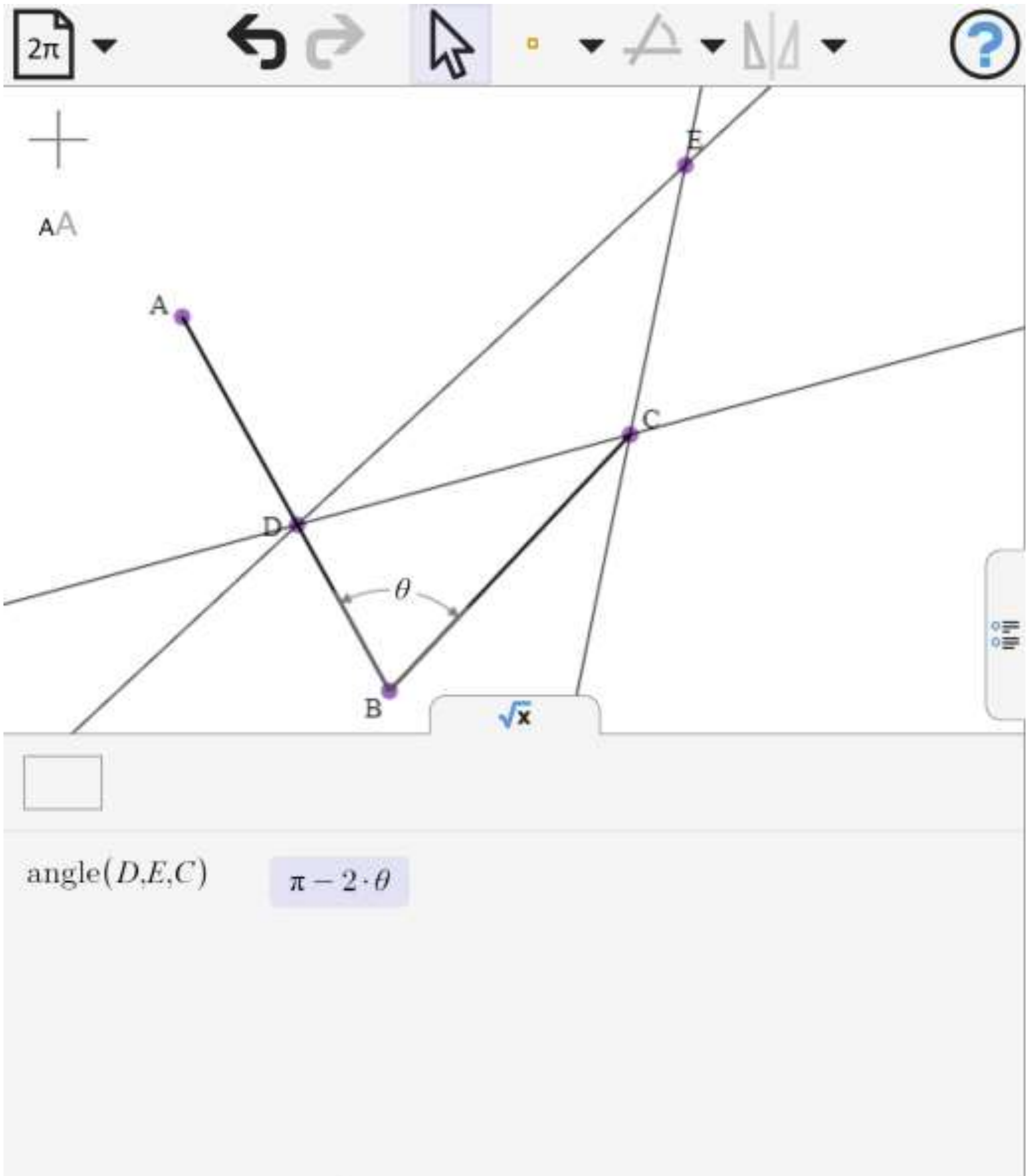
From this along with the previous result, we can deduce that the intersection of the reflected images lies on the cicumcircle of B,C and the circumcenter of ABC.

61. Reflections of angled rays



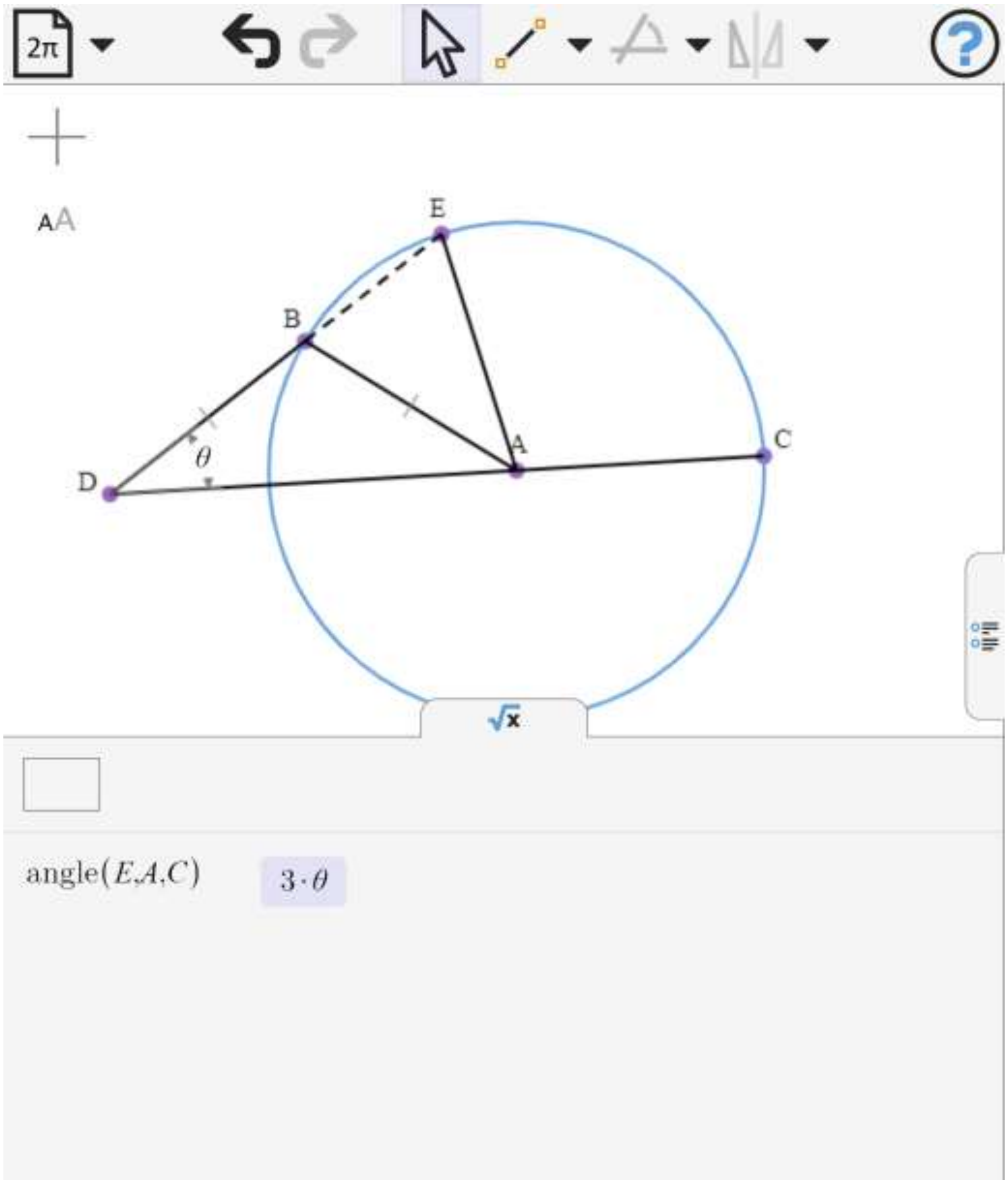
In the above, if the incident rays are at angle ϕ, then the reflected rays are at angle 2π-2θ-ϕ.

62. Reflection in a corner



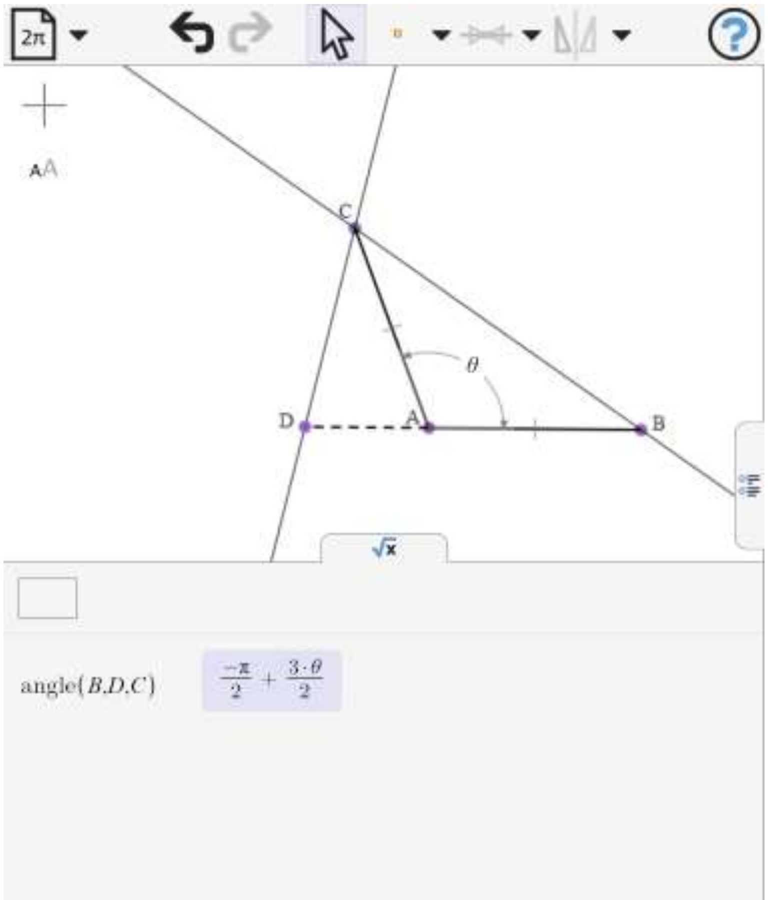
Let angle ABC be θ. A ray incident at C is reflected in BC, then the reflected ray is itself reflected in AB. The angle between the resulting ray and the original ray is $\pi - 2\theta$.

63. Archimedes Angle Trisector



This figure, constructed forward from D triples an angle. Ran backwards, it requires a 'neusis'.. that is a construction not performed with straightedge and compass, but does perform an angle trisection.

64. Reflections in a box lid



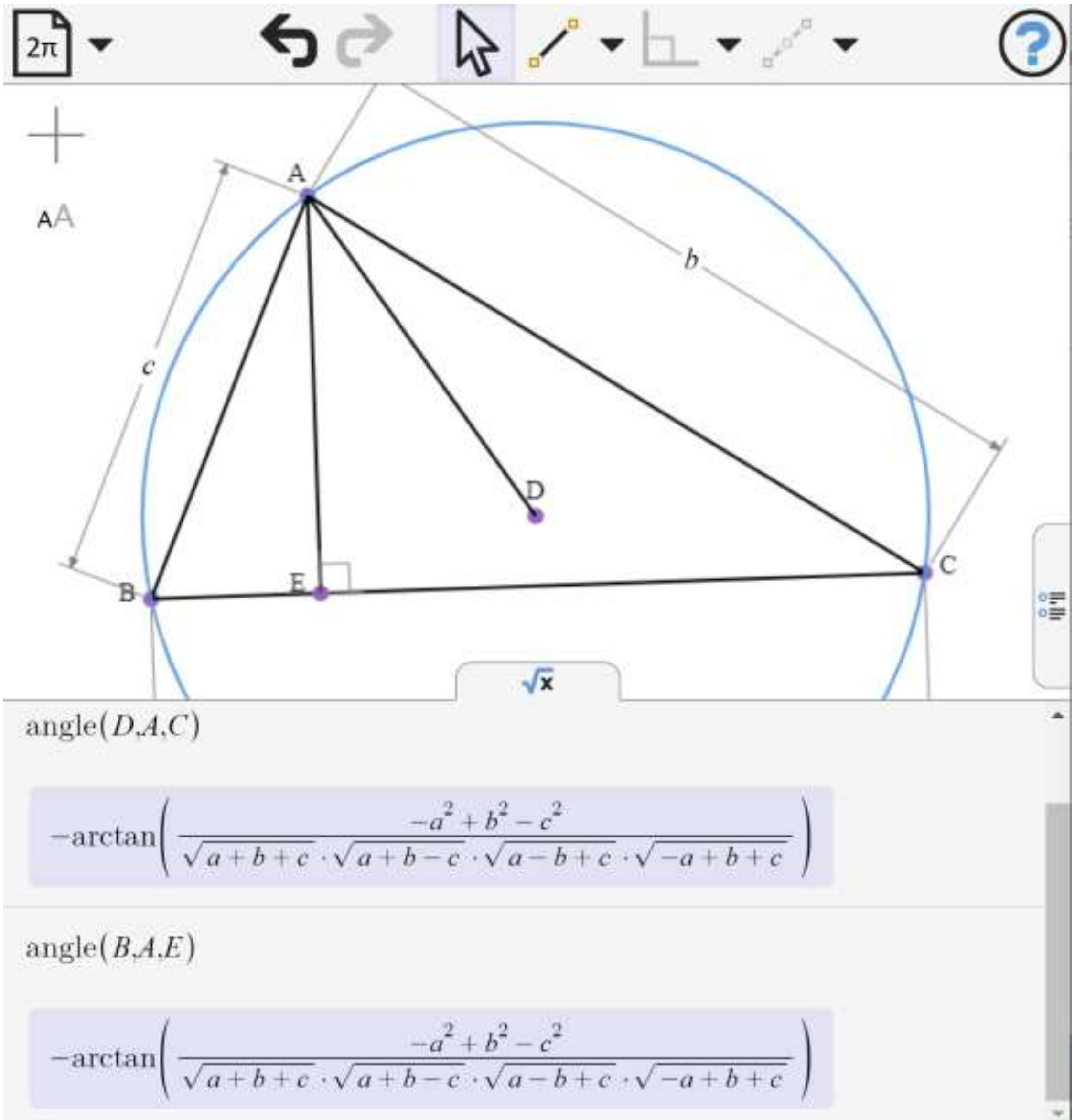
Let ABC be an isosceles triangle with AB and BC equal. Reflect the line BC in AC. The angle between the reflected image and AB is $\frac{3\theta}{2} - \frac{\pi}{2}$

The Circumcircle

65. Angle between circumdiameter and radius

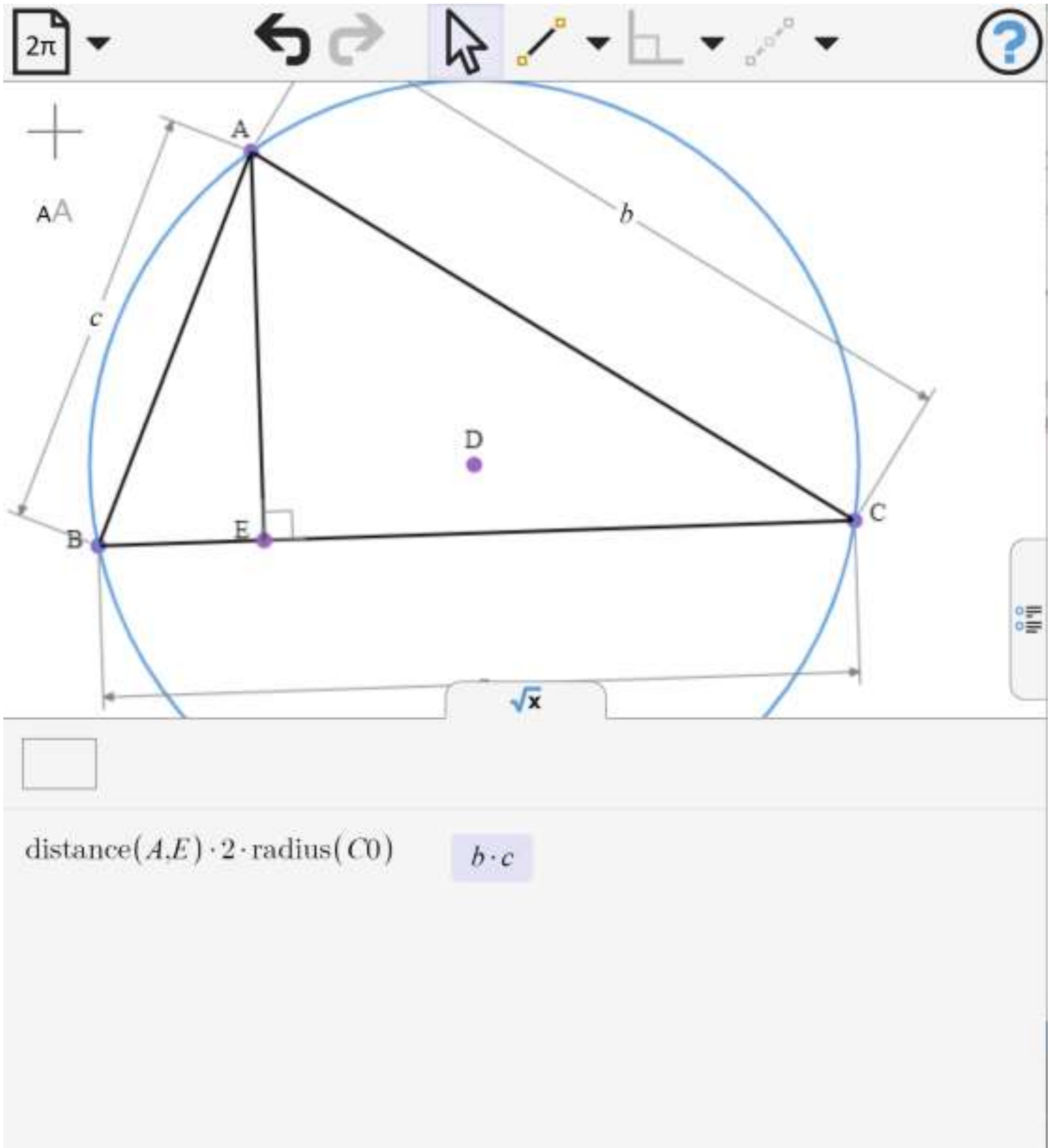
The angle between the circumdiameter and the altitude issued from the same vertex of a triangle is bisected by the bisector of the angle of the triangle at the vertex considered.

We show that angle BAE equals DAC in the diagram, which gives the result.



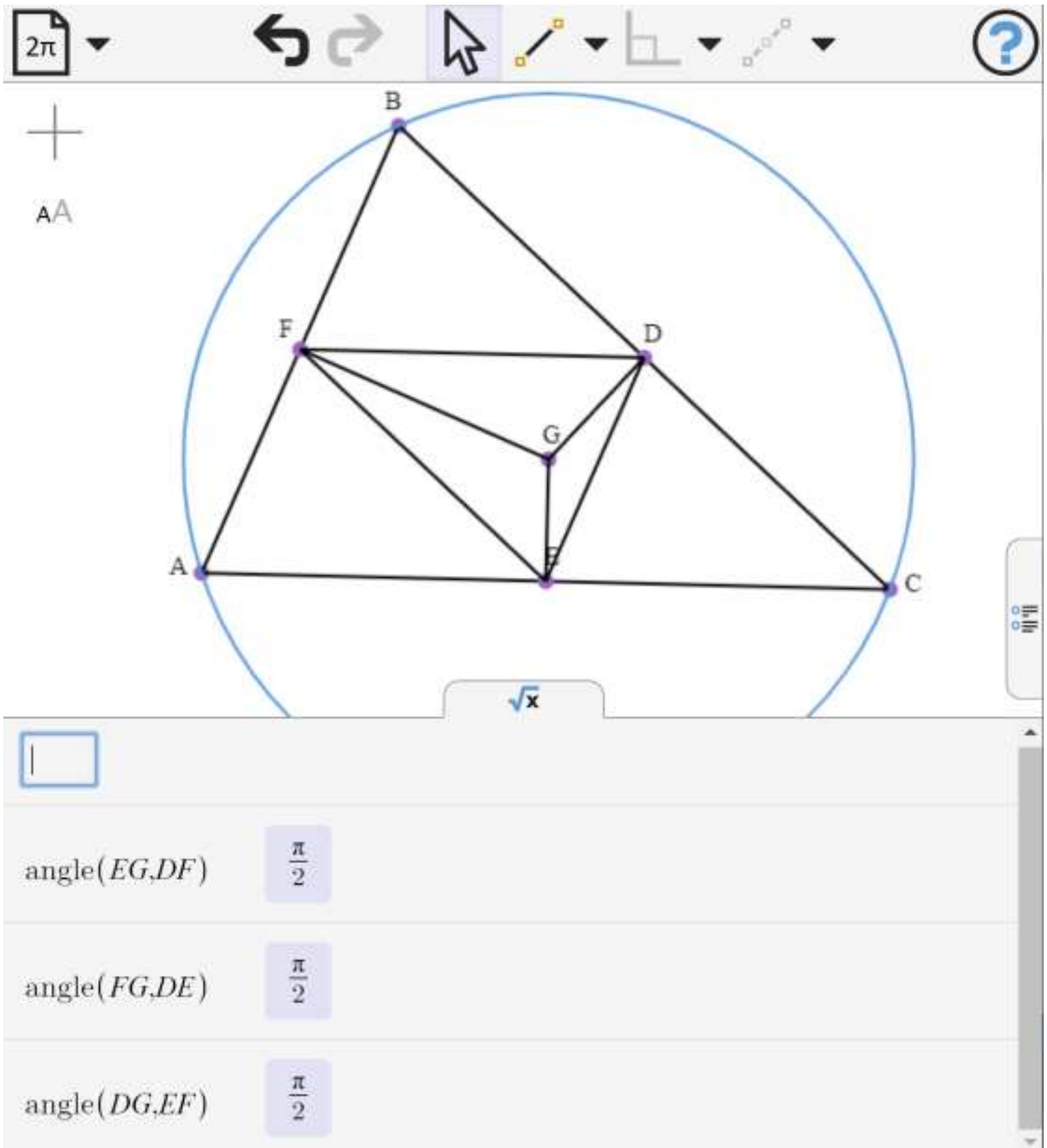
66. Triangle side lengths related to altitude and circumcircle diameter

The product of two sides of a triangle is equal to the altitude to the third side multiplied by the circumdiameter.



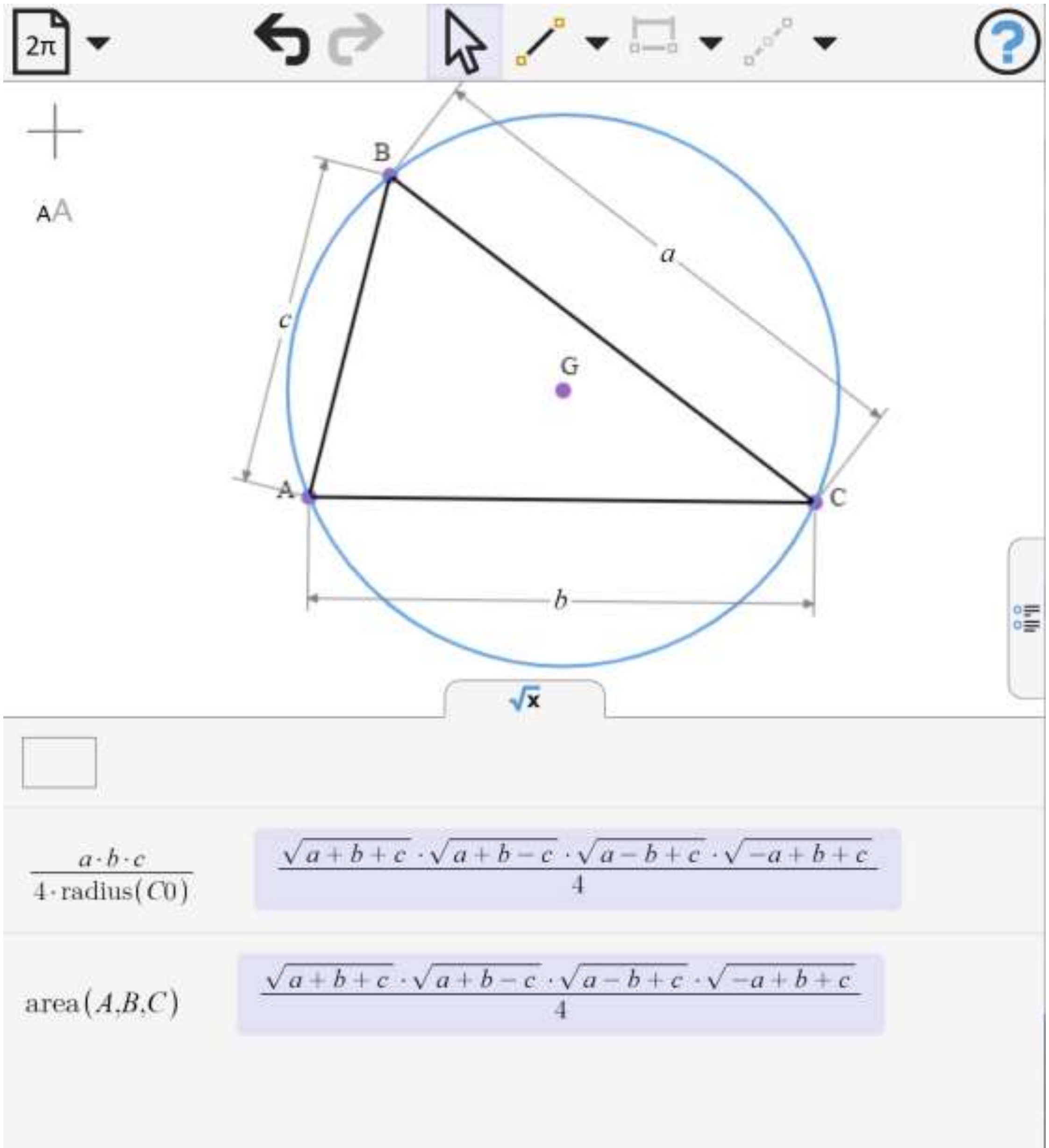
67. Circumcenter is orthocenter of medial triangle

Prove that the circumcenter of a triangle is the orthocenter of its medial triangle



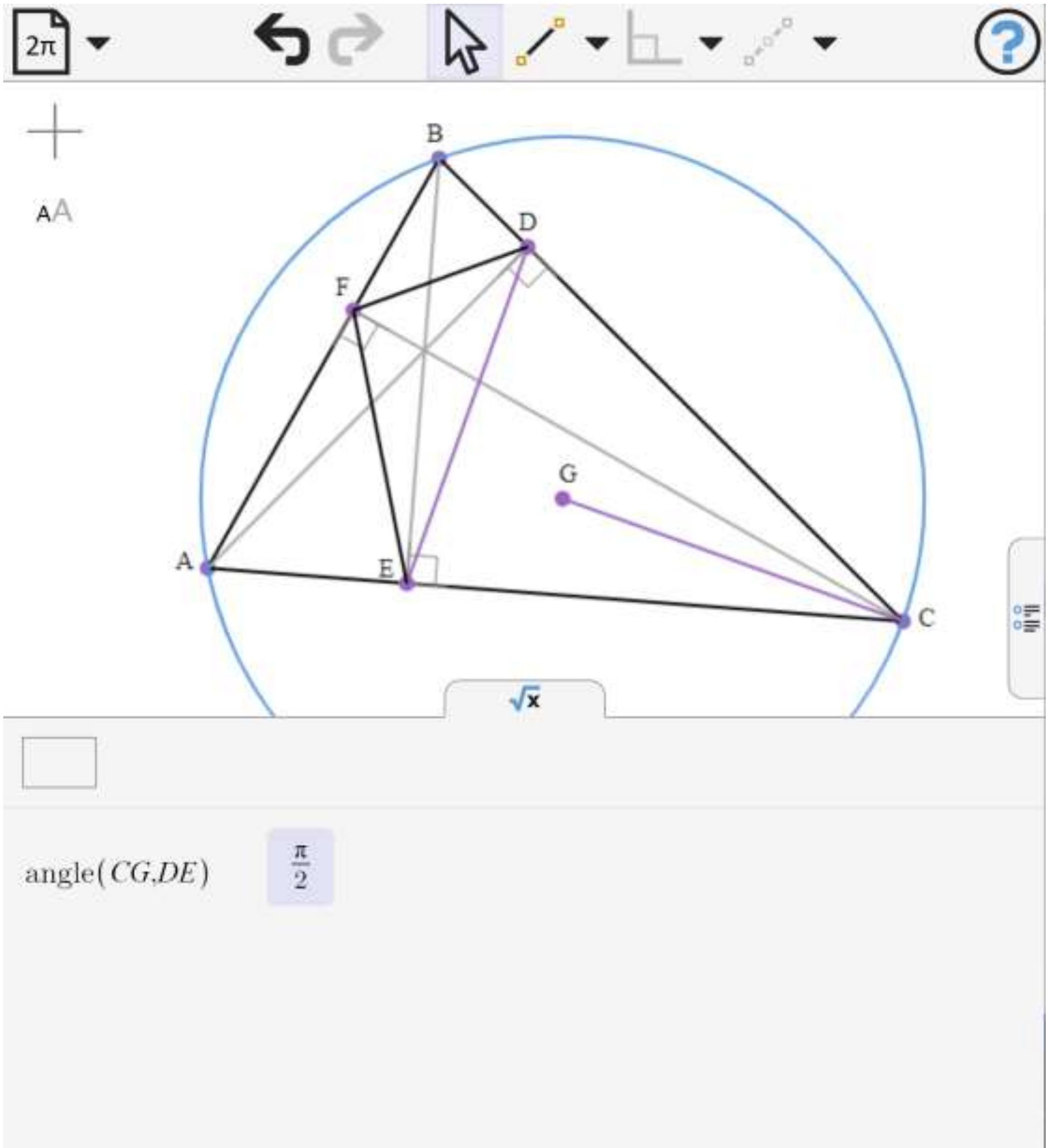
68. Triangle area related to circumcircle radius

The area of a triangle is equal to the product of its three sides divided by the double circumdiameter of the triangle



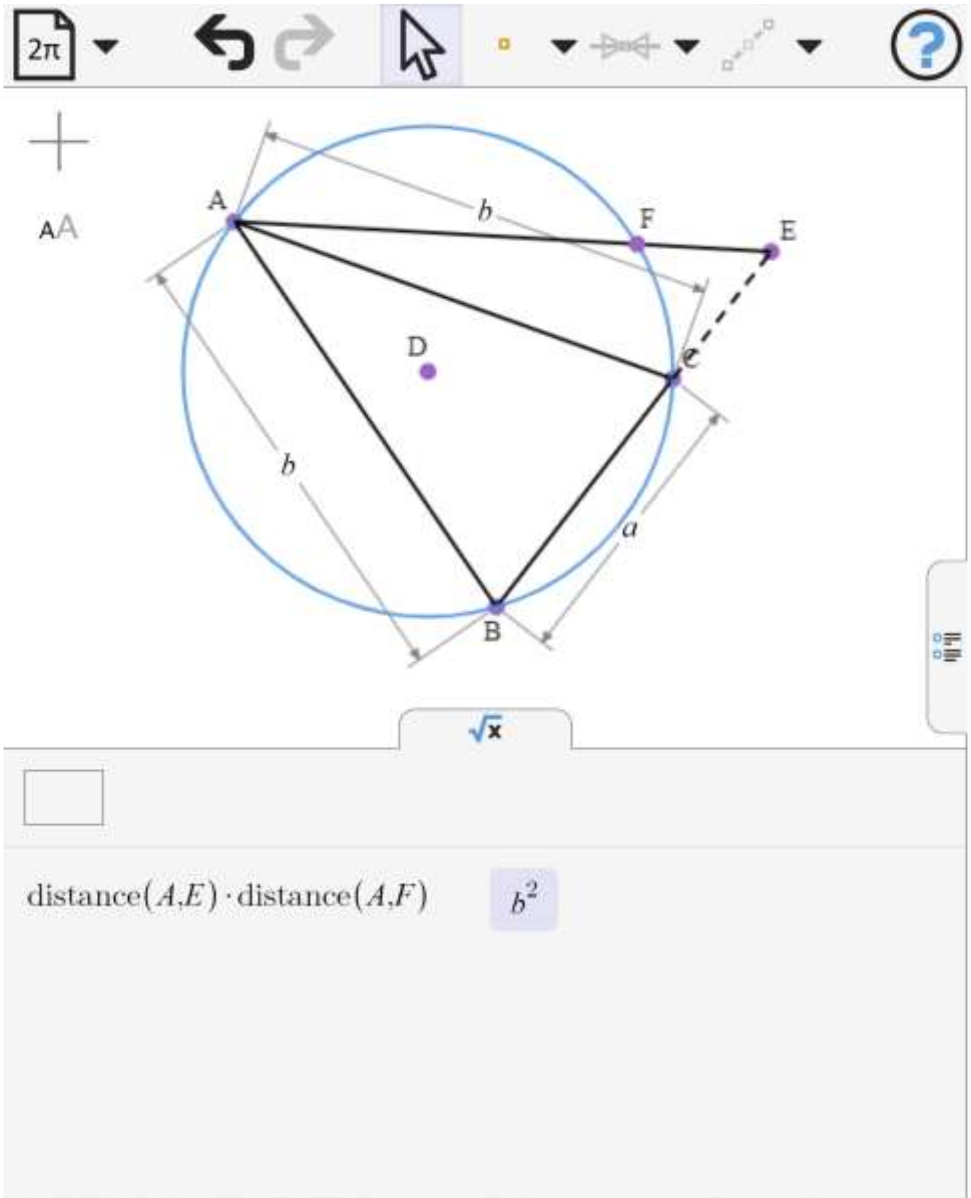
69. Circumcircle and orthic triangle

The radii of the circumcircle passing through the vertices of a triangle are perpendicular to the corresponding sides of the orthic triangle



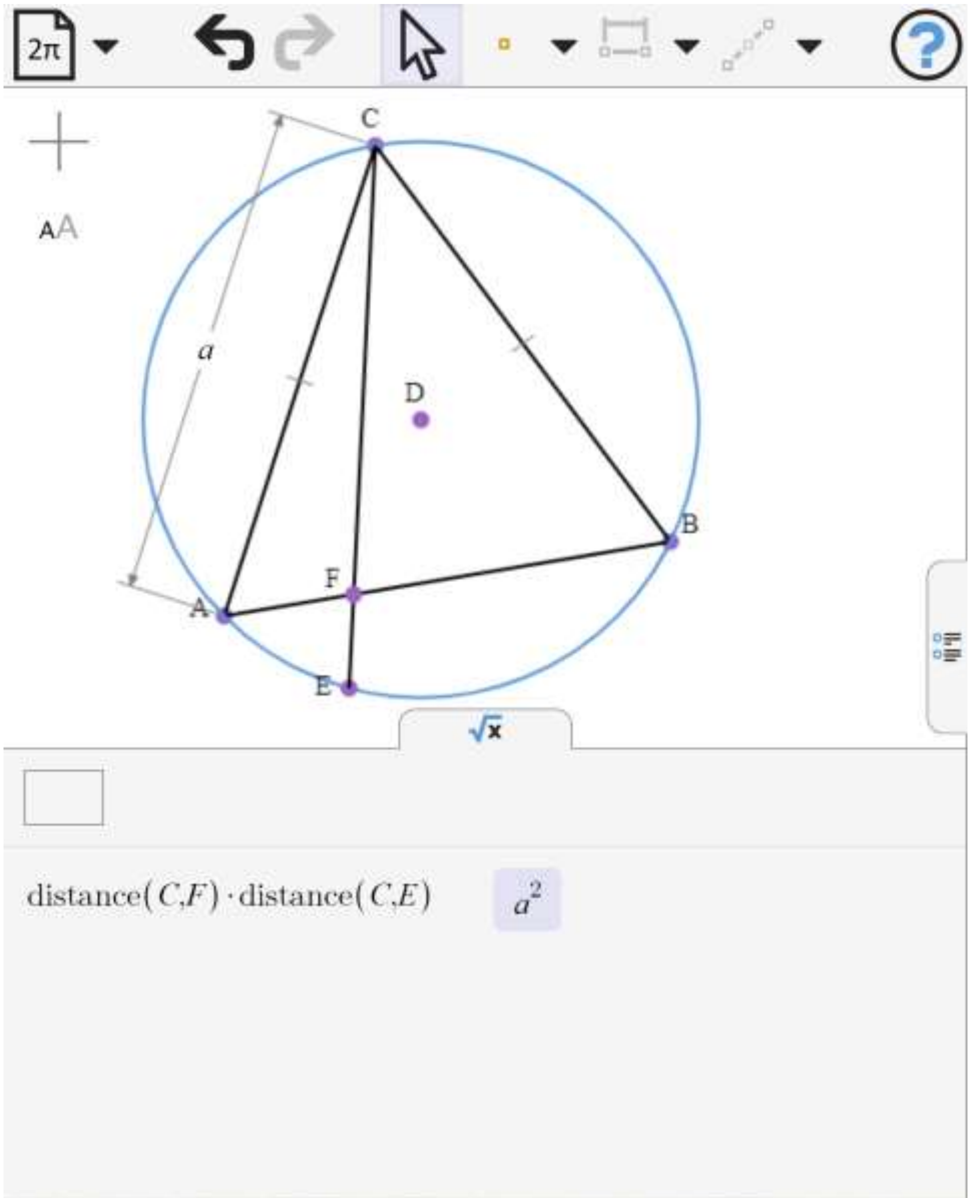
70. An isosceles triangle relationship

Let ABC be a triangle with AC=AB. E is a point on BC. Line AE meets the circumcircle of ABC at F. Show that AB^2=AE.AF



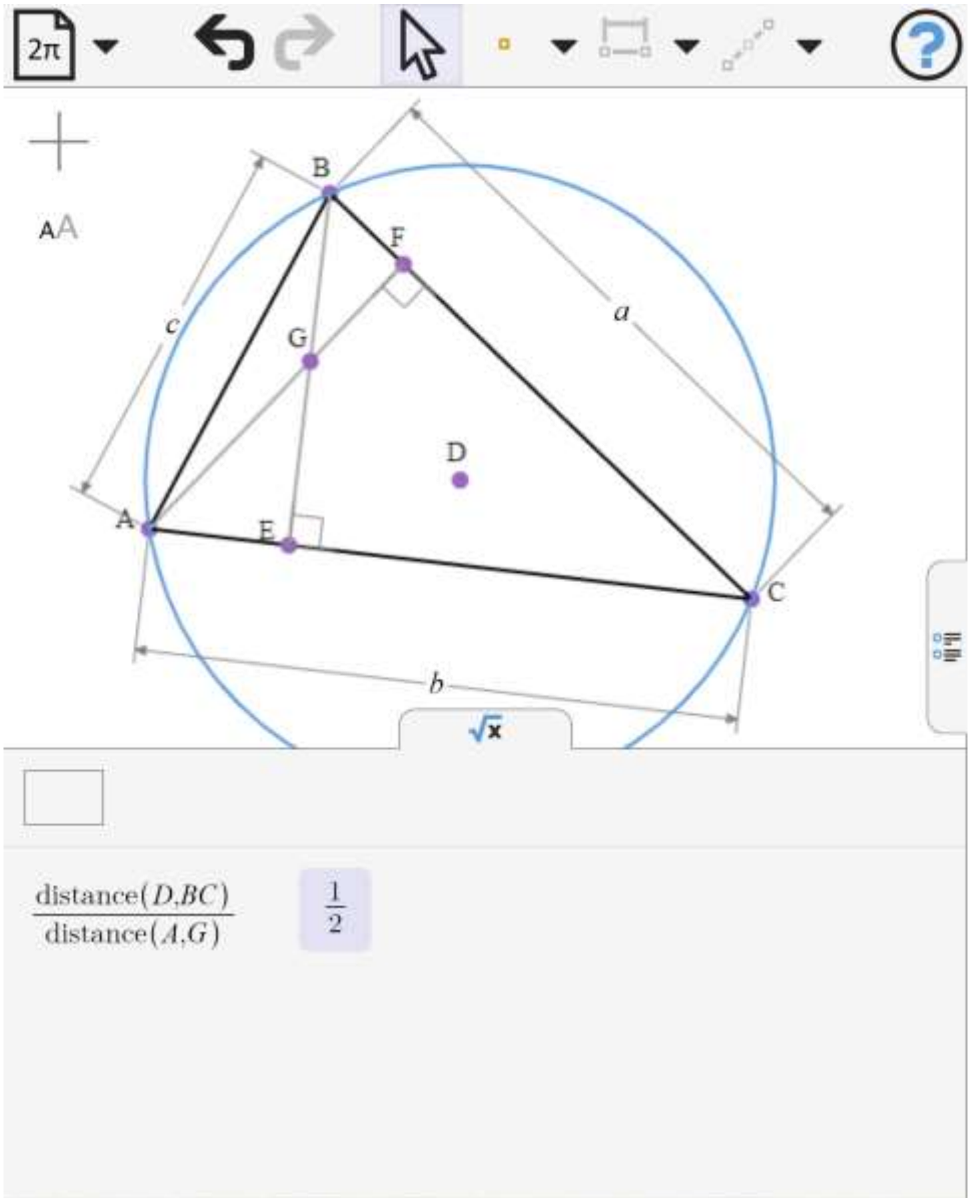
71. Circumcircle of an isosceles triangle

Let C be the midpoint of the arc AB of circle (D). E is a point on the circle. F is the intersection of AB and CE. Show that CA^2=CE.CF.



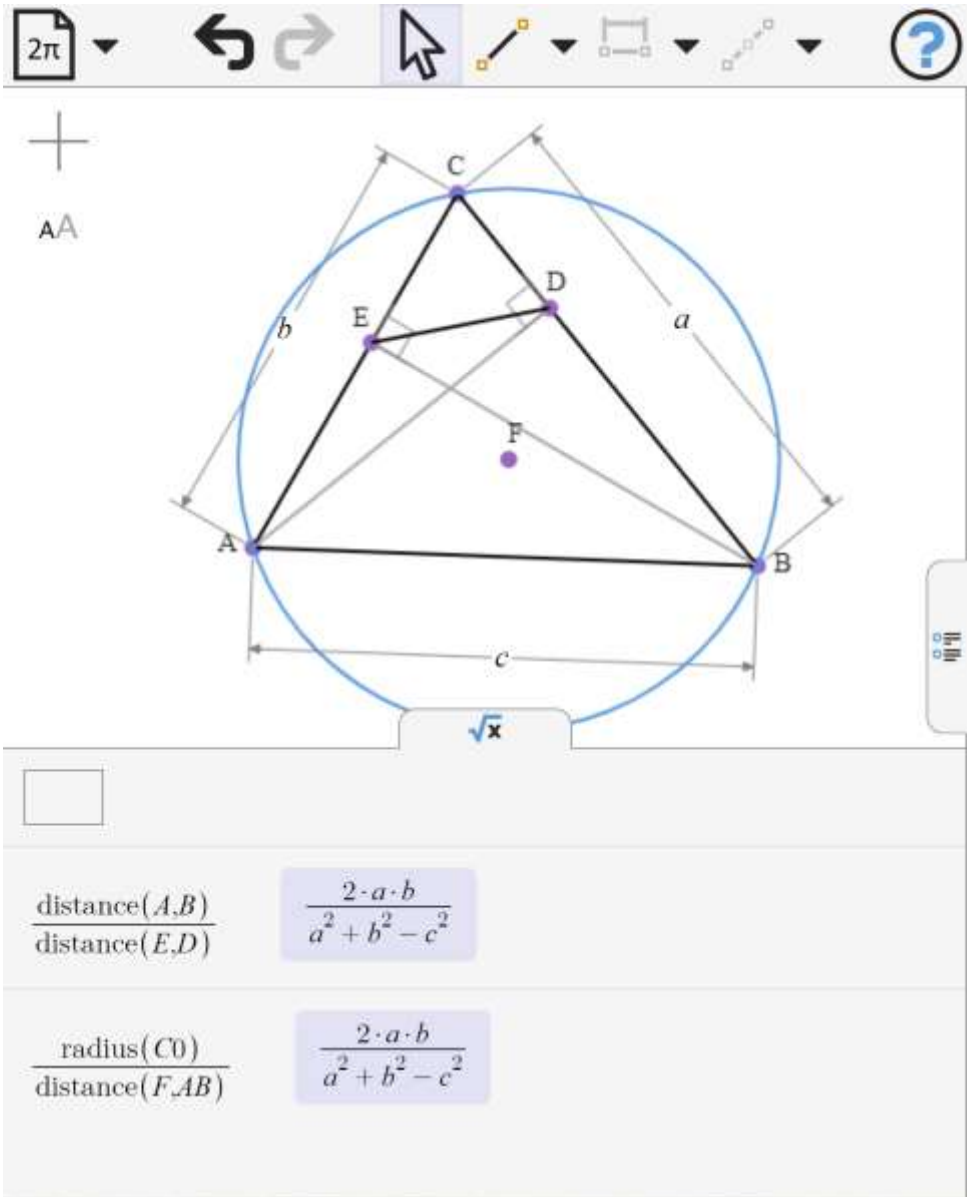
72. Distance to circumcenter and orthocenter

The distance of a side of a triangle from the circumcenter is equal to half the distance of the opposite vertex from the orthocenter.



73. A relationship between the length of a side in the original triangle to distance between the feet of the altitudes

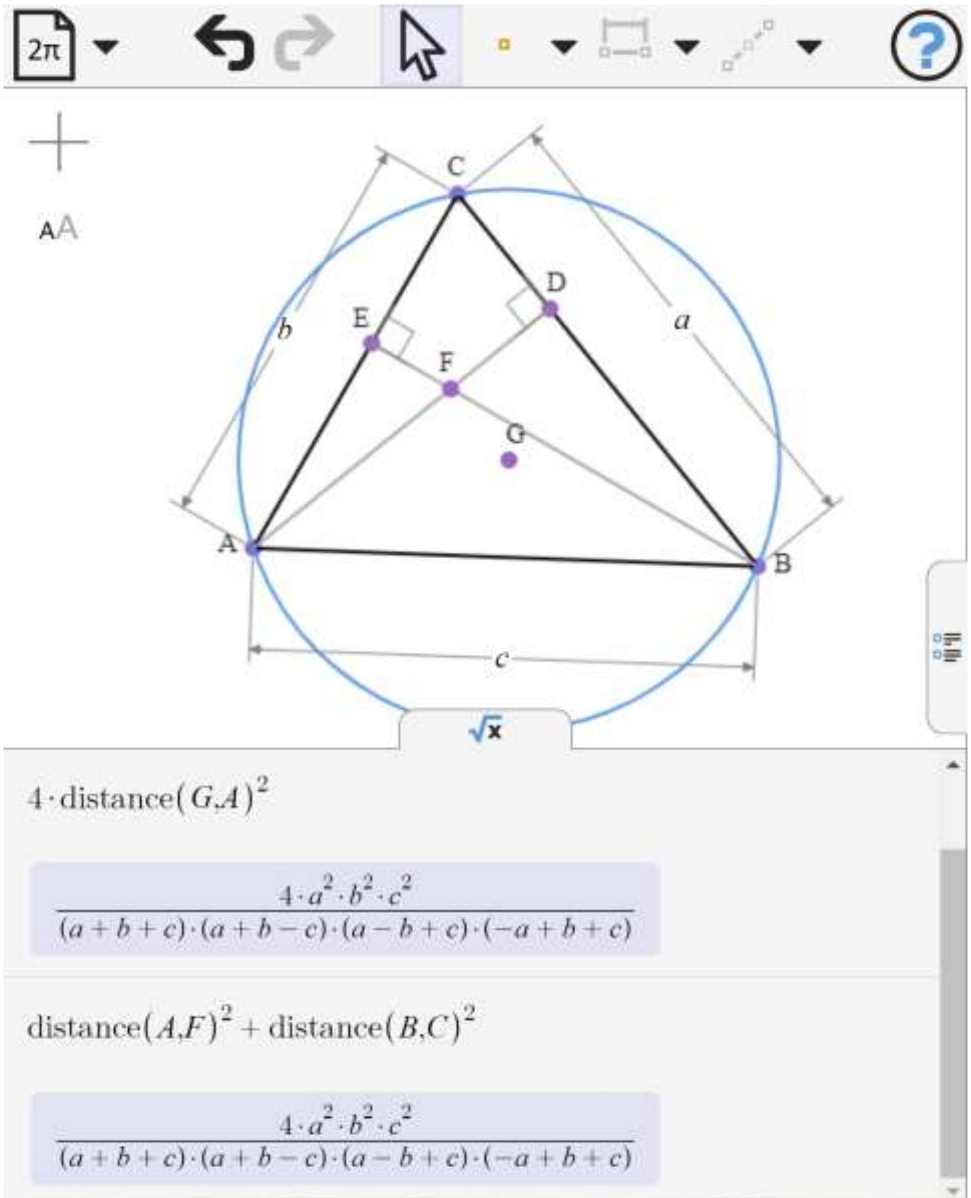
The ratio of a side of a triangle to the corresponding side of the orthic triangle is equal to the ratio of the circumradius to the distance of the side considered from the circumcenter.



74. Distance between orthocenter and circumcenter

F is the orthocenter of triangle ABC and G is the circumcenter.

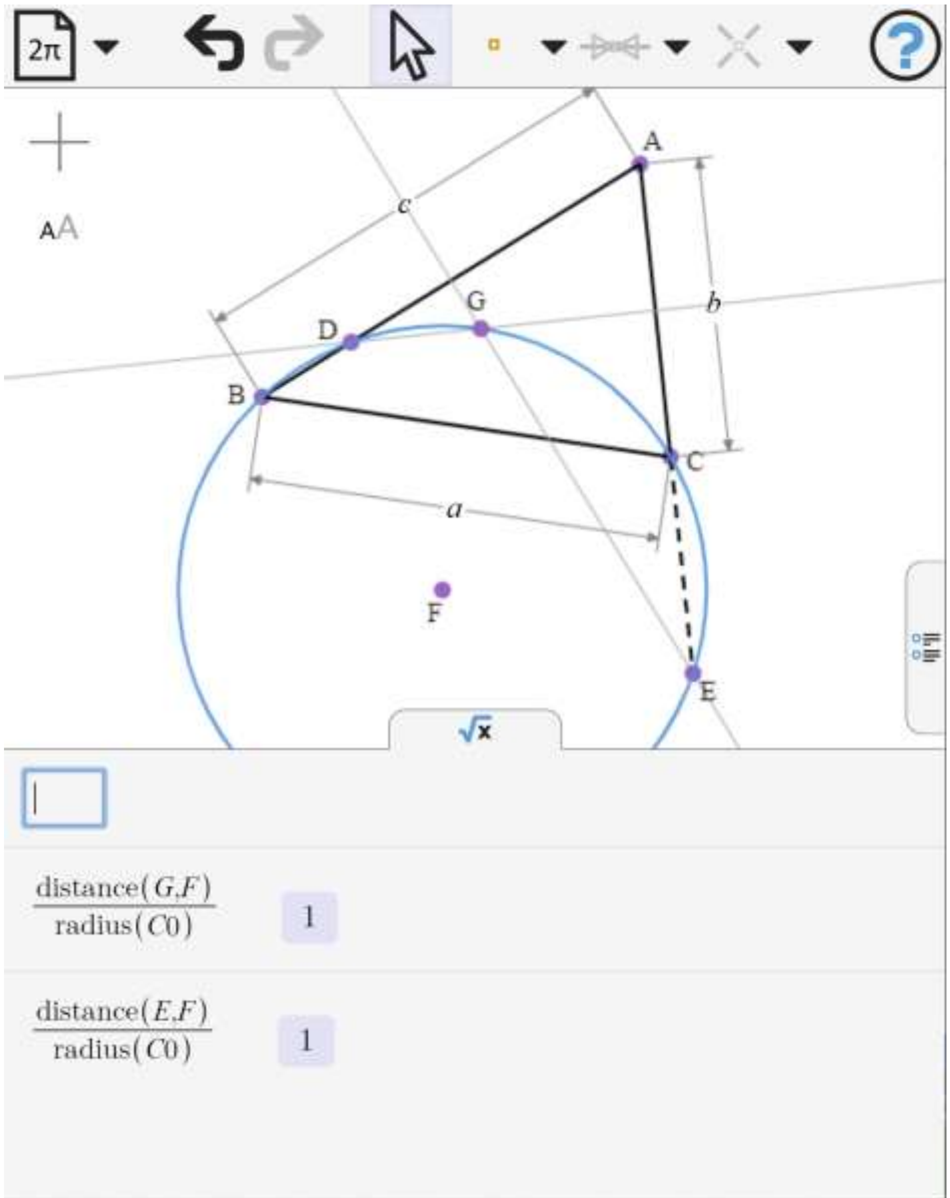
$AF^2+BC^2=4GA^2$



75. Intersections of perpendicular bisectors with neighboring sides

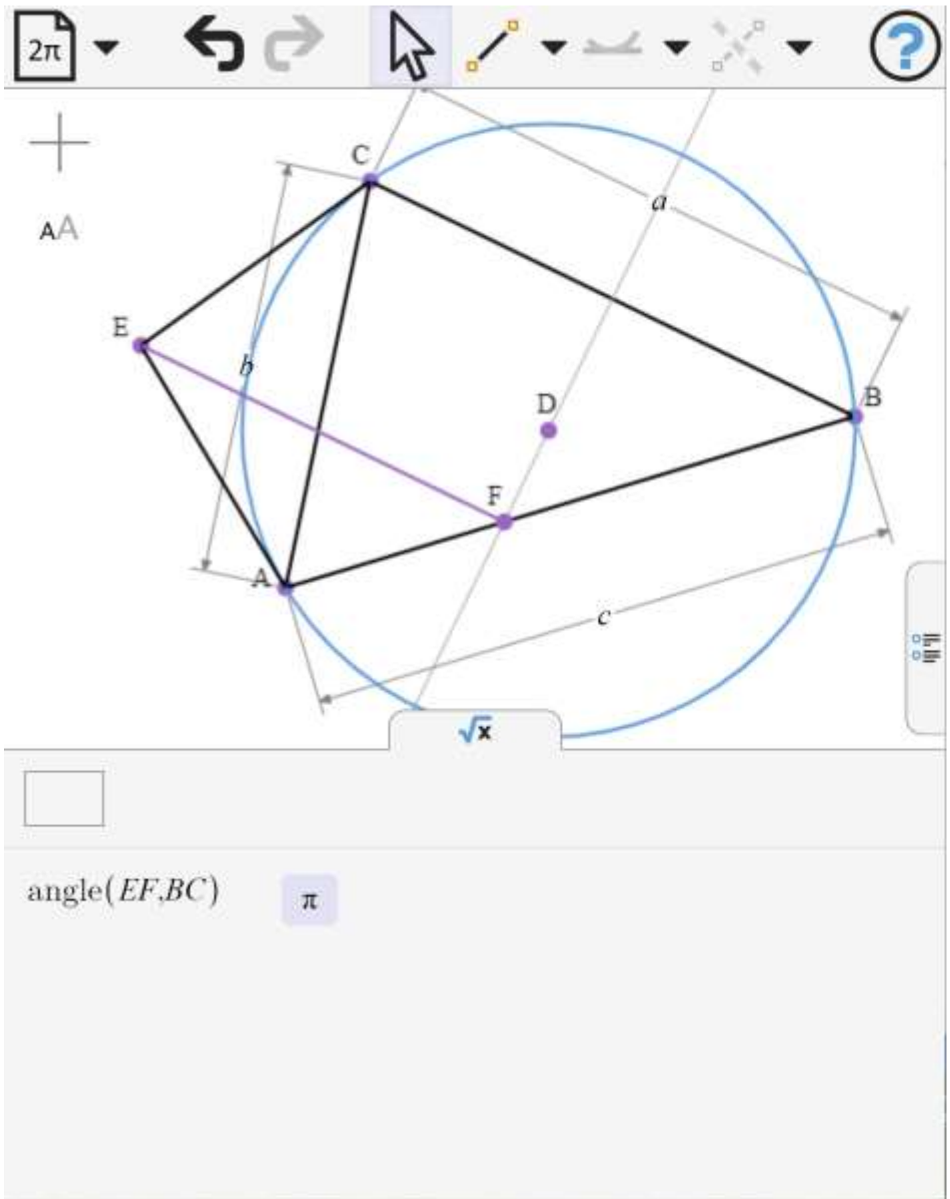
The perpendicular bisectors of the sides AC, AB of the triangle ABC meet the sides AB, AC in D and E. Prove that the points B, C, D, E lie on a circle.

Furthermore the intersection of the perpendicular bisectors lies on the same circle.



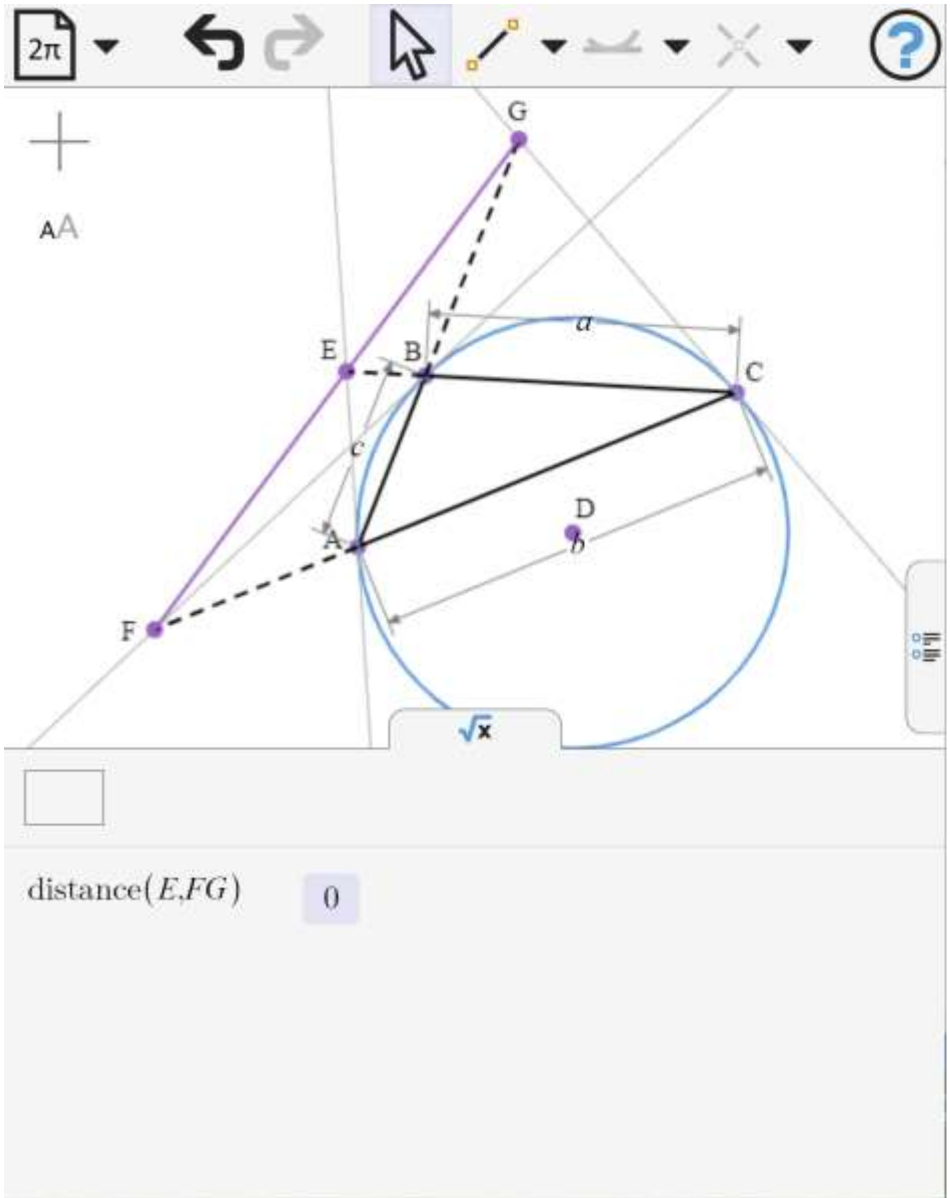
76. Line joining the intersection of two tangents of the circumcircle with the intersection between a perpendicular bisector and the adjacent side

The two tangents to the circumcircle of ABC at A and C meet at E. The perpendicular bisector of BC meets AB at F. Show that EF is parallel to BC.



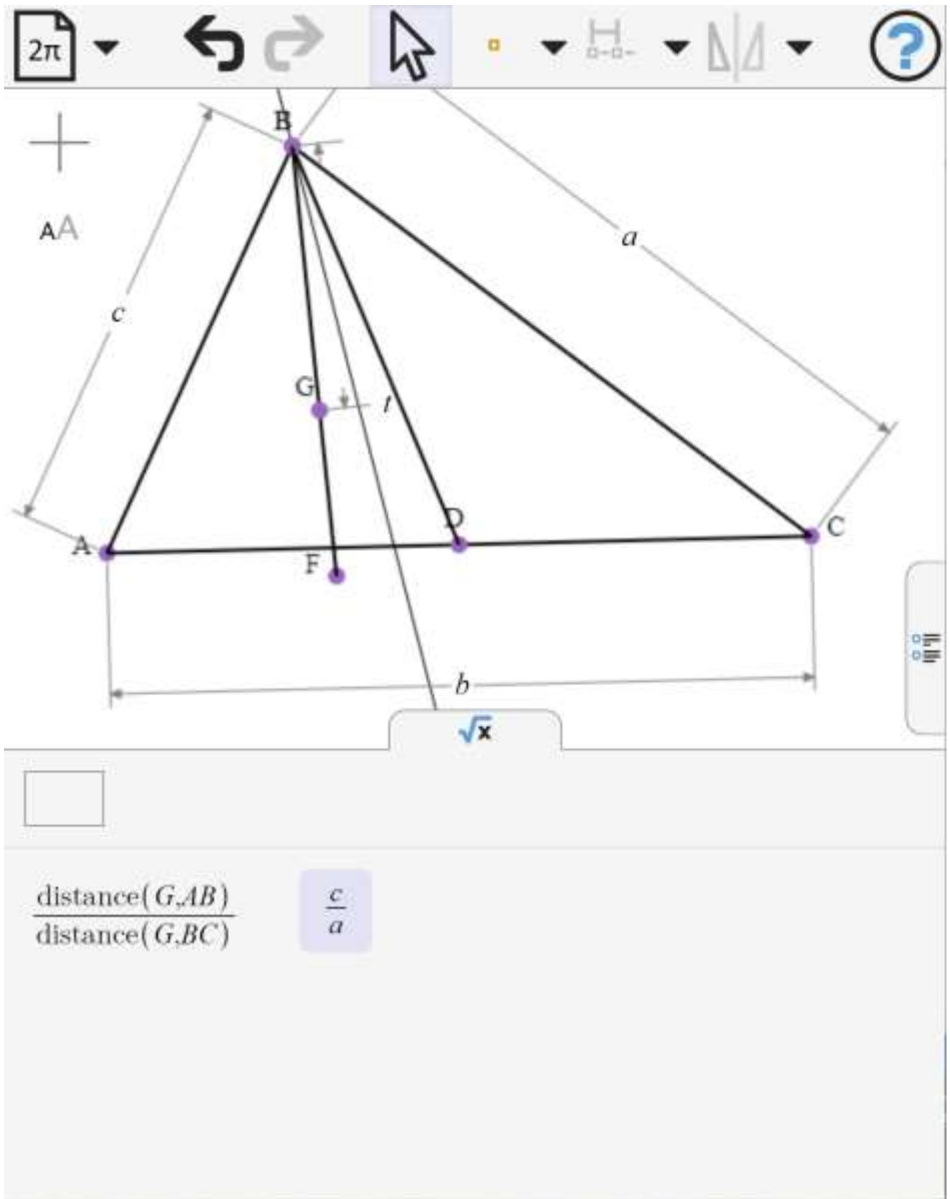
77. Lemoine Axis

The lines tangent to the circumcircle of a triangle at the vertices meet opposite sides in three collinear points (the Lemoine axis of the triangle).



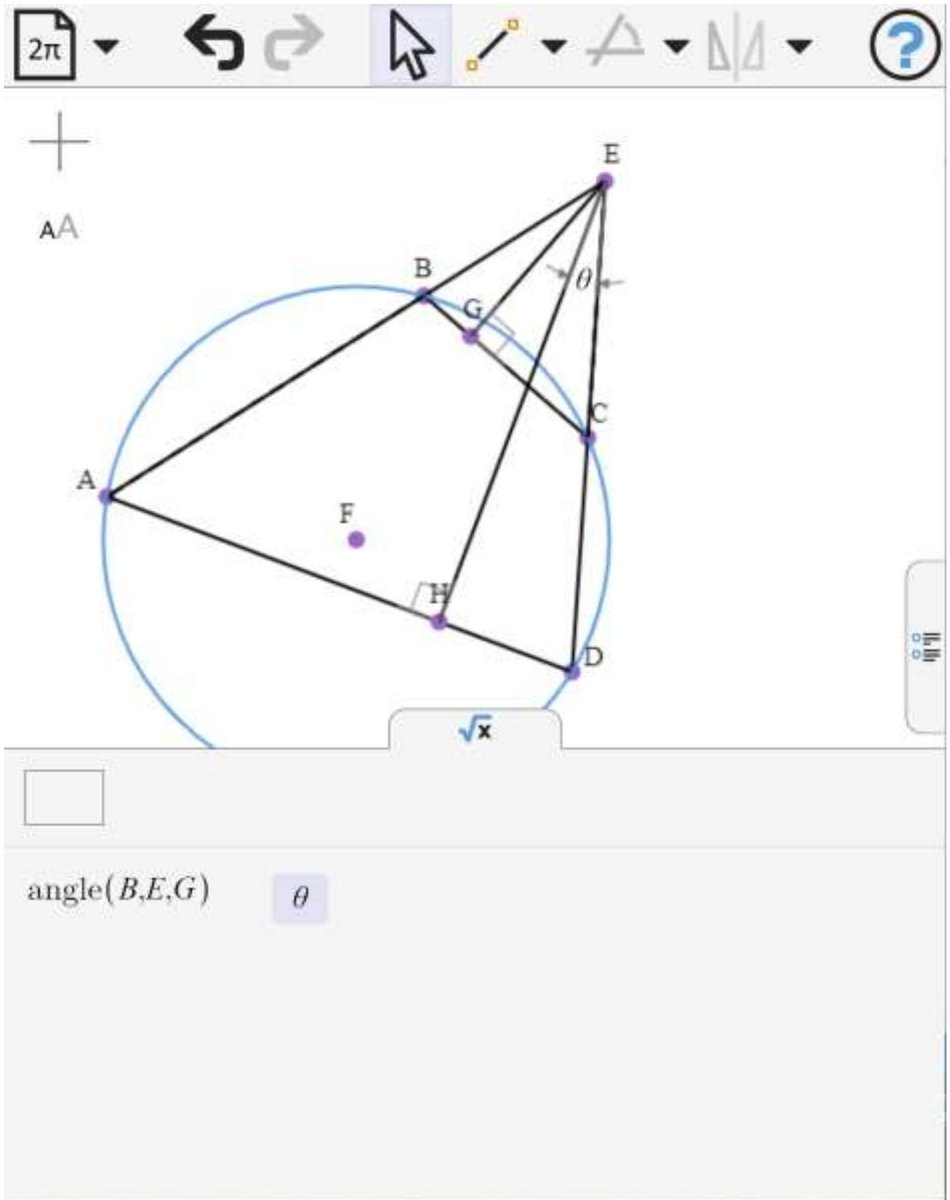
78. Point on the symmedian

The distances from a point on the symmedian of a triangle to the two including sides are proportional to those sides. (The symmedian is the image of the median under reflection in the angle bisector.)



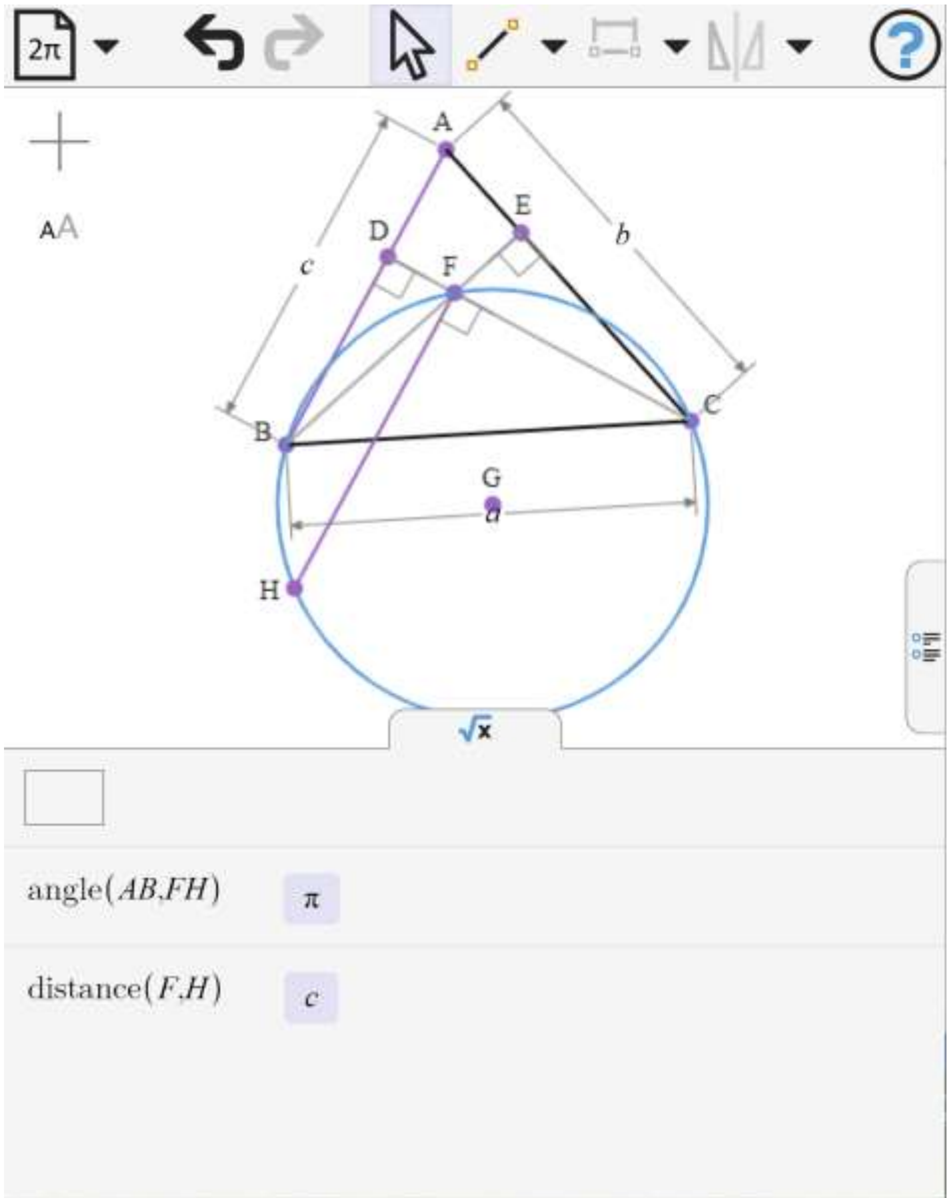
79. Projections onto opposite sides of a cyclic quadrilateral

Let ABCD be a cyclic quadrilateral and E the intersection of sides AB and BC. Let G and H be the projections of E onto BC and AD. Then angles BEG and HED are equal.



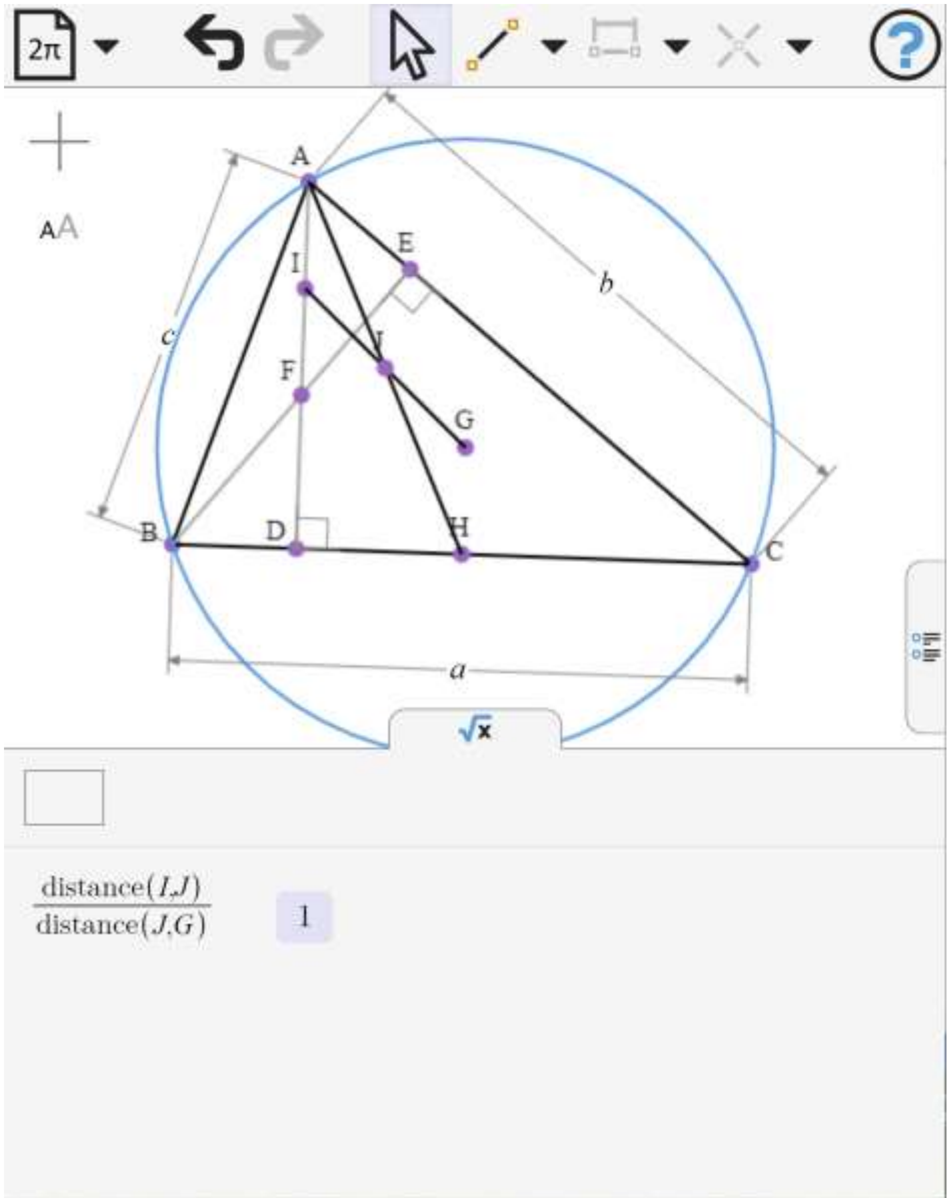
80. A parallelogram defined by the circumcircle of two vertices and the orthocenter of a triangle

The perpendicular at the orthocenter F to the altitude FC of the triangle ABC meets the circumcircle of FBC in H. Show that ABHF is a parallelogram.



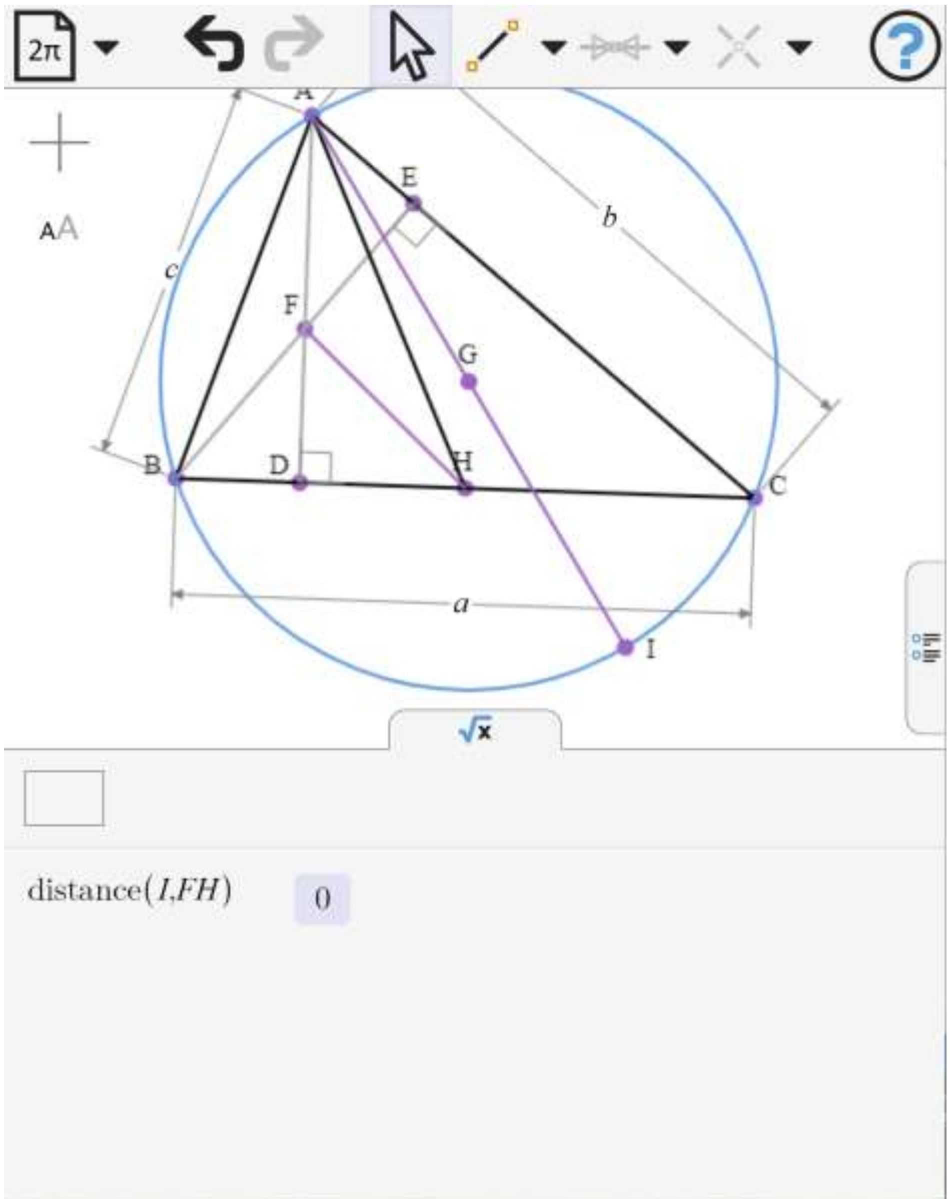
81. Line defined by the orthocenter and the circumcenter is bisected by the median

Let F be the orthocenter of triangle ABC and G its circumcenter. Let I be the midpoint of AF. Show that the segment IG is bisected by the median AH.



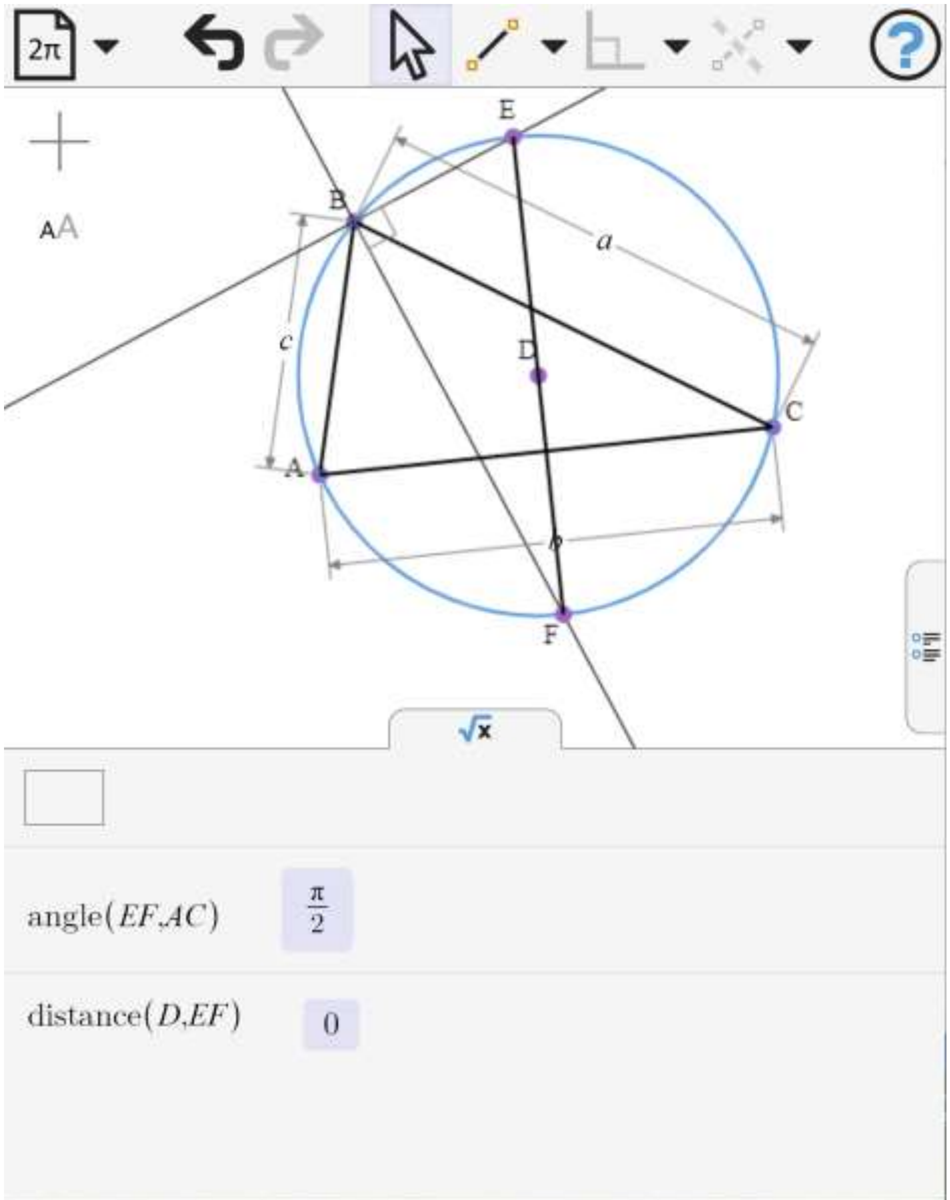
82. Line joining the orthocenter with the midpoint of a side

Prove that FH (see above) passes through the diametric opposite of A on the circumcircle



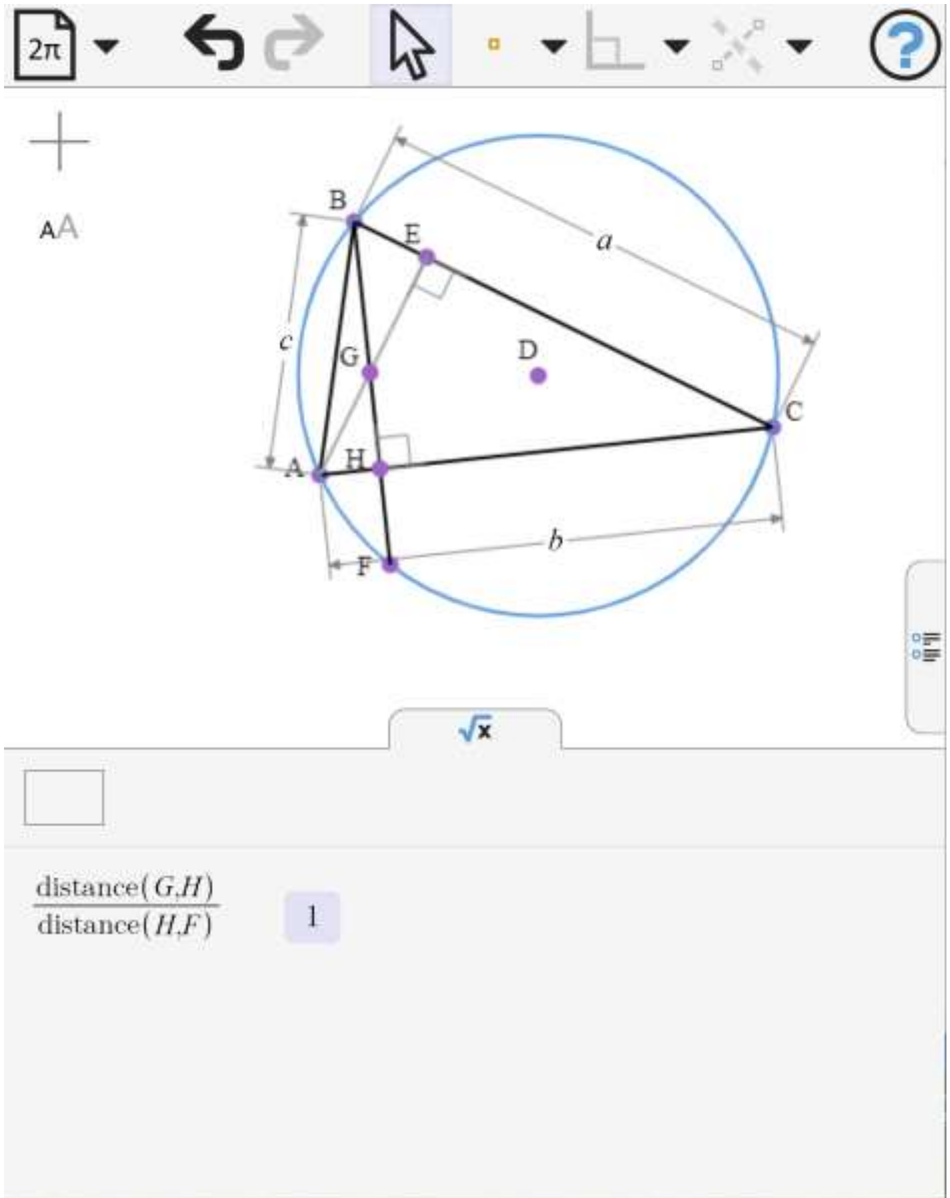
83. Internal and external angle bisectors

The internal and external bisectors of an angle of a triangle pass through the ends of the circumdiameter which is perpendicular to the side opposite the vertex considered



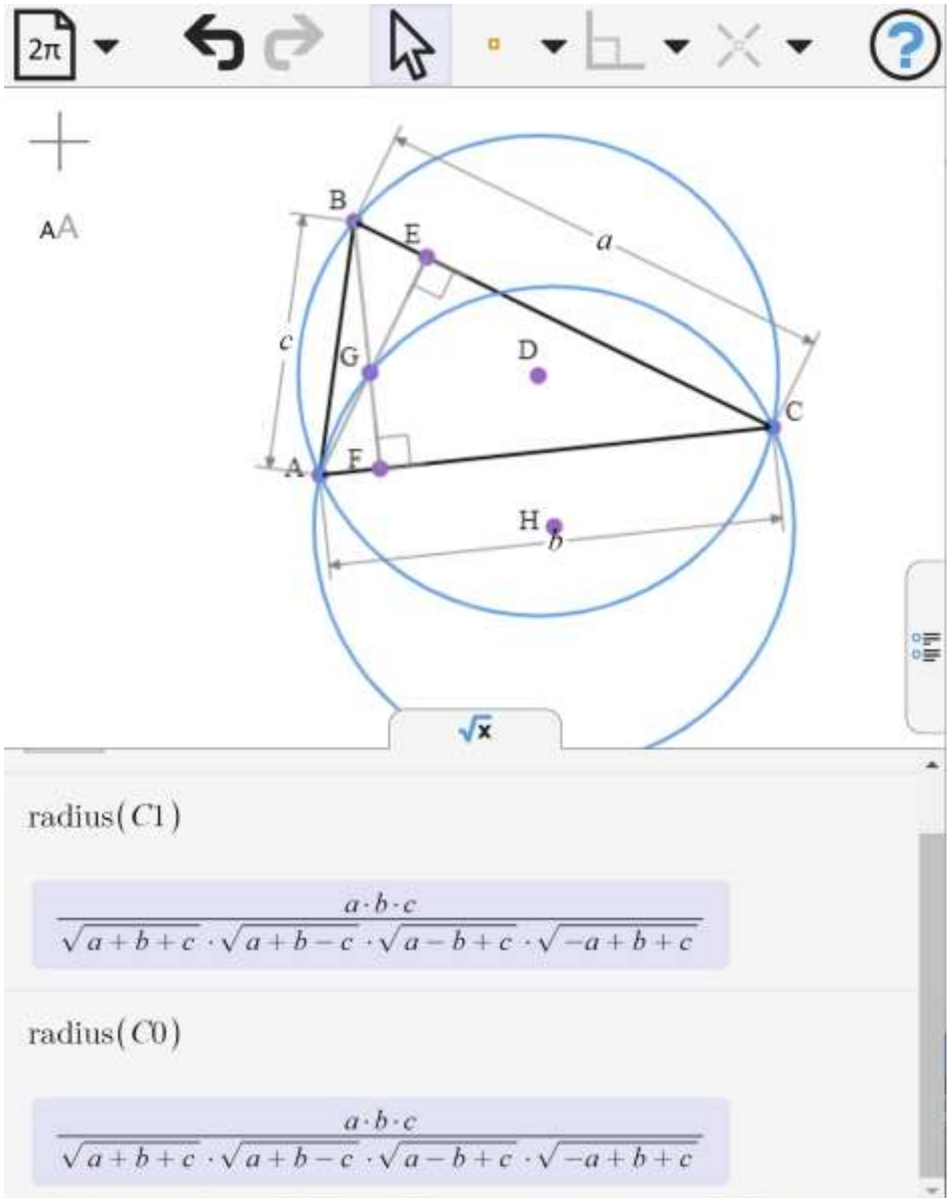
84. An altitude intersecting the circumcircle

The segment of the altitude extended between the orthocenter and the second point of intersection with the circumcircle is bisected by the corresponding side of the triangle



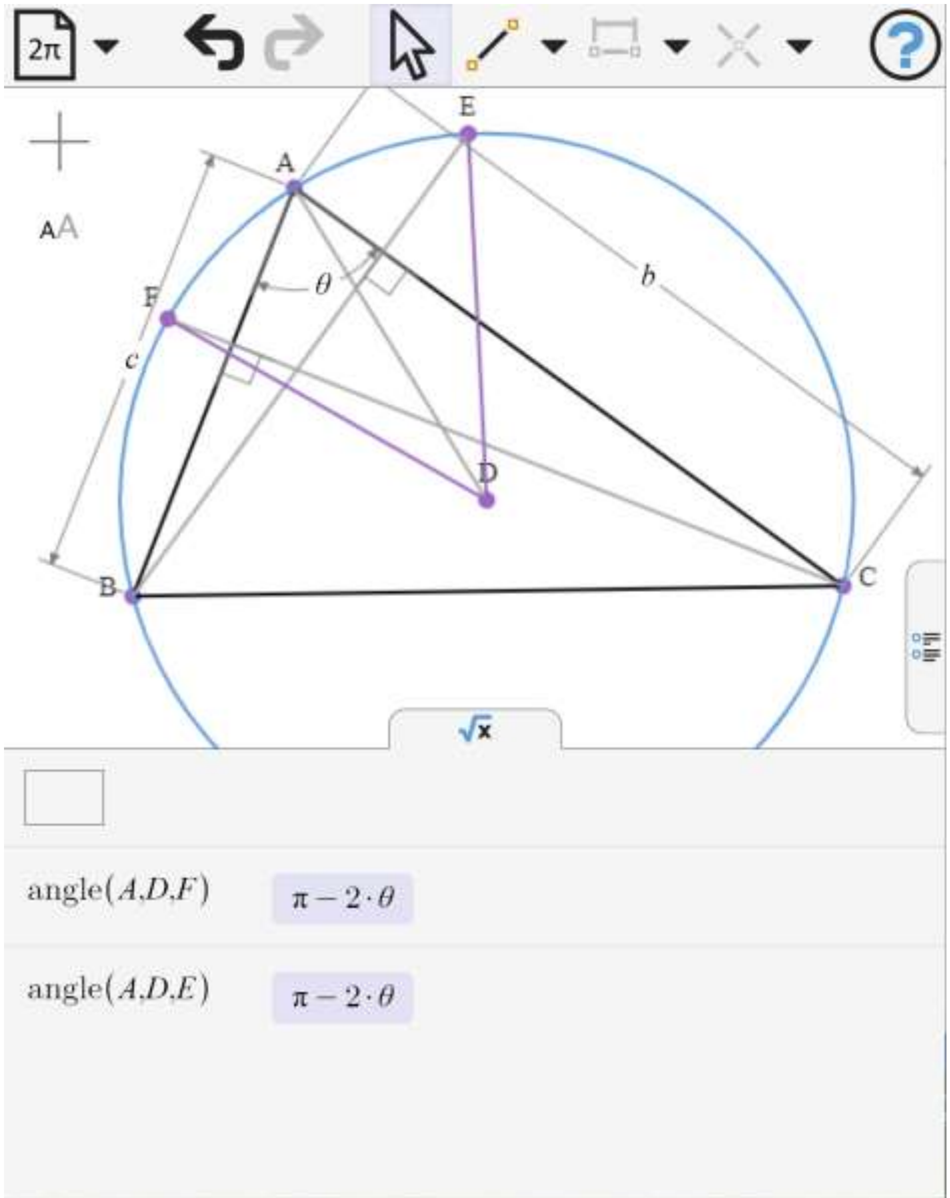
85. Circumcircle of two points and the orthocenter

The circumcircle of the triangle formed by two vertices and the orthocenter of a given triangle is equal to the circumcircle of the given triangle



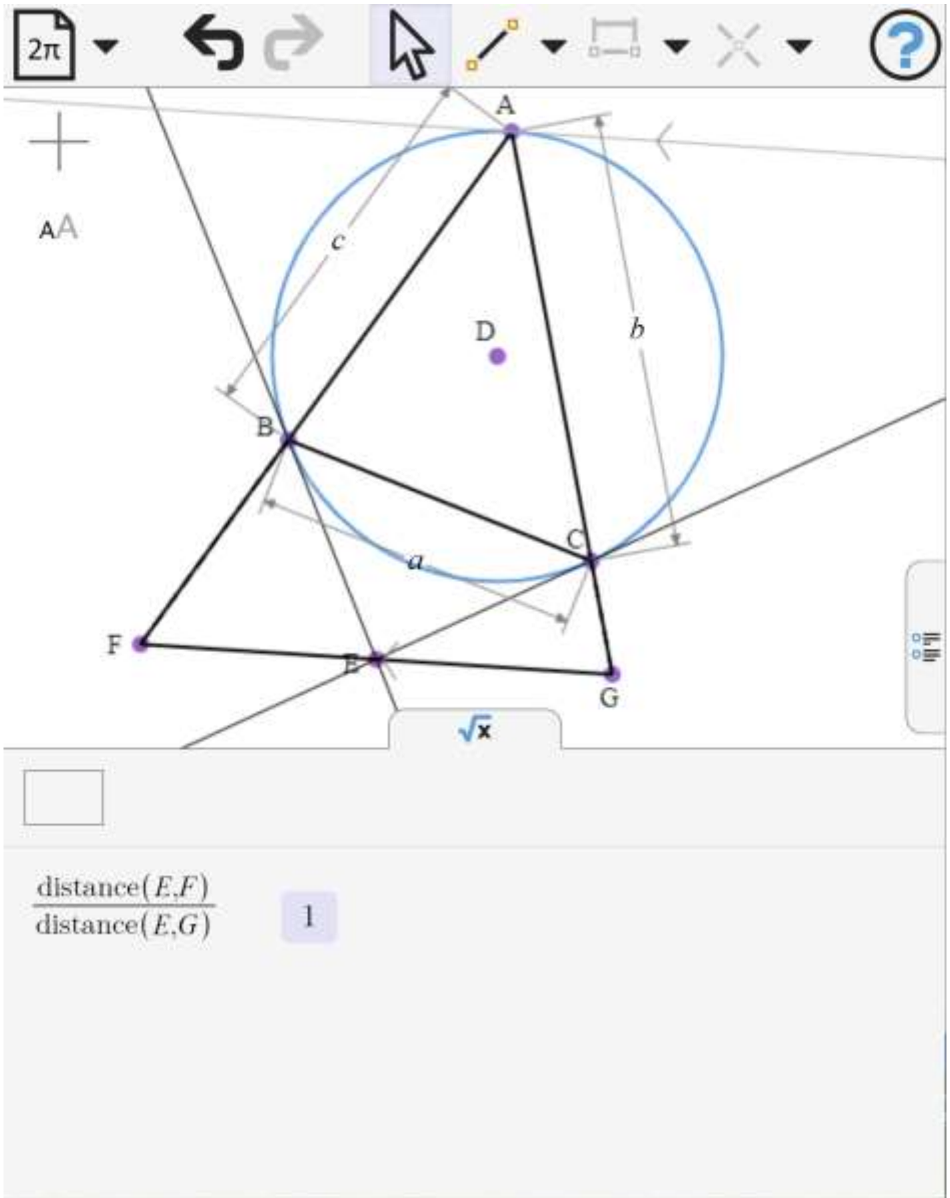
86. Arc defined by intersection of altitudes with circumcircle

A vertex of a triangle is the midpoint of the arc determined on its circumcircle by the two altitudes, produced, issued from the two other vertices



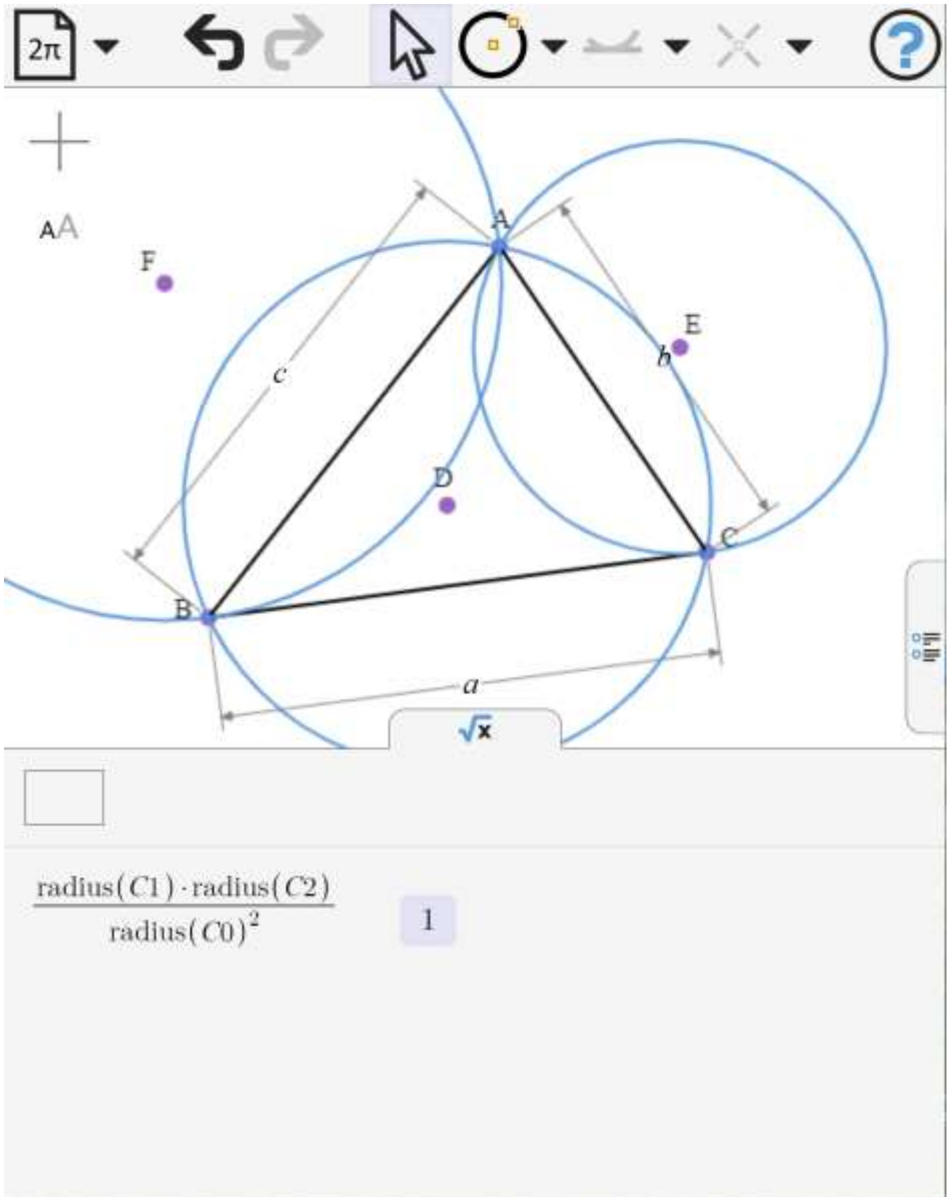
87. A line parallel to one circumcircle tangent through the intersection of the others

Through the point of intersection of the tangents DB, DC to the circumcircle (O) of the triangle ABC a parallel is drawn to the line touching (O) at A. If this parallel meets AB, AC in E, F show that D bisects EF.



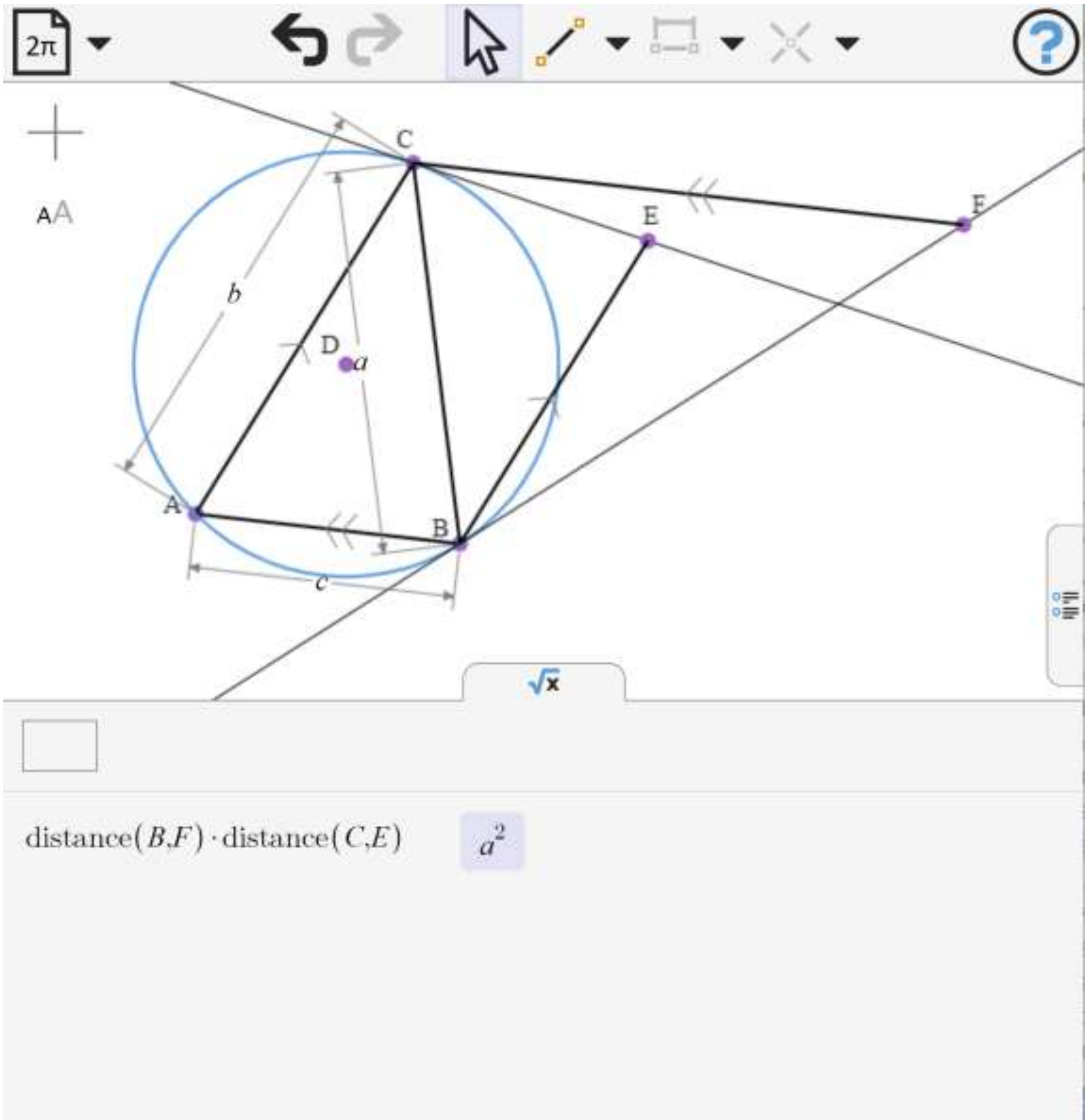
88. Relationship between mix-linear circles and circumcircle

In a triangle ABC let p and q be the radii of two circles through A touching side BC at B and C respectively. Then p.q=R^2 (where R is the circumradius).



89. Intersections between parallels and tangents

The parallel to the side AC through the vertex B of the triangle ABC meets the tangent to the circumcircle (O) of ABC at C in B1, and the parallel through C to AB meets the tangent to (O) at B in C1. Prove that BC^2=BC1.B1C

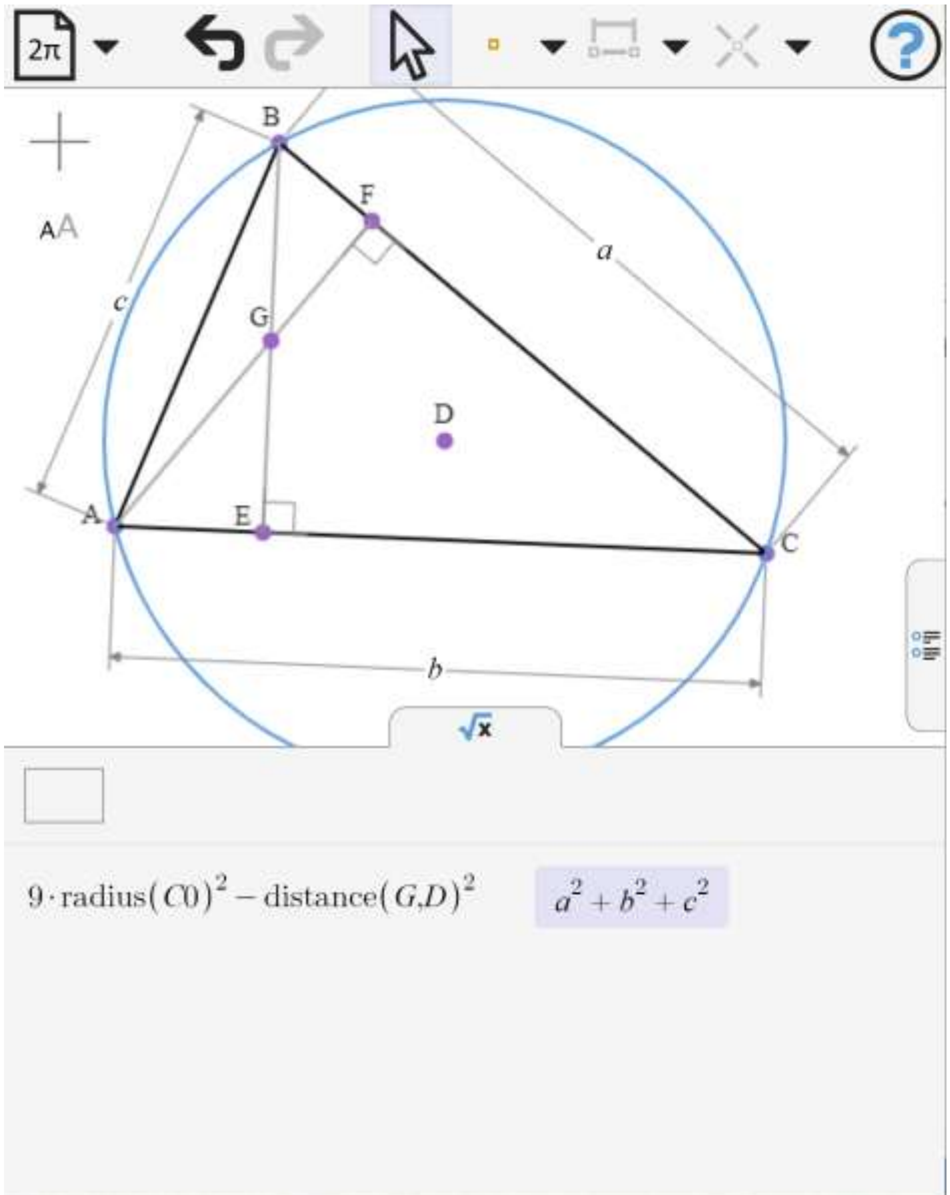


The Euler Line

The circumcenter O, orthocenter H and centroid G of a given triangle are collinear and the line is called the Euler Line of the triangle.

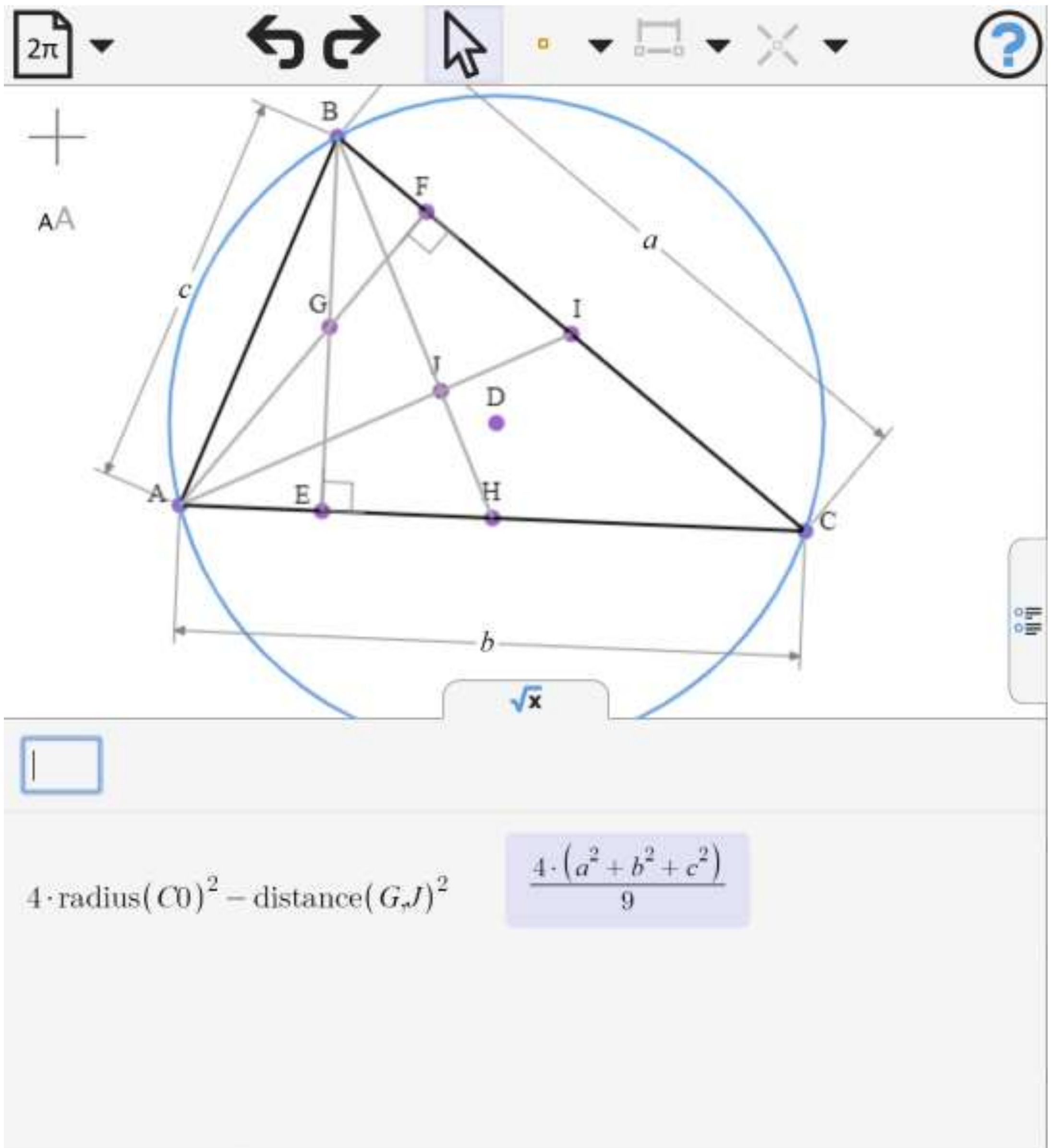
90. Distance between orthocenter and circumcenter

Let D and G be the circumcenter and orthocenter of a triangle ABC. Let R be the radius of the circumcircle. Show that $DG^2=9R^2-a^2-b^2-c^2$



91. Distance between orthocenter and centroid

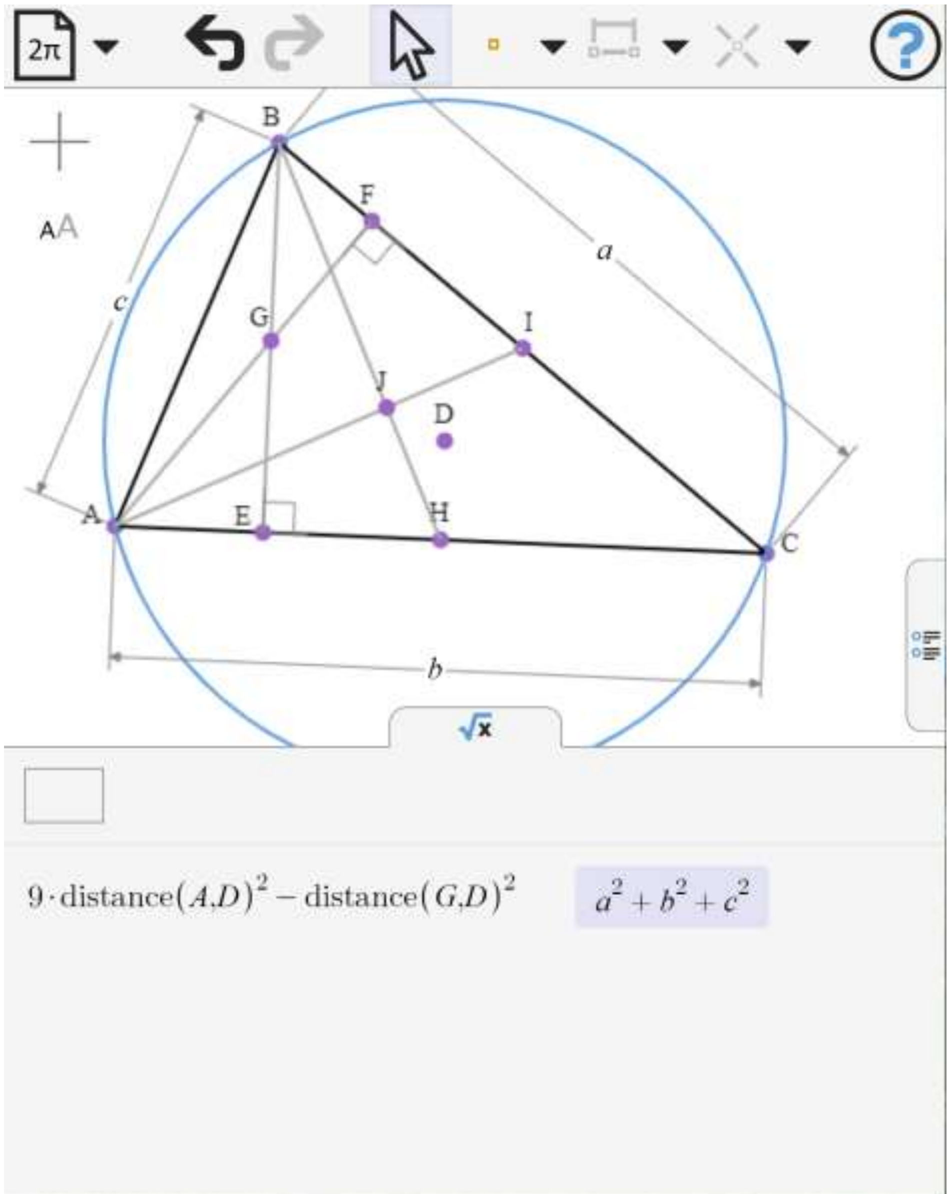
With the same notations for the triangle ABC we have: $4AD^2 = \frac{4}{9}AB^2 + \frac{4}{9}AC^2 + \frac{4}{9}BC^2 + GH^2$



92. Circumradius related to side lengths and distance between orthocenter and circumcenter

With the usual notations for the triangle ABC we have:

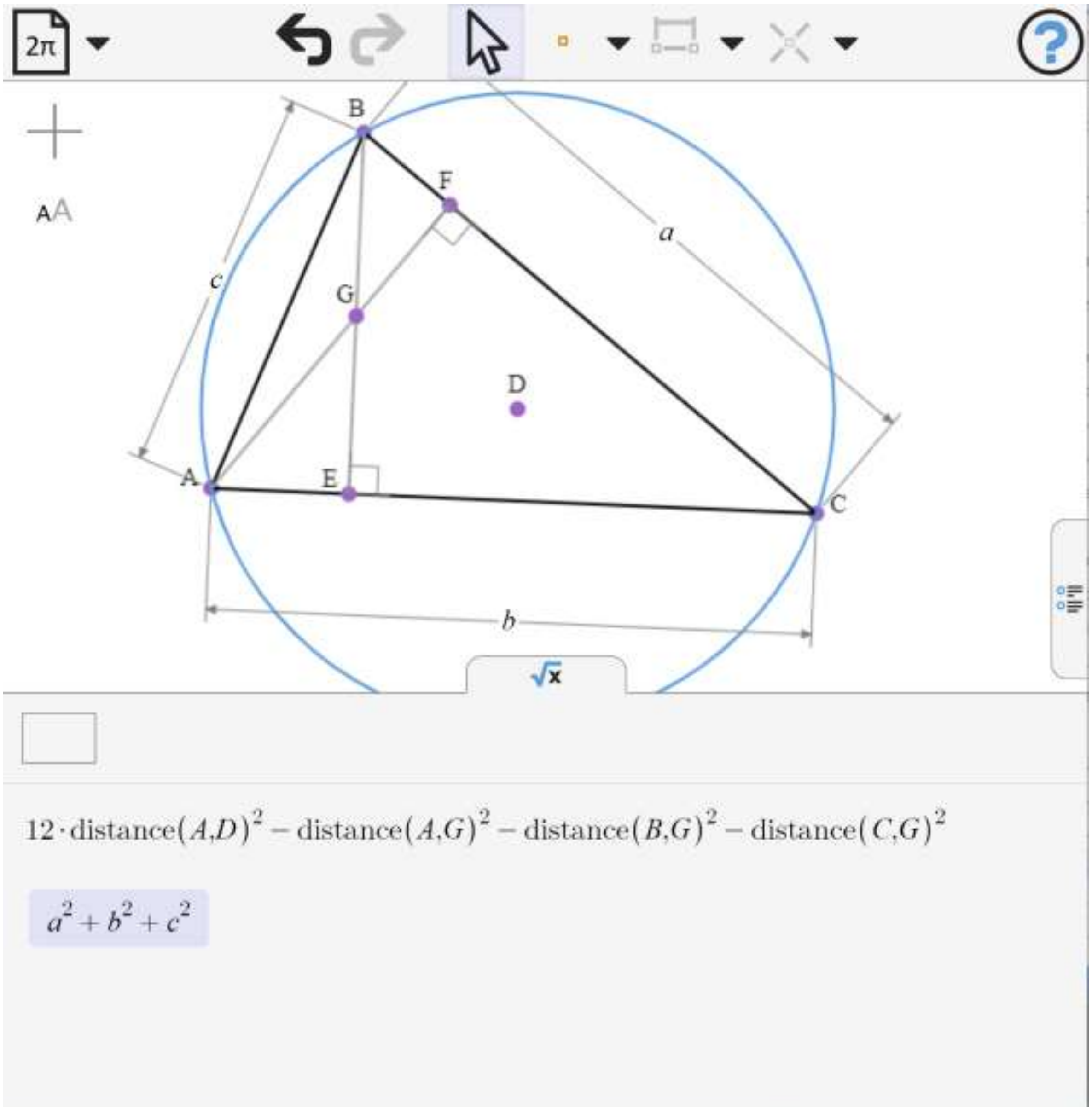
$9AD^2 = AB^2 + AC^2 + BC^2 + GD^2$



93. Circumradius in terms of distances between vertices and orthocenter

With the notation of the previous example,

$12AD^2 = AB^2+AC^2+BC^2+AG^2+BG^2+CG^2$

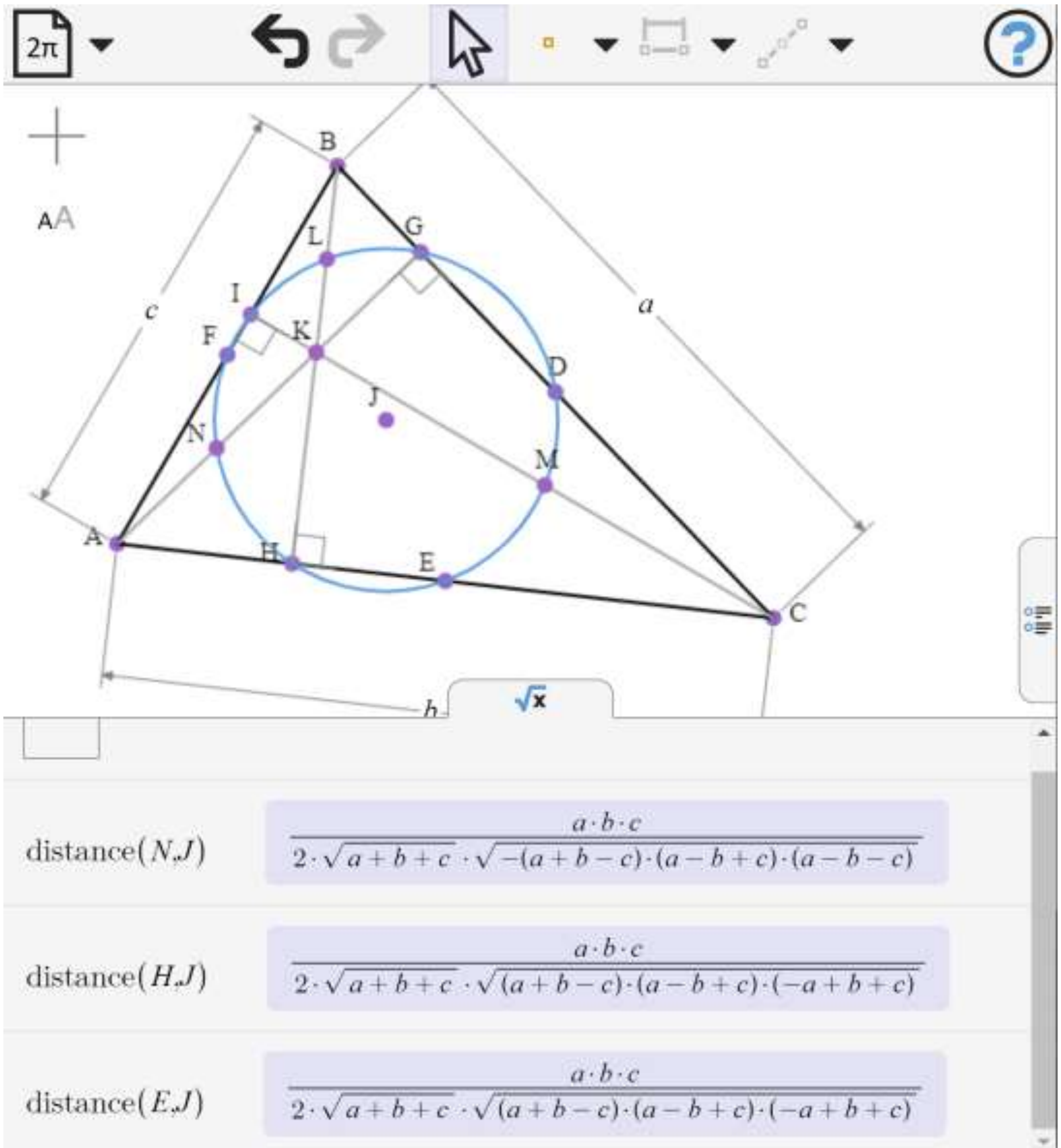


The Nine Point Circle

The midpoints of the segments joining the orthocenter of a triangle to its vertices are called the Euler Points of the triangle. The three Euler Points determine the Euler Triangle.

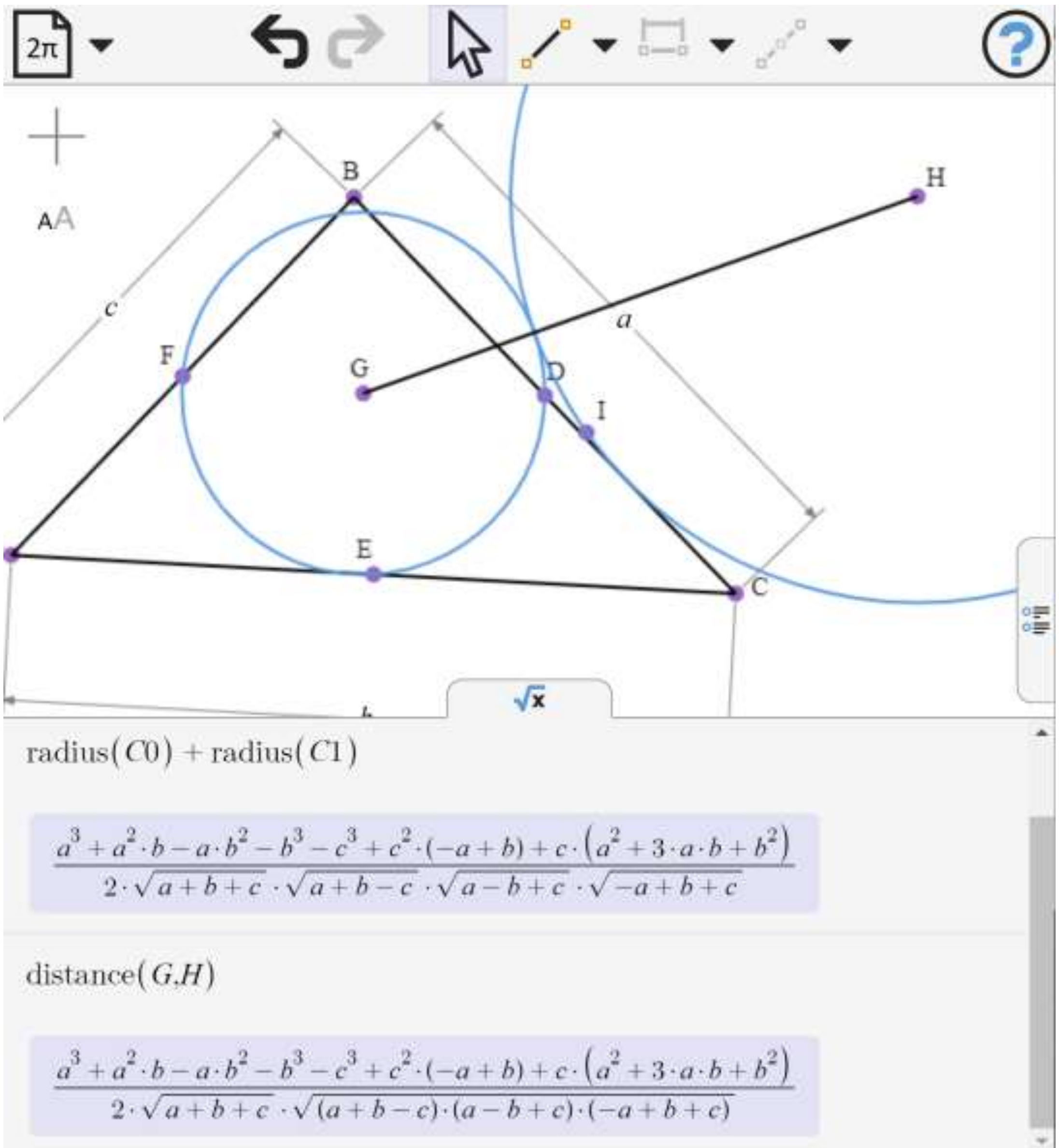
94. The Nine Point Circle Theorem

In a triangle, the midpoints of the sides, the feet of the altitudes and the Euler Points lie on the same circle. (The Euler Points are the midpoints of the segments joining the vertices to the orthocenter).



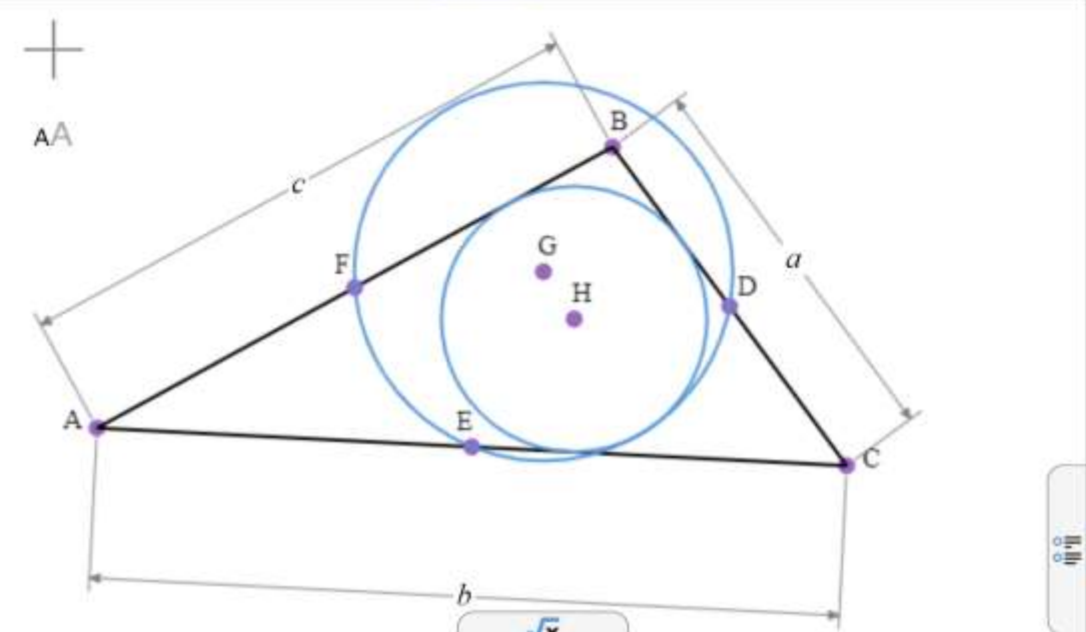
95. Feuerbach’s Theorem

The nine-point circle of a triangle touches each of the four tritangent circles of the triangle.



For an excircle, we show that the distance between the circle centers is the sum of their radii.

For the incircle, we show that the distance between the circle centers is the difference in the radii.



radius$(C0)$ − radius$(C1)$

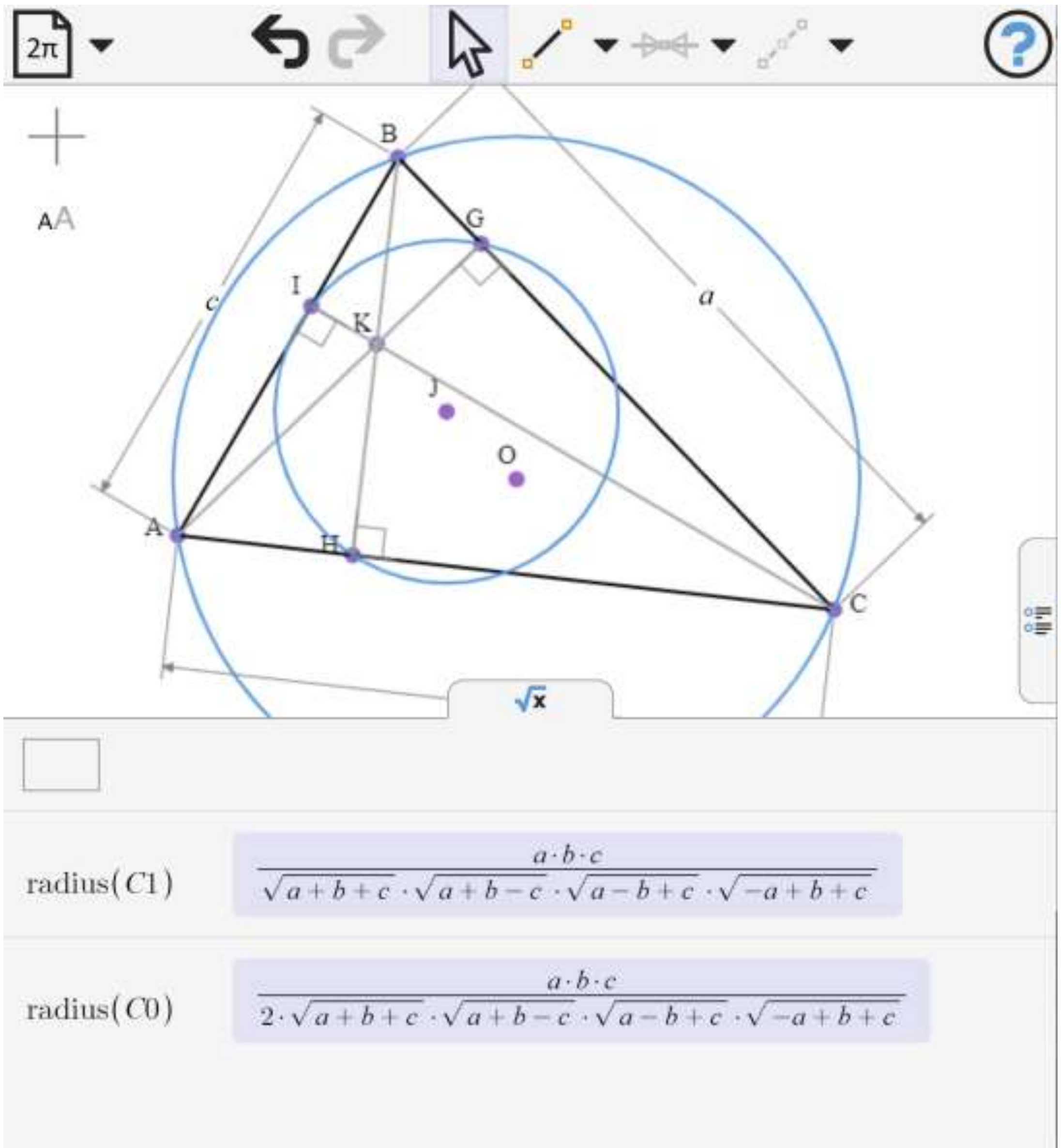
$$\frac{a^3 - a^2 \cdot b - a \cdot b^2 + b^3 + c^3 + c^2 \cdot (-a-b) + c \cdot \left(-a^2 + 3 \cdot a \cdot b - b^2\right)}{2 \cdot \sqrt{a+b+c} \cdot \sqrt{a+b-c} \cdot \sqrt{a-b+c} \cdot \sqrt{-a+b+c}}$$

distance(G,H)

$$\frac{a^3 - a^2 \cdot b - a \cdot b^2 + b^3 + c^3 + c^2 \cdot (-a-b) + c \cdot \left(-a^2 + 3 \cdot a \cdot b - b^2\right)}{2 \cdot \sqrt{a+b+c} \cdot \sqrt{(a+b-c) \cdot (a-b+c) \cdot (-a+b+c)}}$$

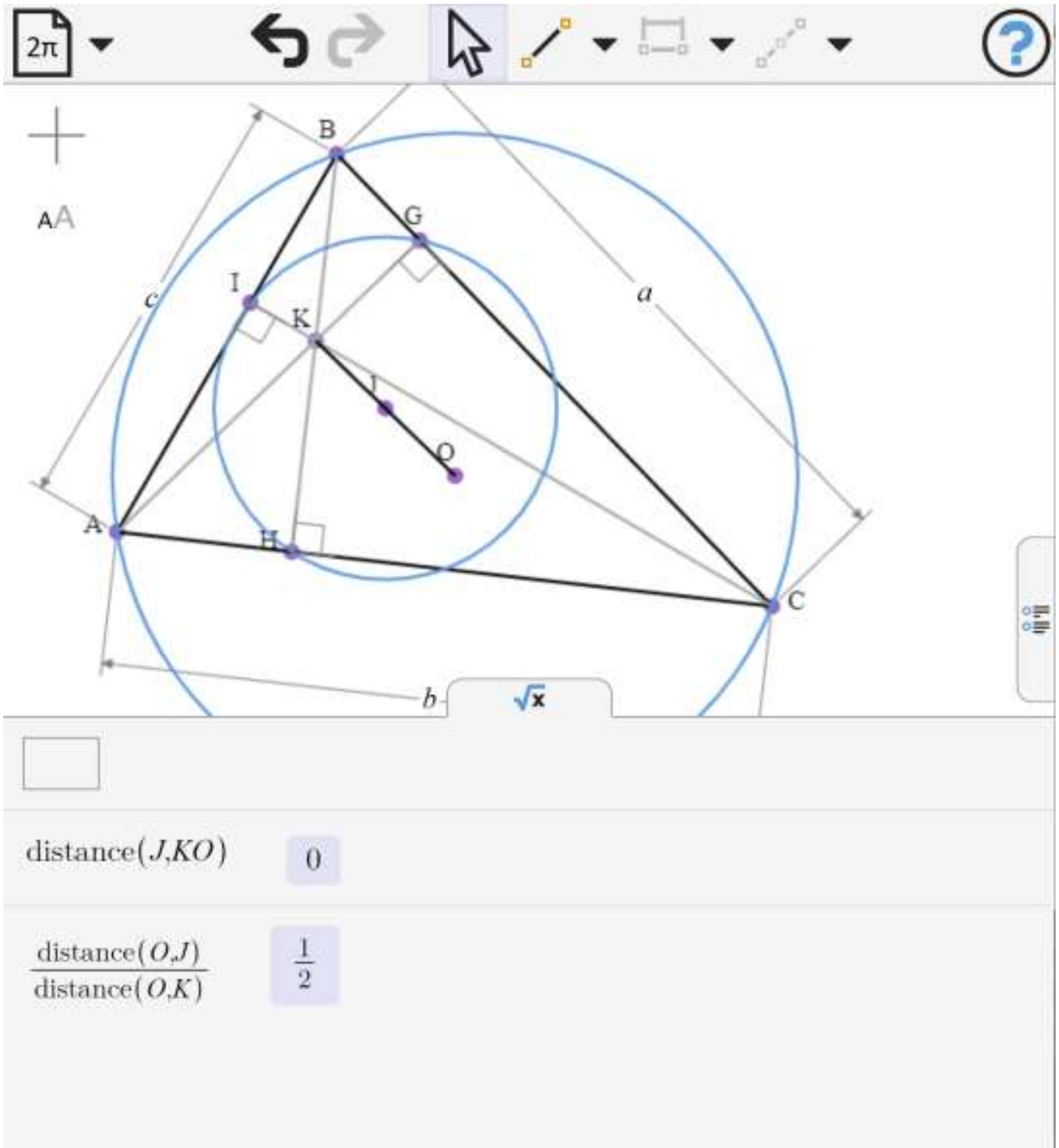
96. Nine Point Circle and Circumcircle

The radius of the nine-point circle is equal to half the circumradius of the triangle



97. Nine Point circle center

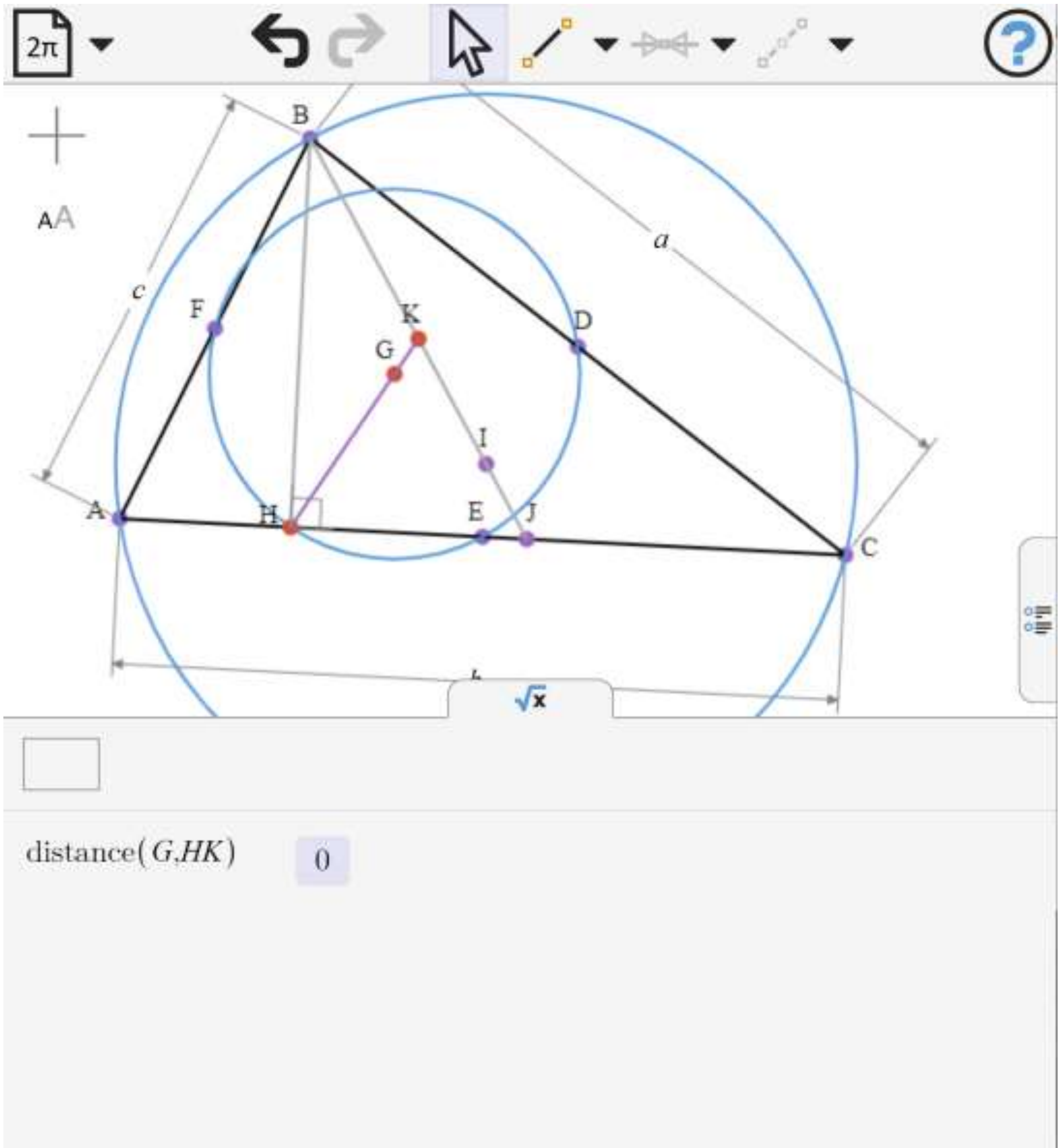
The nine-point circle center lies on the Euler line midway between the circumcenter and the orthocenter.



We show that J lies on HO and that OK = 2.OJ.

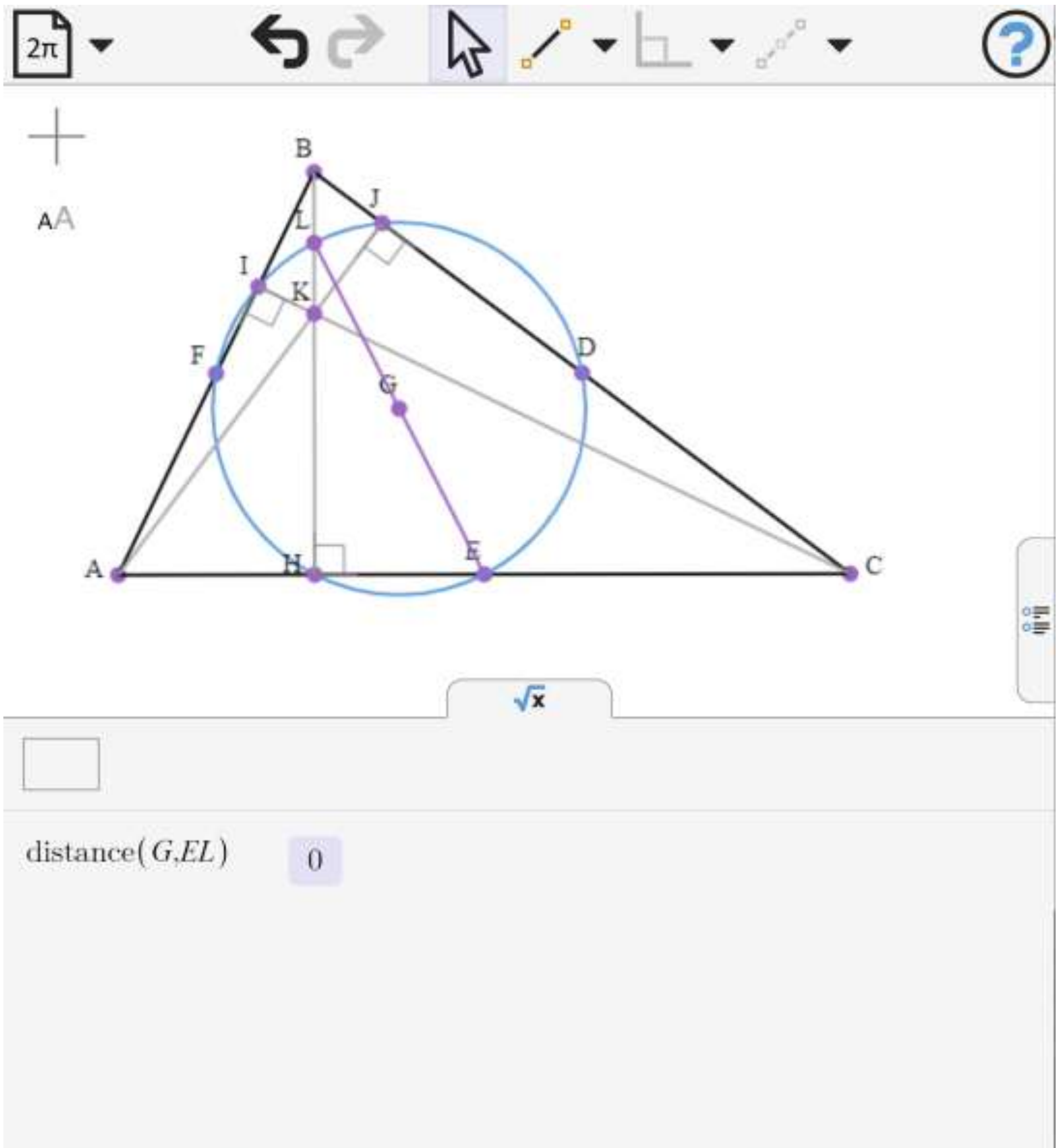
98. Nine point circle center collinearity

Show that the foot of the altitude of a triangle on a side, the midpoint of the segment of the circumdiameter between this side and the opposite vertex and the nine point center are collinear



99. Nine point circle center halfway between Euler point and opposite side midpoint

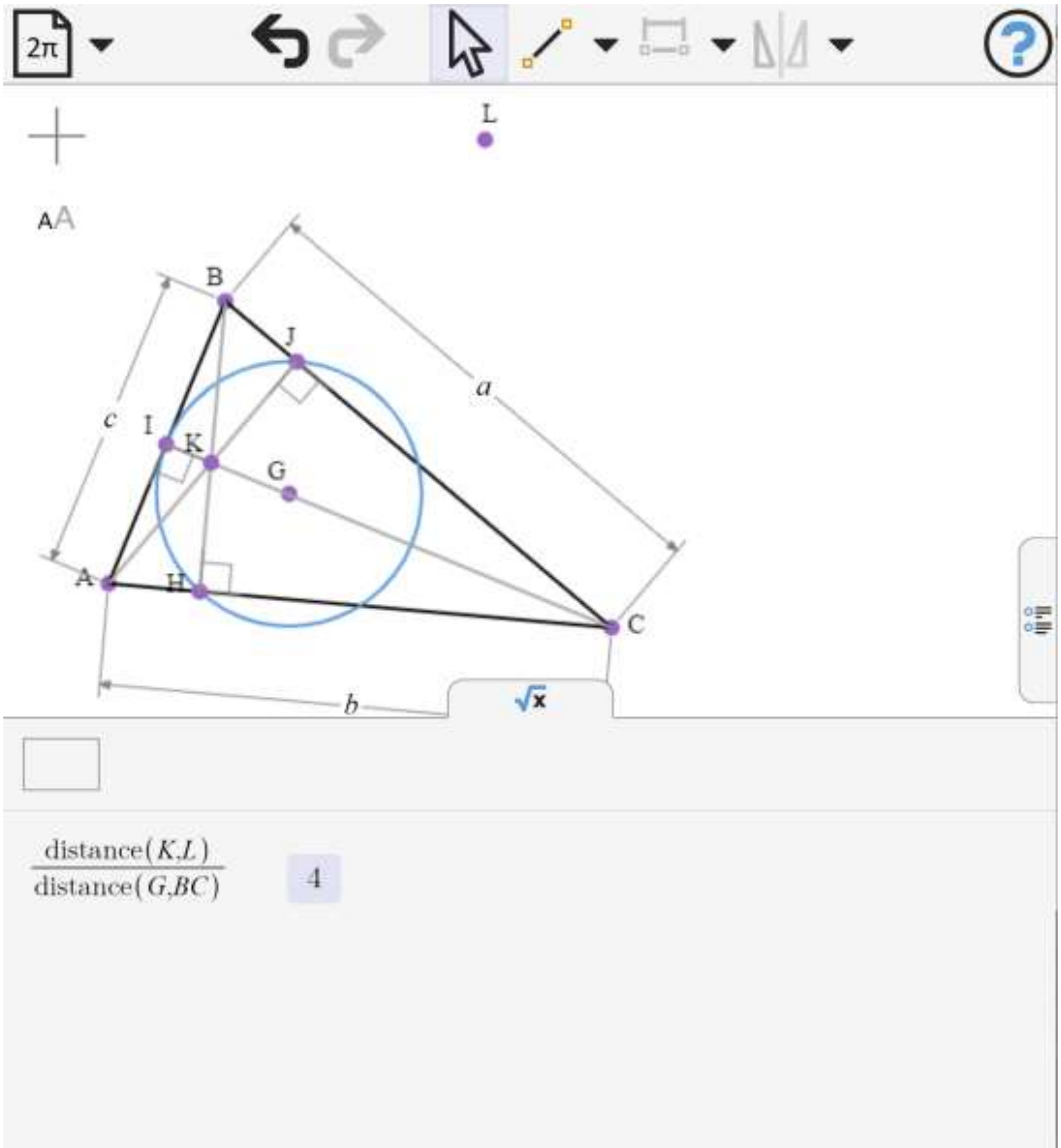
The center of the nine-point circle is the midpoint of a Euler point and the midpoint of the opposite side.



Both points lie on the 9-point circle, so all we need to show is that the center lies on the chord between the points.

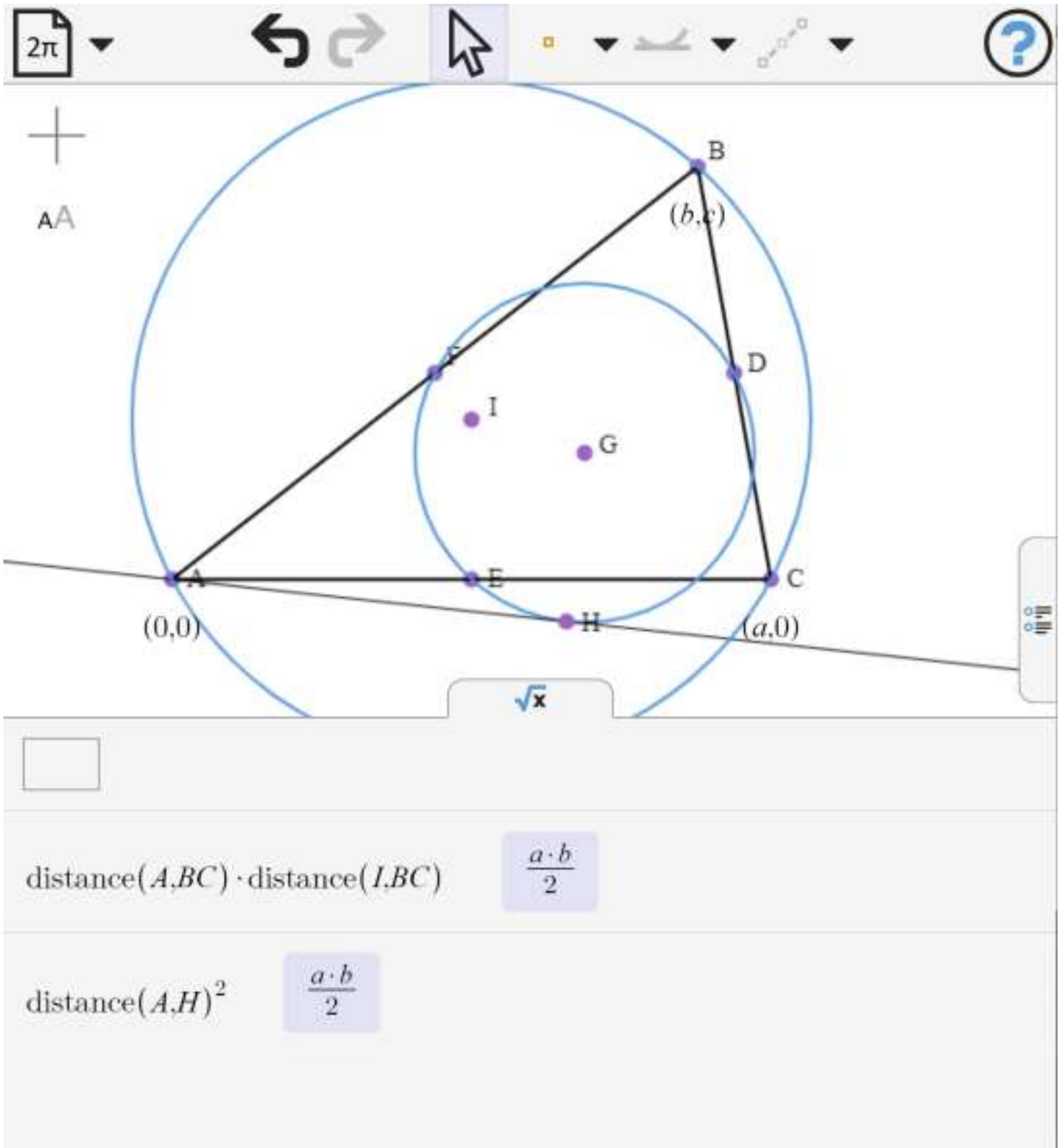
100. Distance between the reflection of a vertex in the opposite side of a triangle and its orthocenter

Let K be the orthocenter of triangle ABC, and G the center of the nine point circle. If L is the reflection of the vertex A in the opposite side BC, show that KL is equal to 4 times the distance of the nine-point center from BC



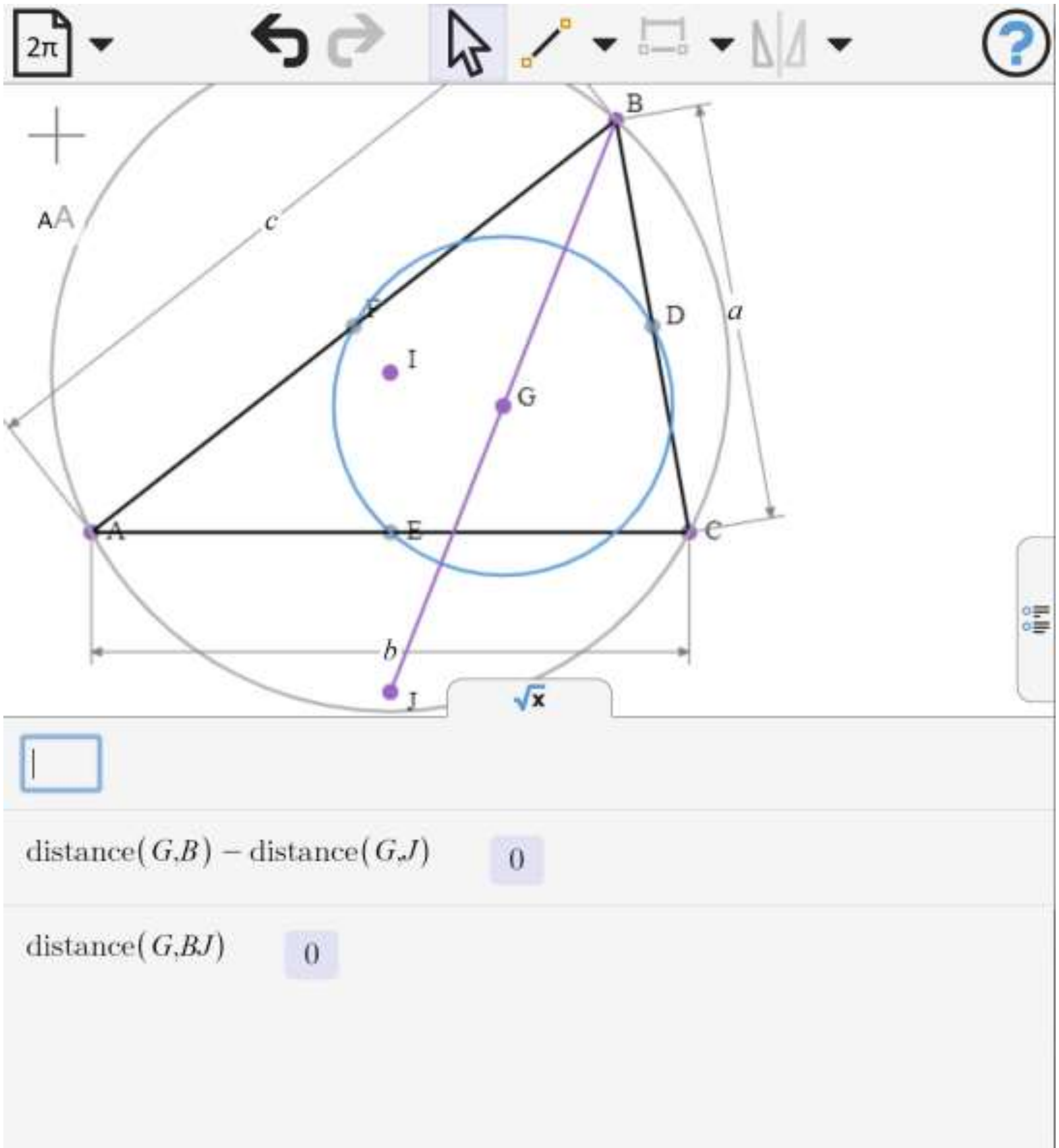
101. Length of the tangent from a vertex to the nine point circle

Show that the square of the tangent from a vertex of a triangle to the nine-point circle is equal to the altitude issued from that vertex multiplied by the distance of the opposite side from the circumcenter.



102. Reflection of the circumcenter in the side of a triangle

Show that the reflection of the circumcenter in the side of a triangle coincides with the reflection of the vertex opposite the side in the nine-point center of the triangle.

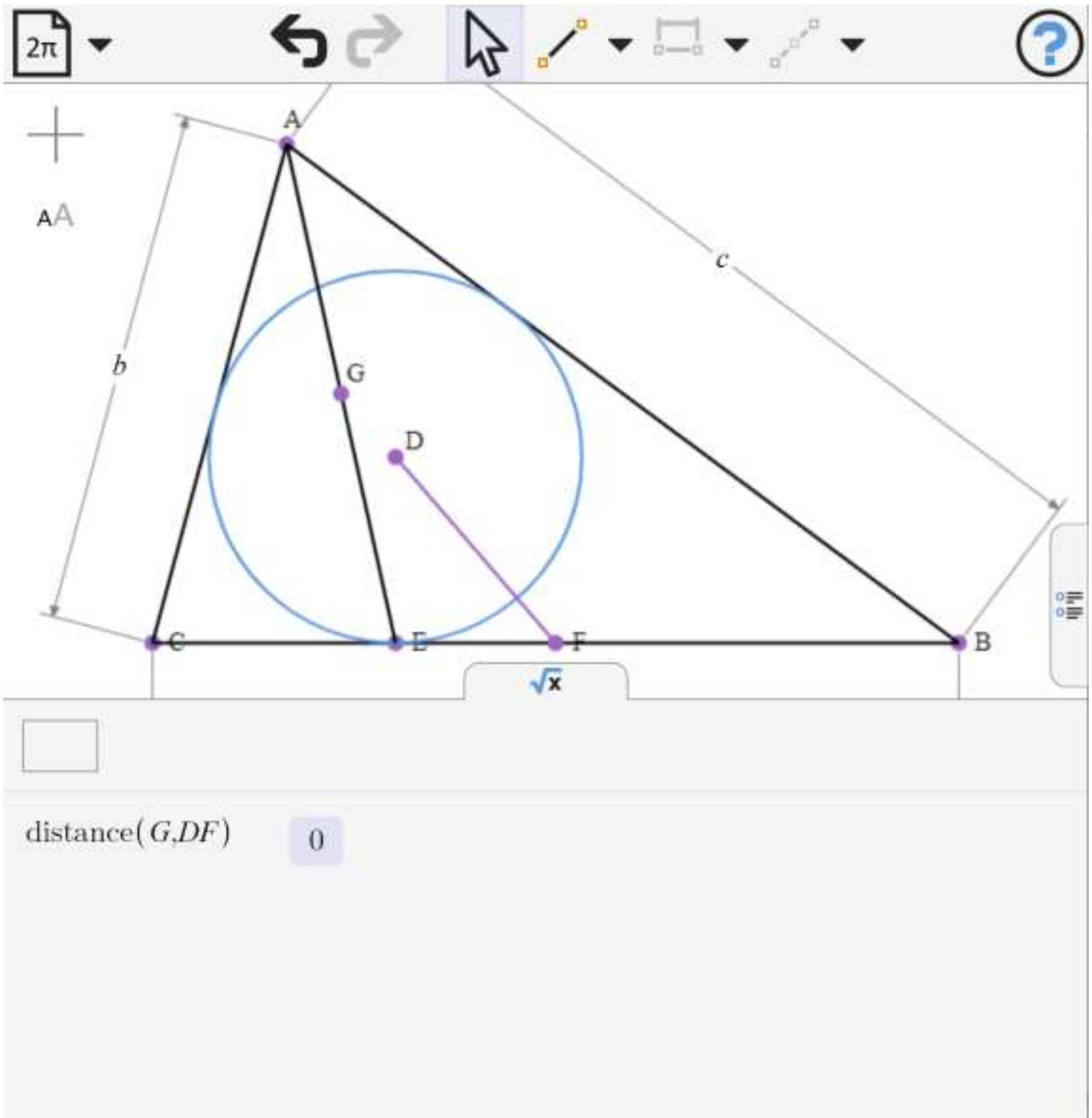


P is the image of O under reflection in AC. We show that N lies on BP and that BN=PN.

Incircles and Excircles

103. The midpoint of line joining a vertex with the point of contact of the incircle on the opposite side

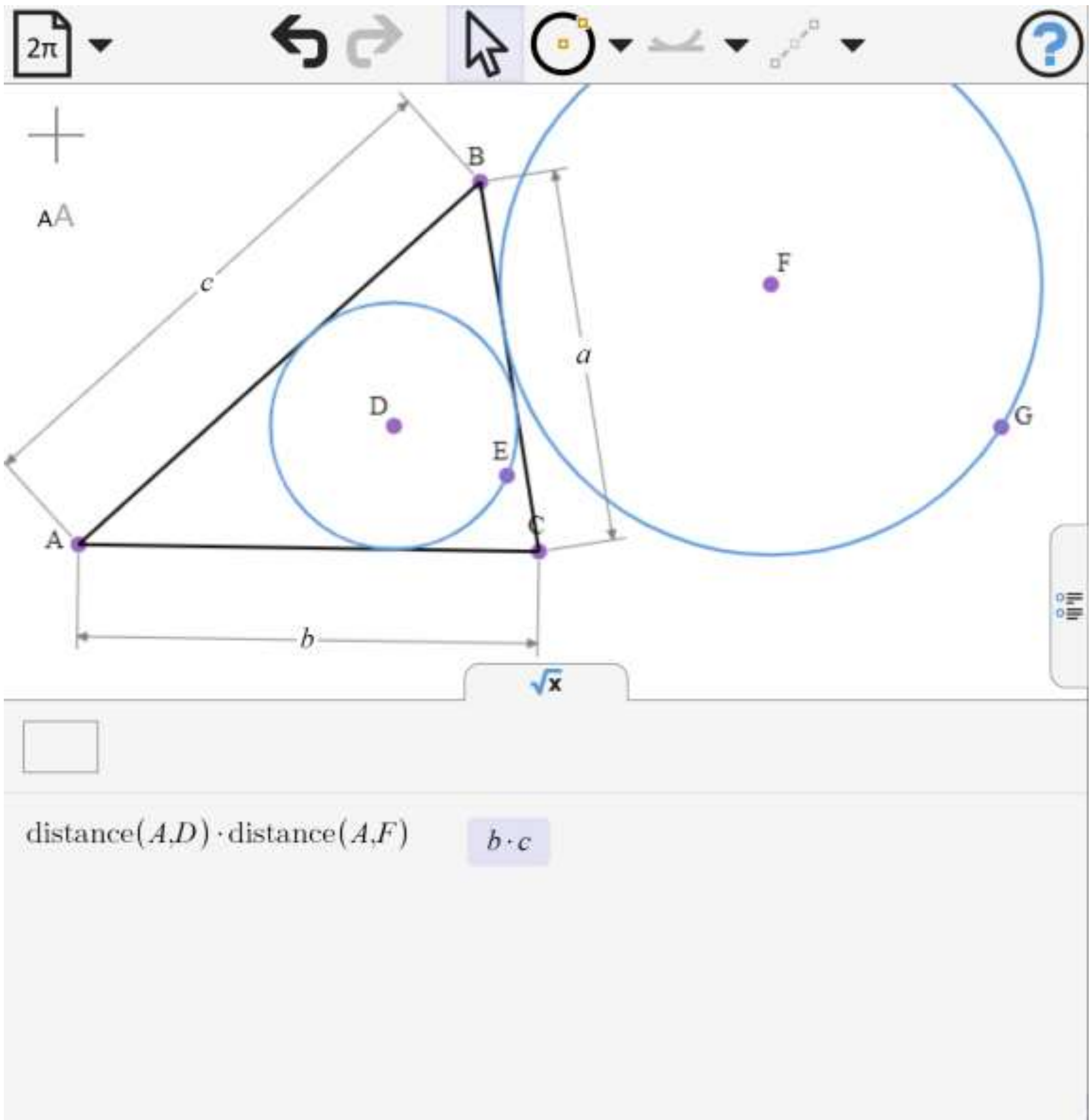
Let incircle (with center I) of triangle ABC touch the side BC at X and M be the midpoint of this side. Then line MI bisects AX.



G is the midpoint of AE, we check that it lies on the line between F and D

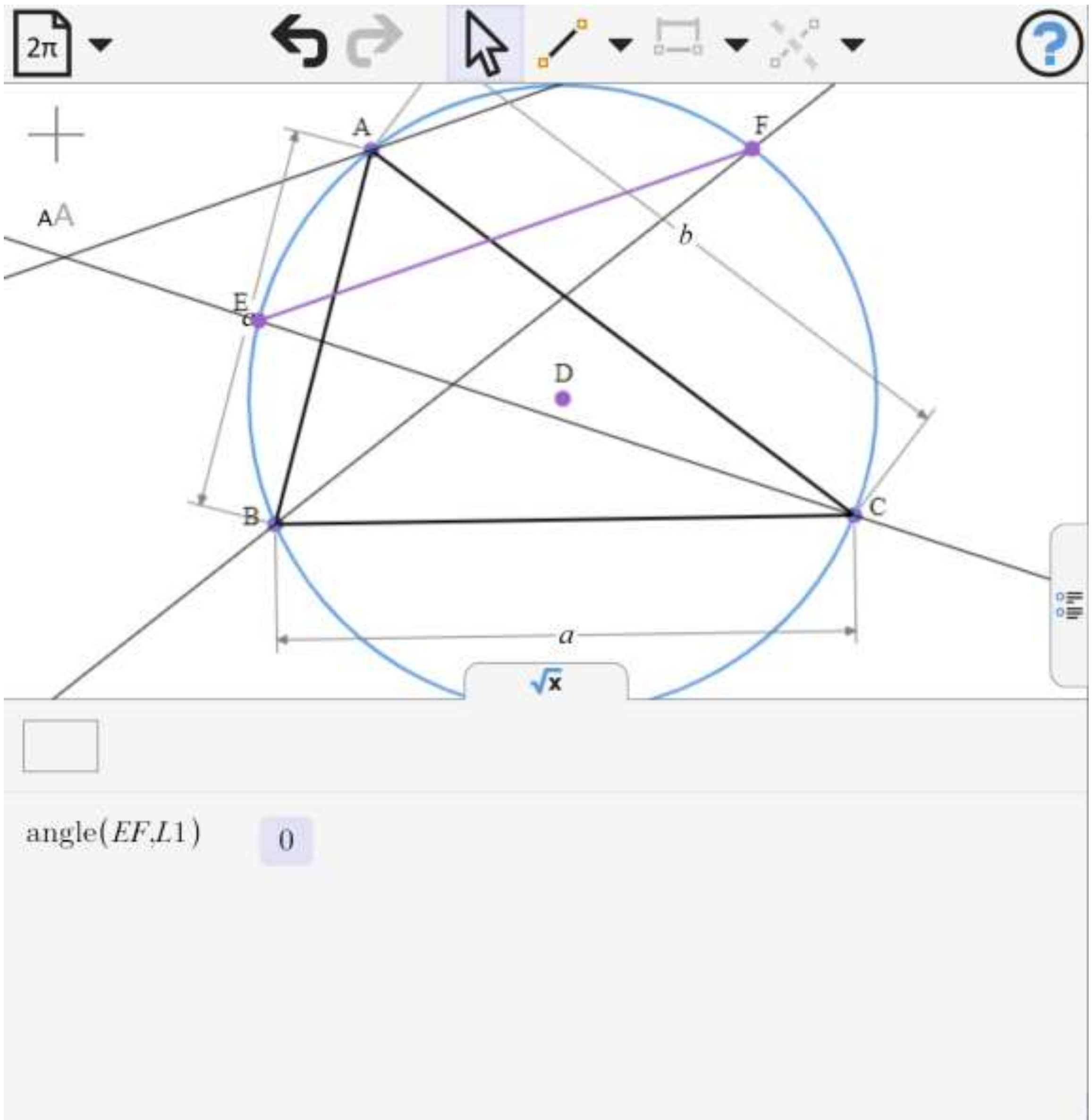
104. Product of the distances from a vertex to the incenter and the center of the opposite excircle

The product of the distances of two tritangent centers of a triangle from the vertex of the triangle collinear with them is equal to the product of the two sides of the triangle passing through the vertex considered.



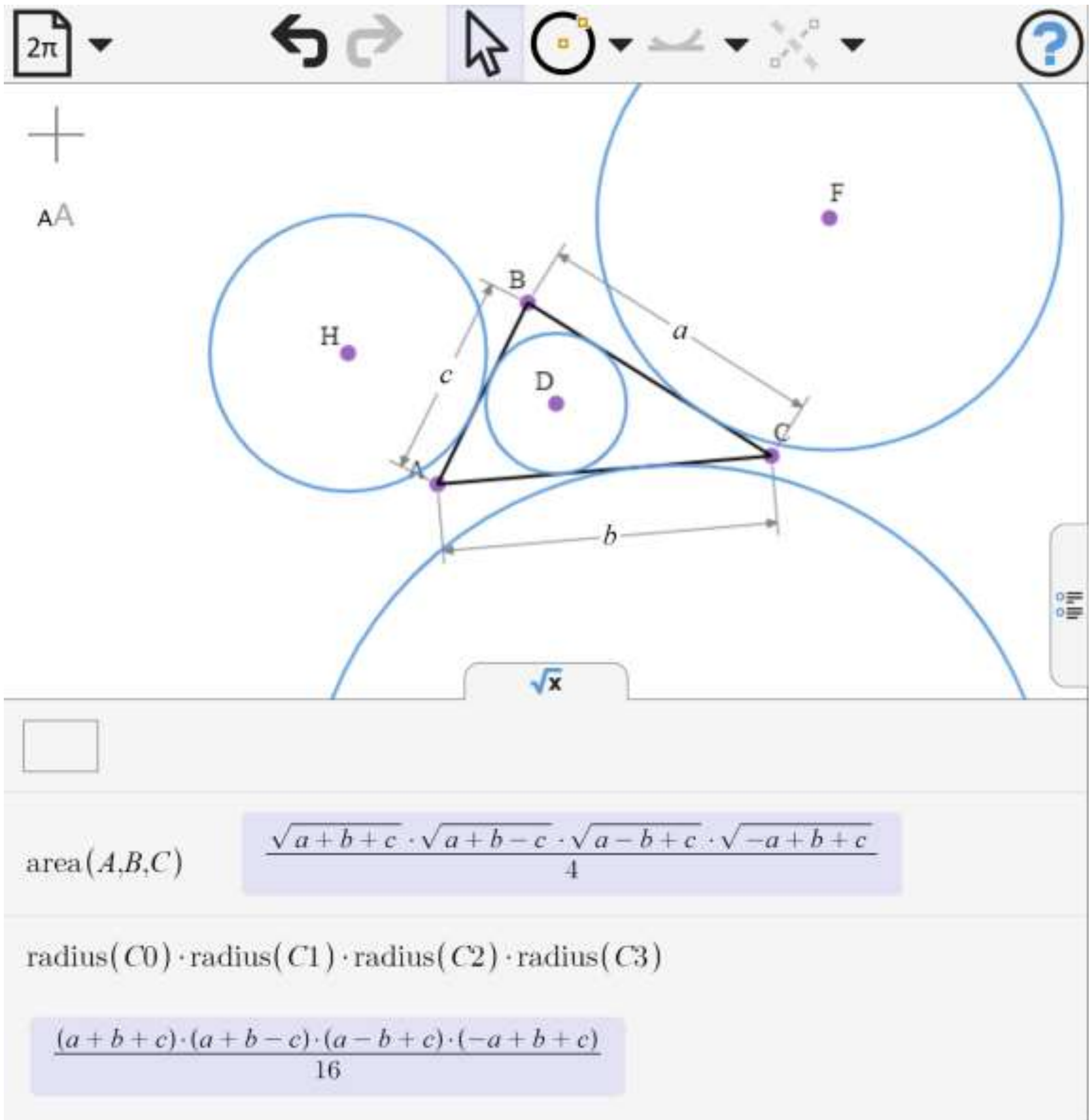
105. A relationship between angle bisectors and the circumcircle

Show that the external bisector of an angle of a triangle is parallel to the line joining the points where the circumcircle is met by the external (internal) bisectors of the other two angles of the triangle



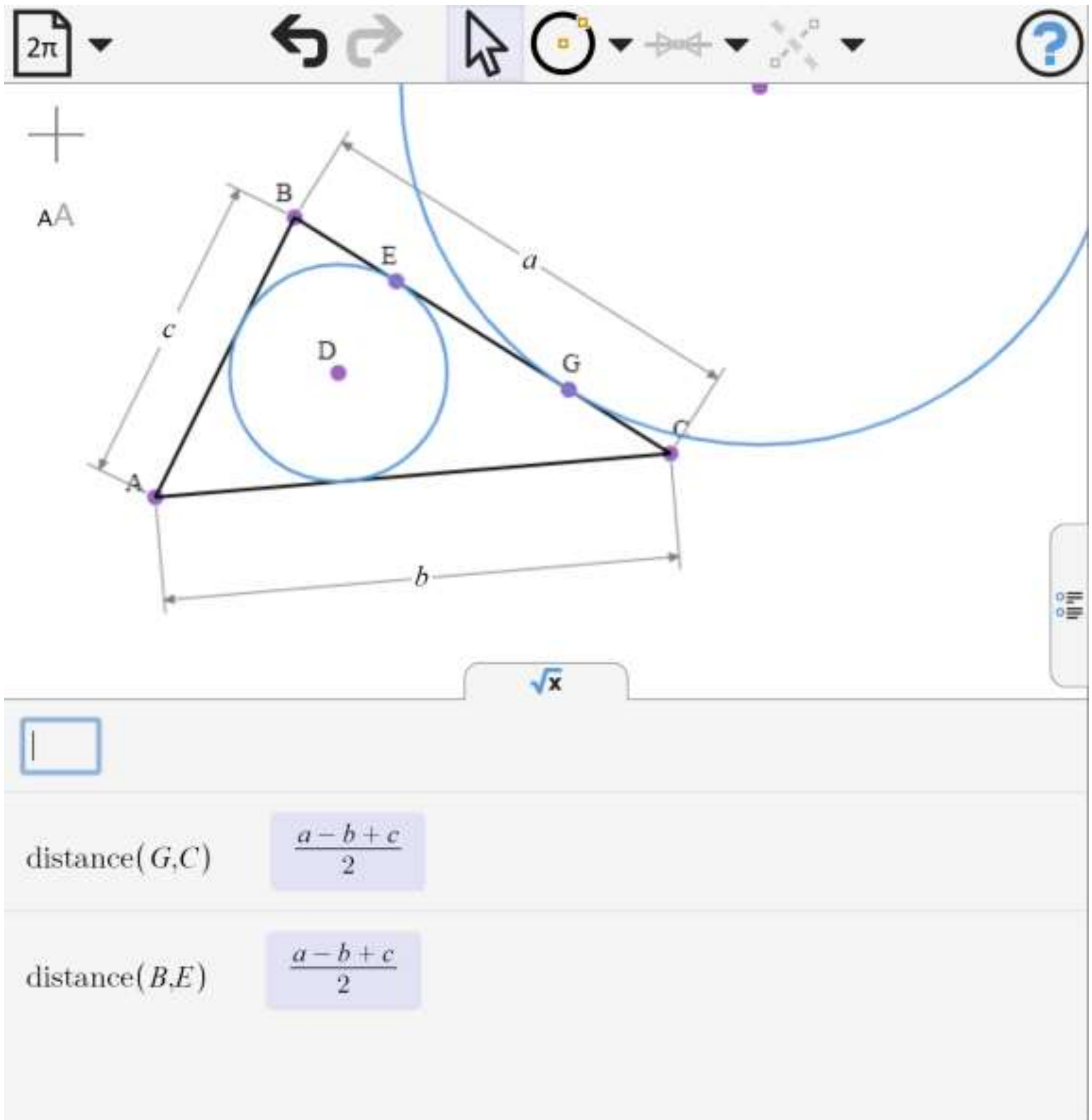
106. Product of incircle and excircle radii

The product of the four tritangent radii of a circle is equal to the square of its area



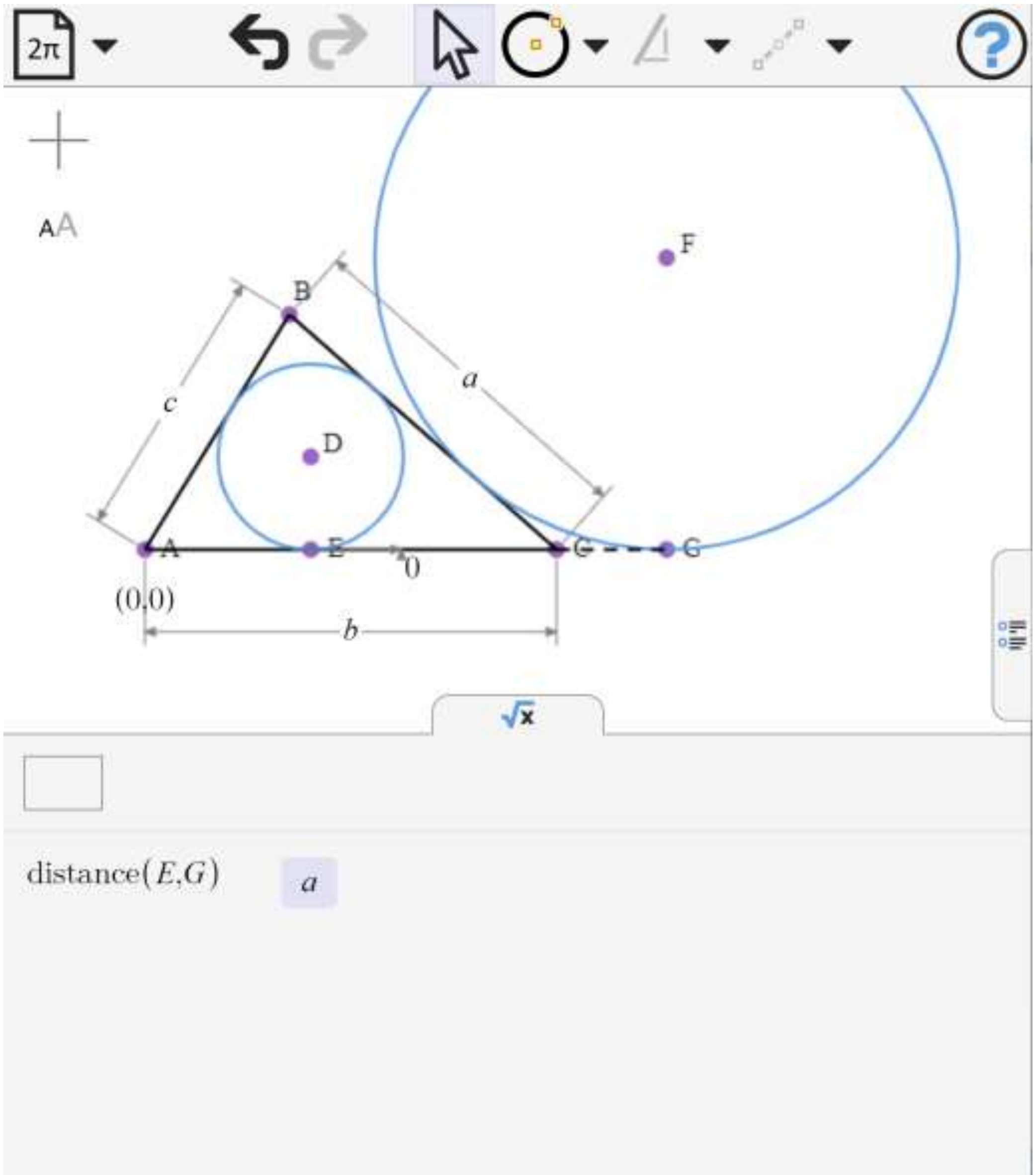
107. Points of contact of incircle and excircle

The points of contact of a side of a triangle with the incircle and the excircle relative to this side are two isotomic points



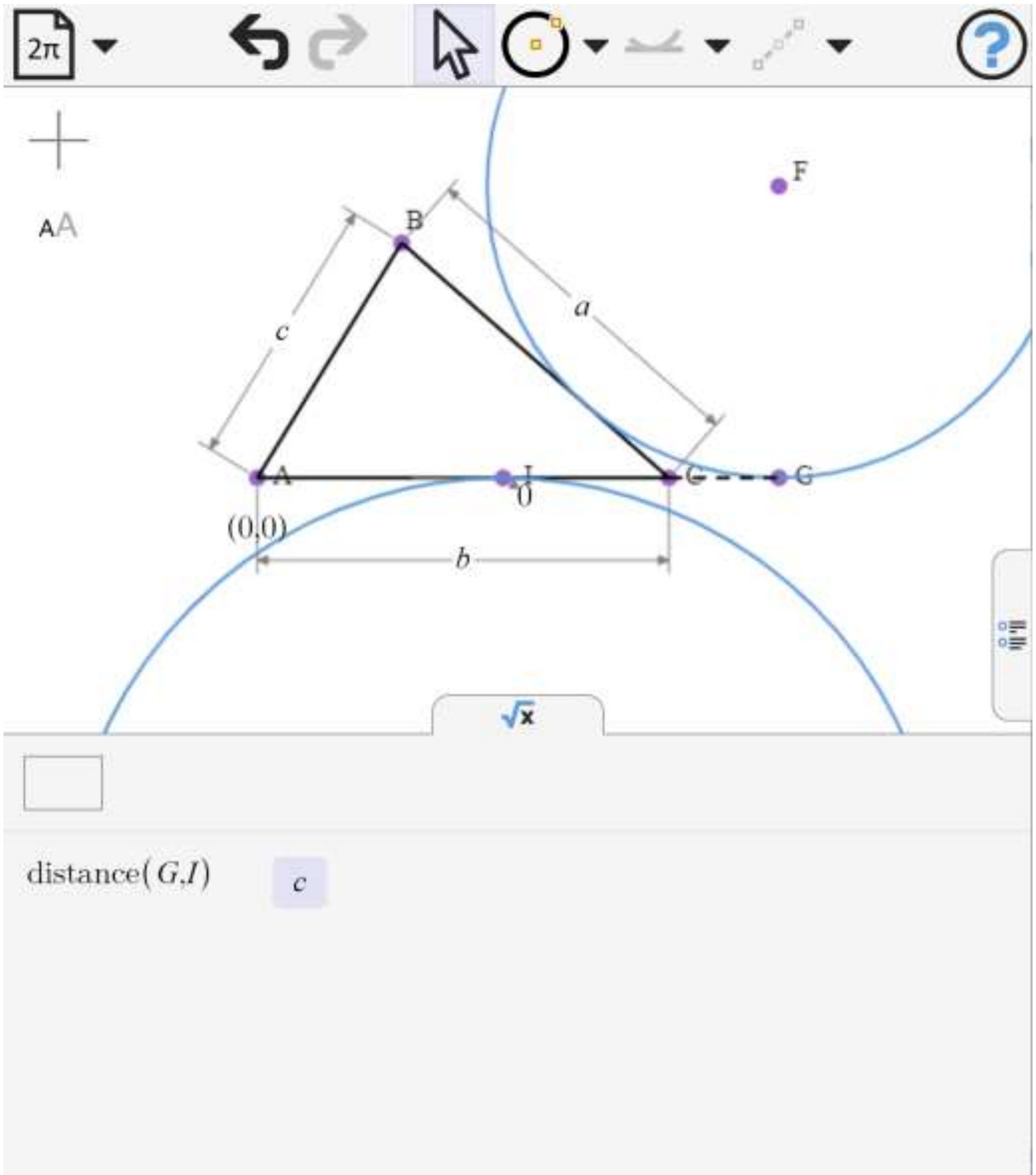
108. Distance between point of contact of incircle and excircle

(I) is the incircle, (J) the excircle defined by side BC. The distance between the points of contact of (I) and (J) with AC (extended) is the same as the length of BC.



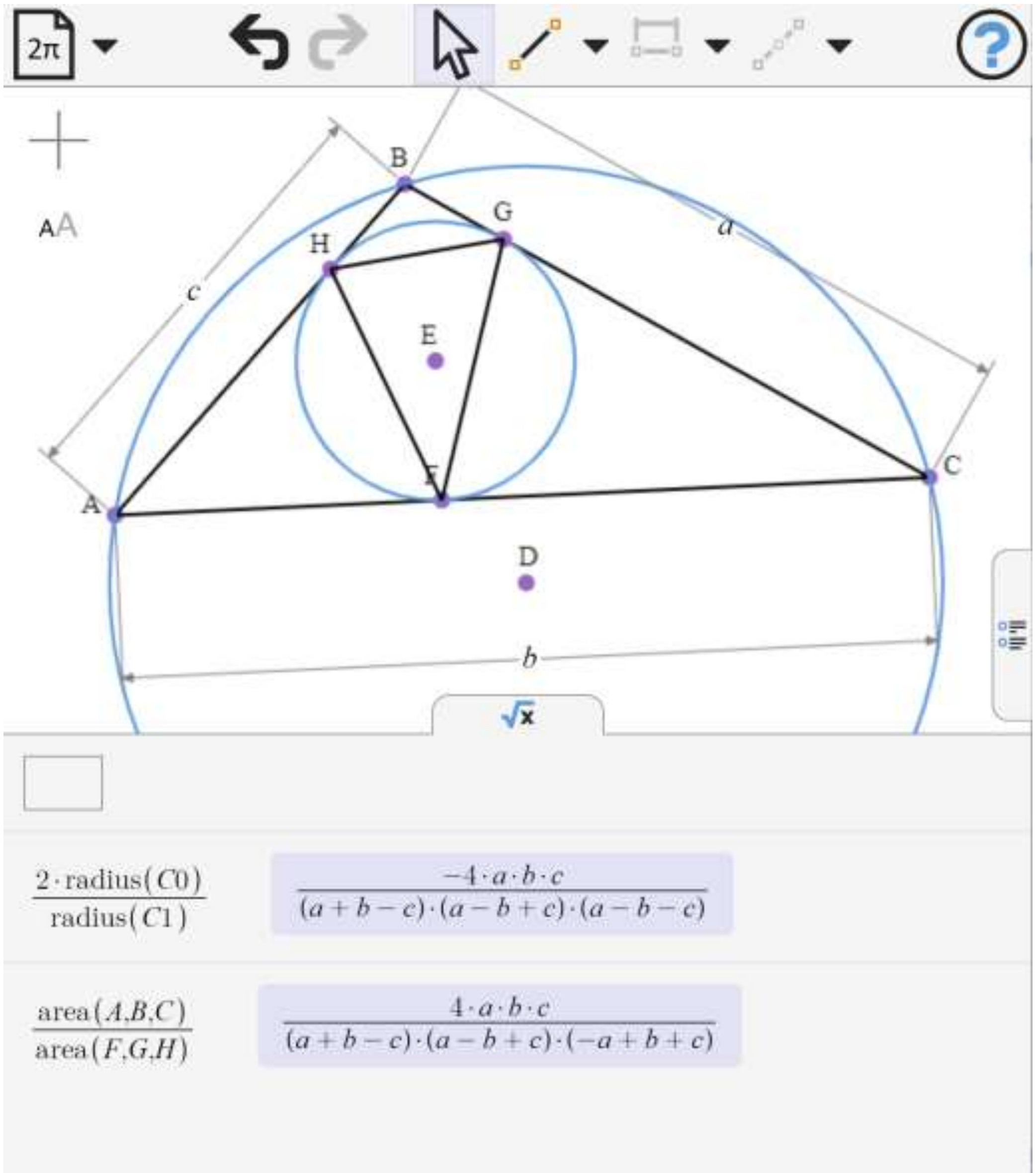
109. Distance between points of contact of two excircles

(I) is the excircle defined by AB, (J) the excircle defined by side BC. The distance between the points of contact of (I) and (J) with BC (extended) is the same as the length of AC.



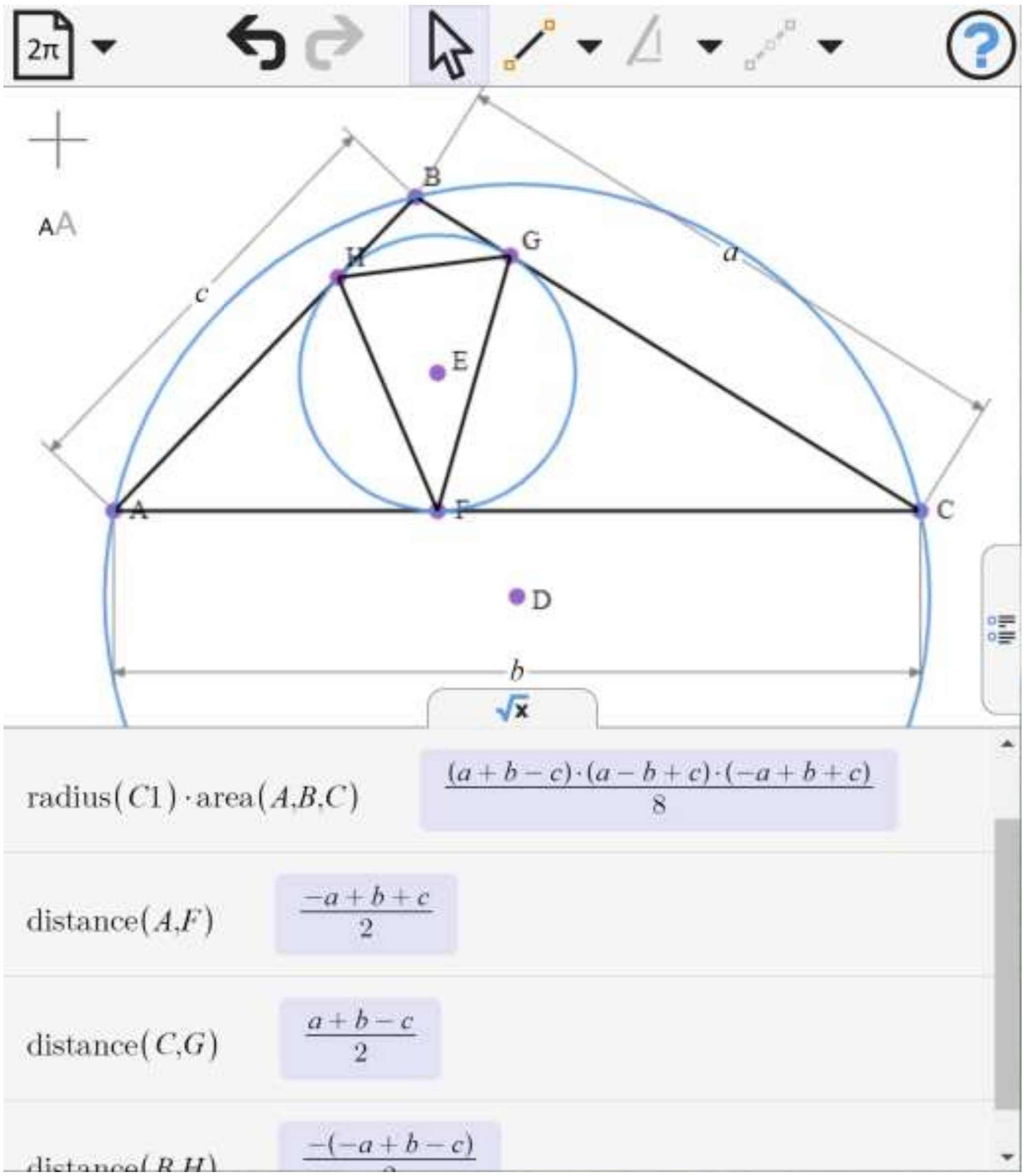
110. Ratio of areas of a triangle and the triangle formed by the points of contact of its incircle

The ratio of the area of a triangle to the area of the triangle determined by the points of contact of the sides with the incircle is equal to the ratio of the circumdiameter of the given triangle with its inradius.



111. Distances between vertices and points of contact with the incircle

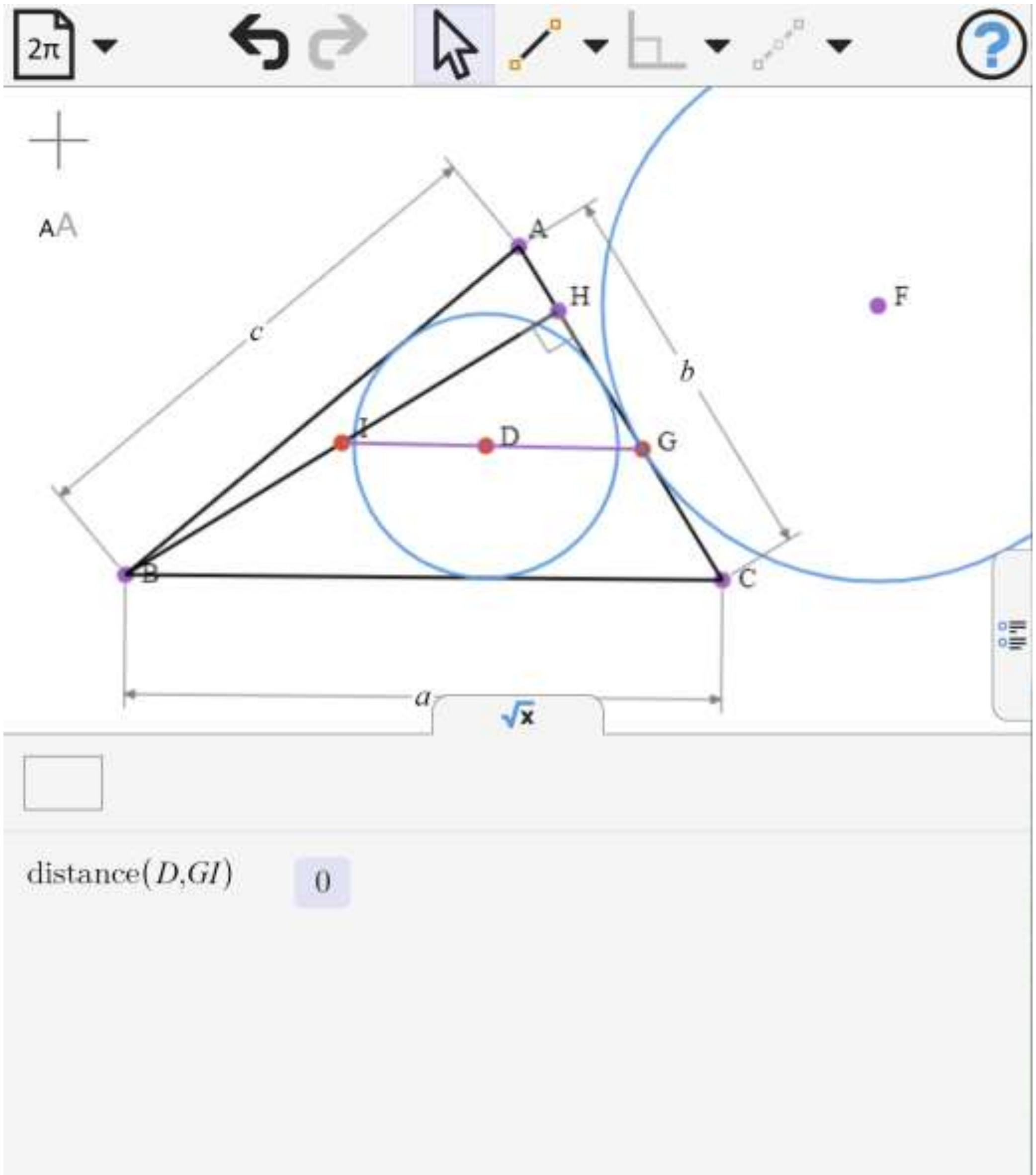
If X, Y and Z are the points of contact between the incircle and the triangle opposite A, B, C respectively. Show that AZ.BX.CY = r times the area of the triangle



Visually we can see that the identity holds, but when we ask Geometry Expressions to compute the product of the lengths, because it goes back to the unsimplified expression, it does not give the good answer!

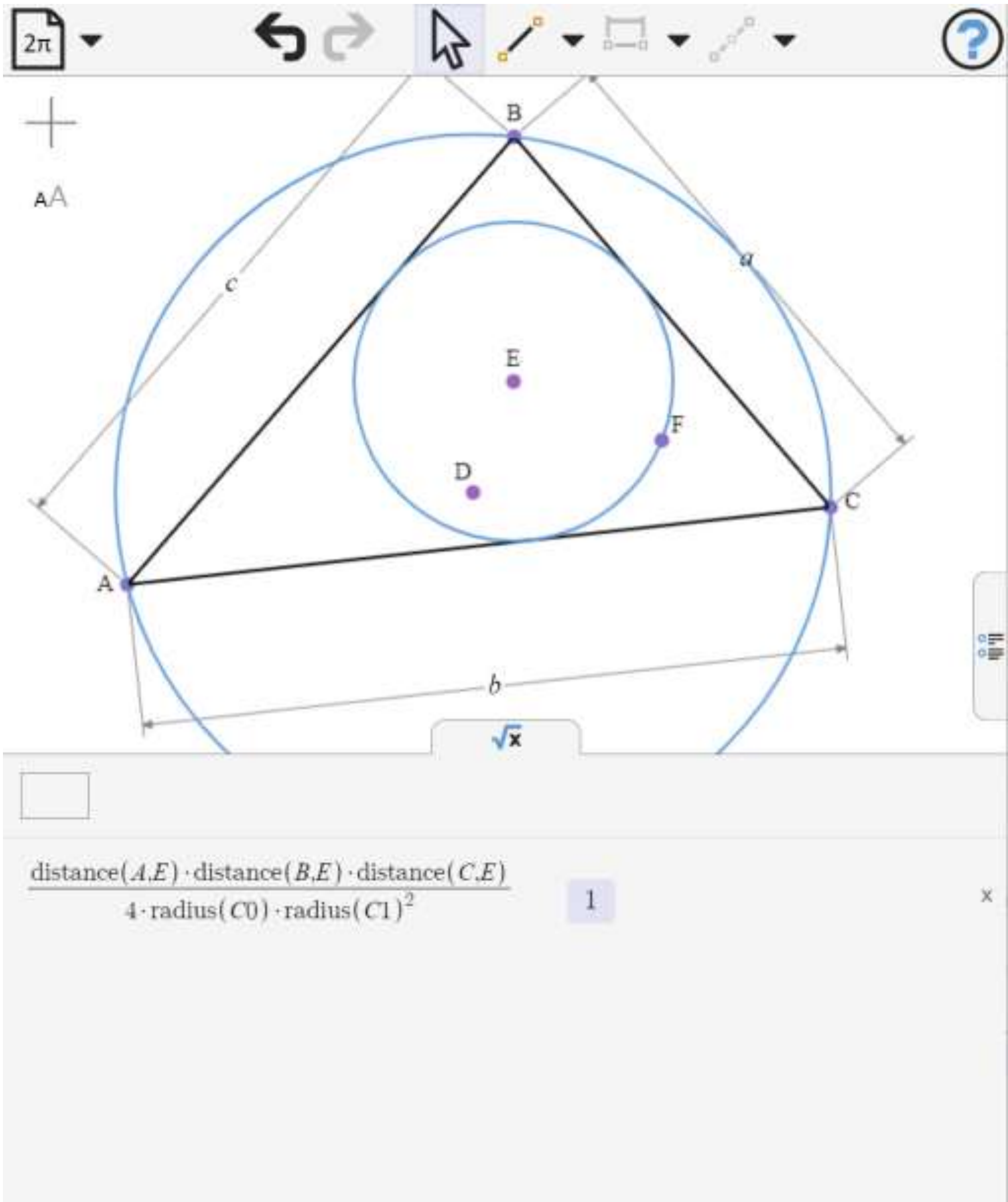
112. Incentre, excircle point of contact and altitude midpoint are collinear

Show that the midpoint of an altitude of a triangle, the point of contact of the corresponding side with the excircle relative to that side and the incenter of the triangle are collinear



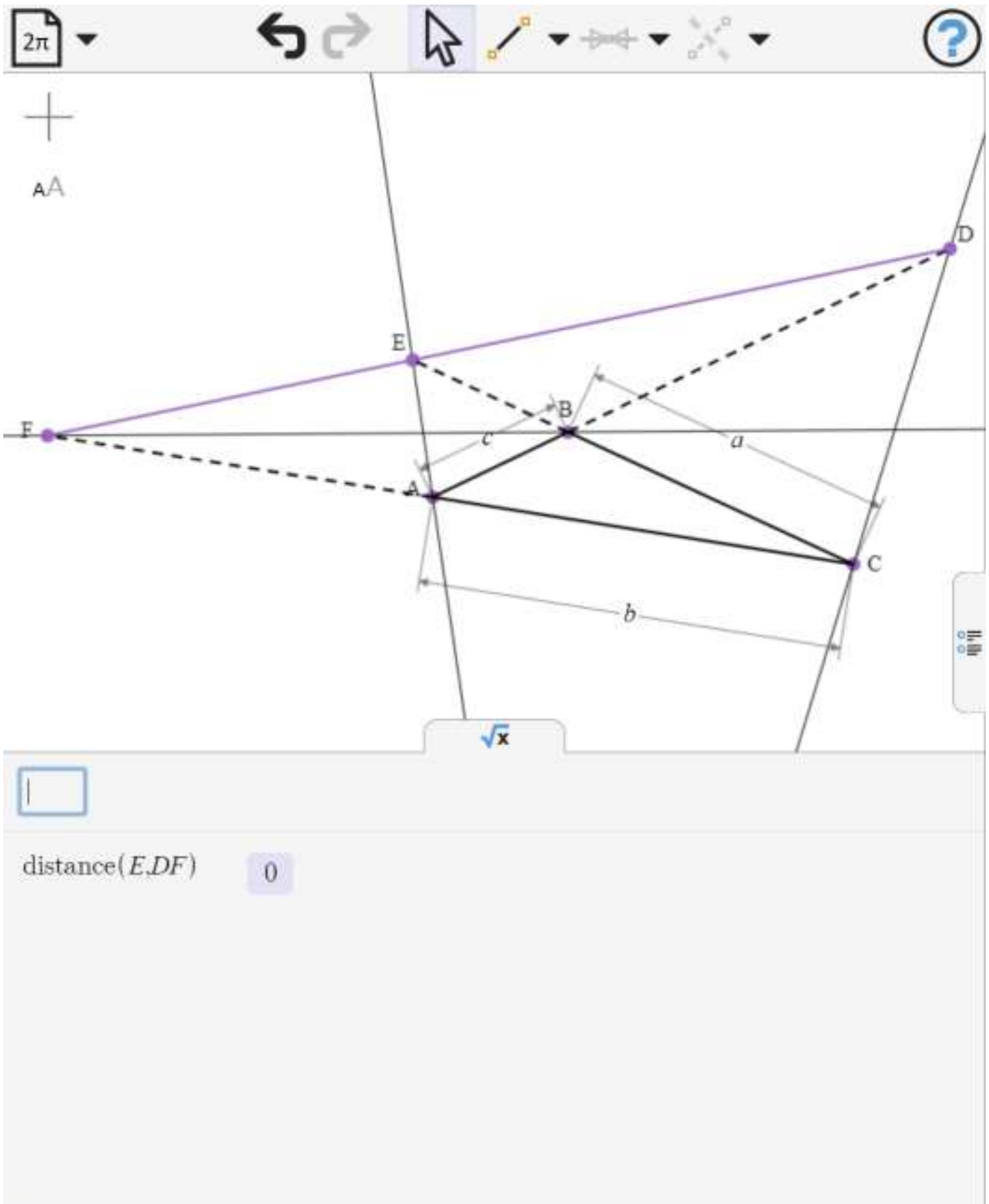
113. Product of the distances from the vertices to the incenter

Show that the product of the distances of the incenter of a triangle from the three vertices of the triangle is equal to $4Rr^2$, where R is the radius of the circumcircle, and r is the radius of the incircle.



114. External angle bisectors

The external bisectors of the angles of a triangle meet the opposite sides in 3 collinear points



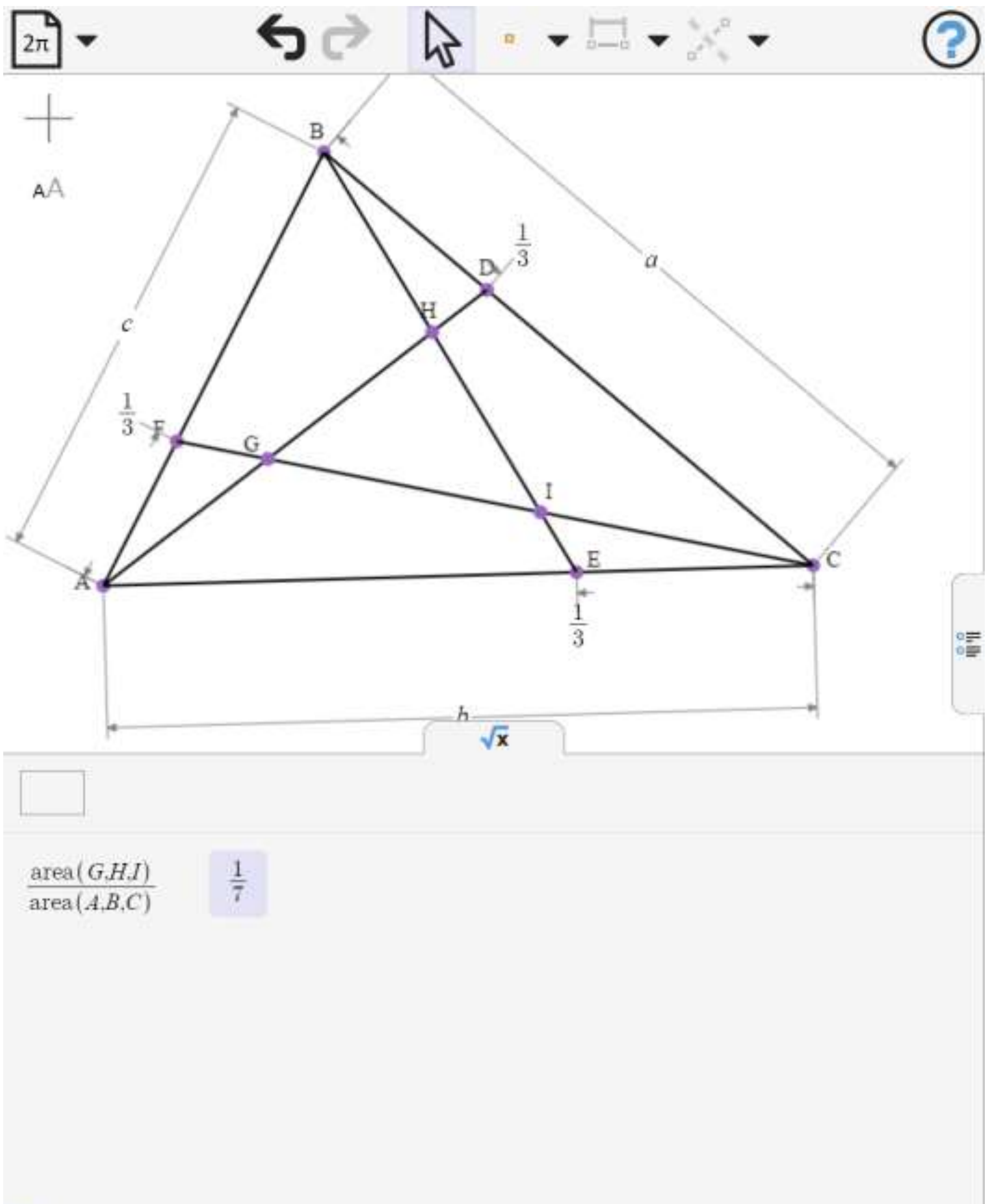
Intercept Triangles

Let L, M, N be three points on the sides BC, CA, AB of triangle ABC. Then triangle LMN and the triangle determined by lines AL, BM and CN are called the intercept triangles of triangle ABC for points L, M, N.

115. Feinman's triangle

Let D,E,F be points on the sides BC, CA, AB of a triangle ABC such that BD/BC = CE/CA = AF/AB = 1/3.

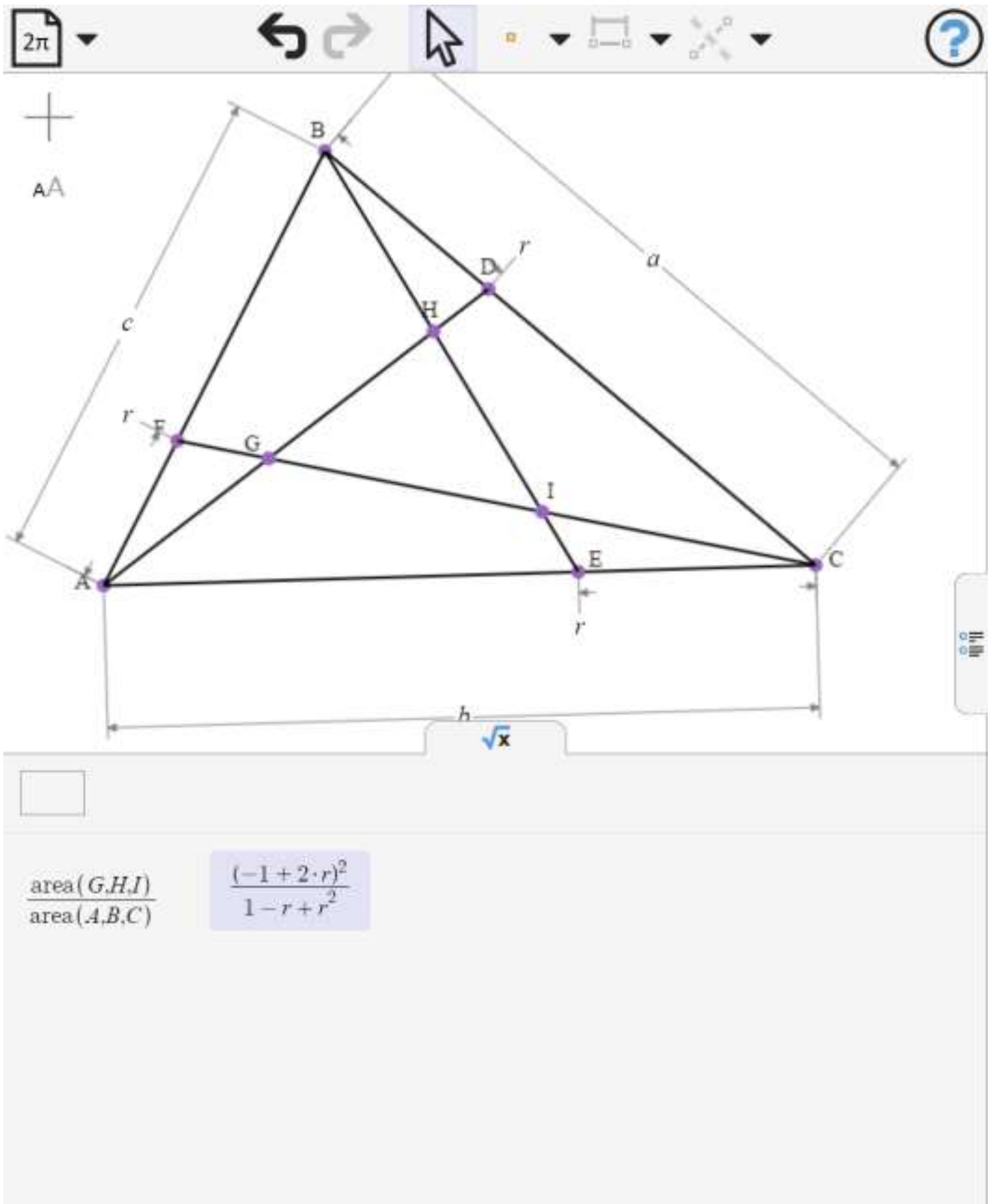
Show the area of the triangle determined by the lines AD, BE, CF is one seventh the area of triangle ABC.



116. Steiner's triangle with same ratio on each side

Let D,E,F be points on the sides BC, CA, AB of a triangle ABC such that BD/BC = CE/CA = AF/AB = r.

Show the ratio of area of the triangle determined by the lines AD, BE, CF to the area of triangle ABC is $\frac{(2 \cdot r-1)^2}{r^2-r+1}$

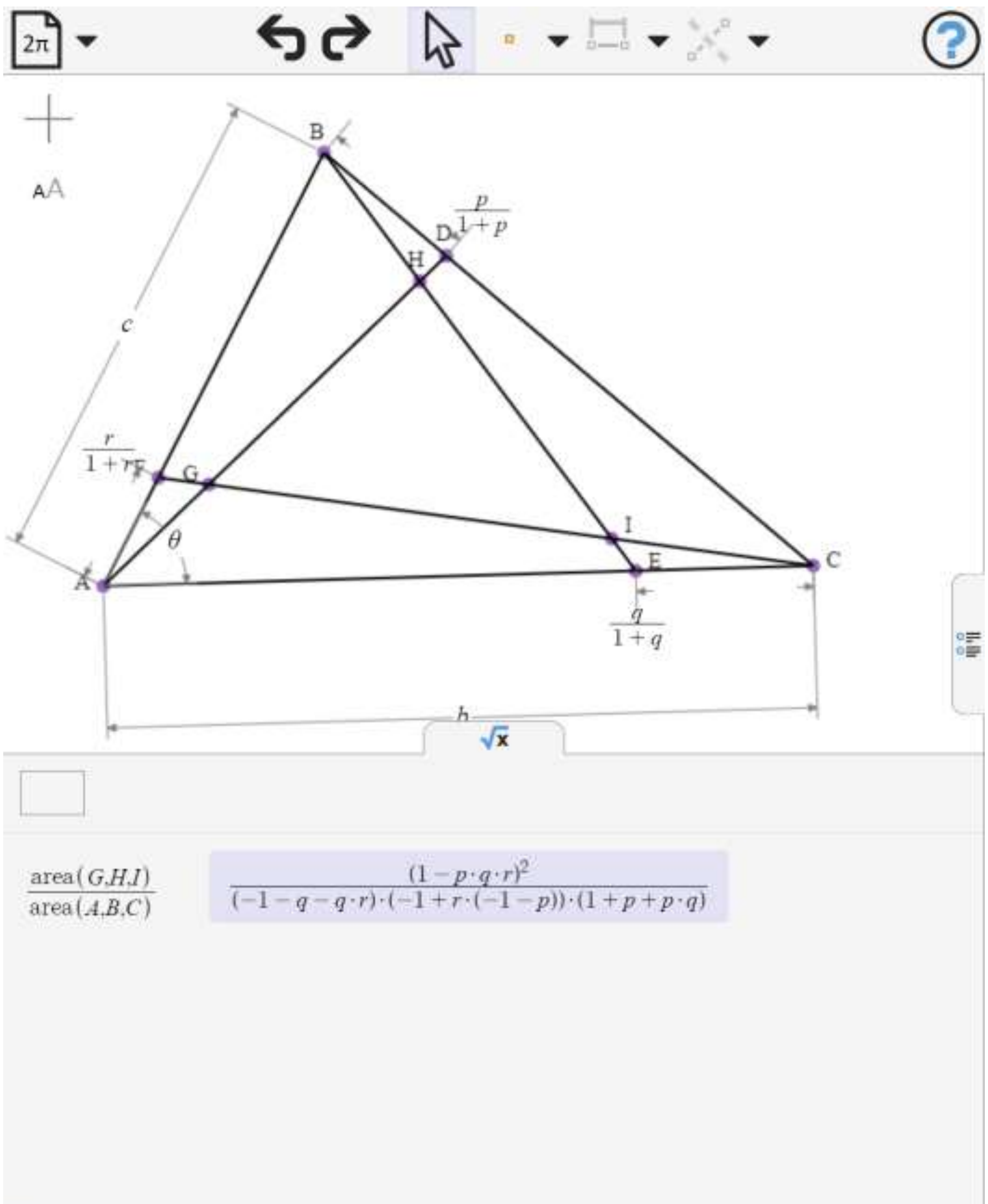


117. Steiner's triangle

Using the above notation if $\frac{BD}{DC}=p, \frac{CE}{EA}=q, \frac{AF}{FB}=r$, then the ratio of areas is $\frac{(pqr-1)^2}{(qp+p+1)(rp+r+1)(rq+q+1)}$

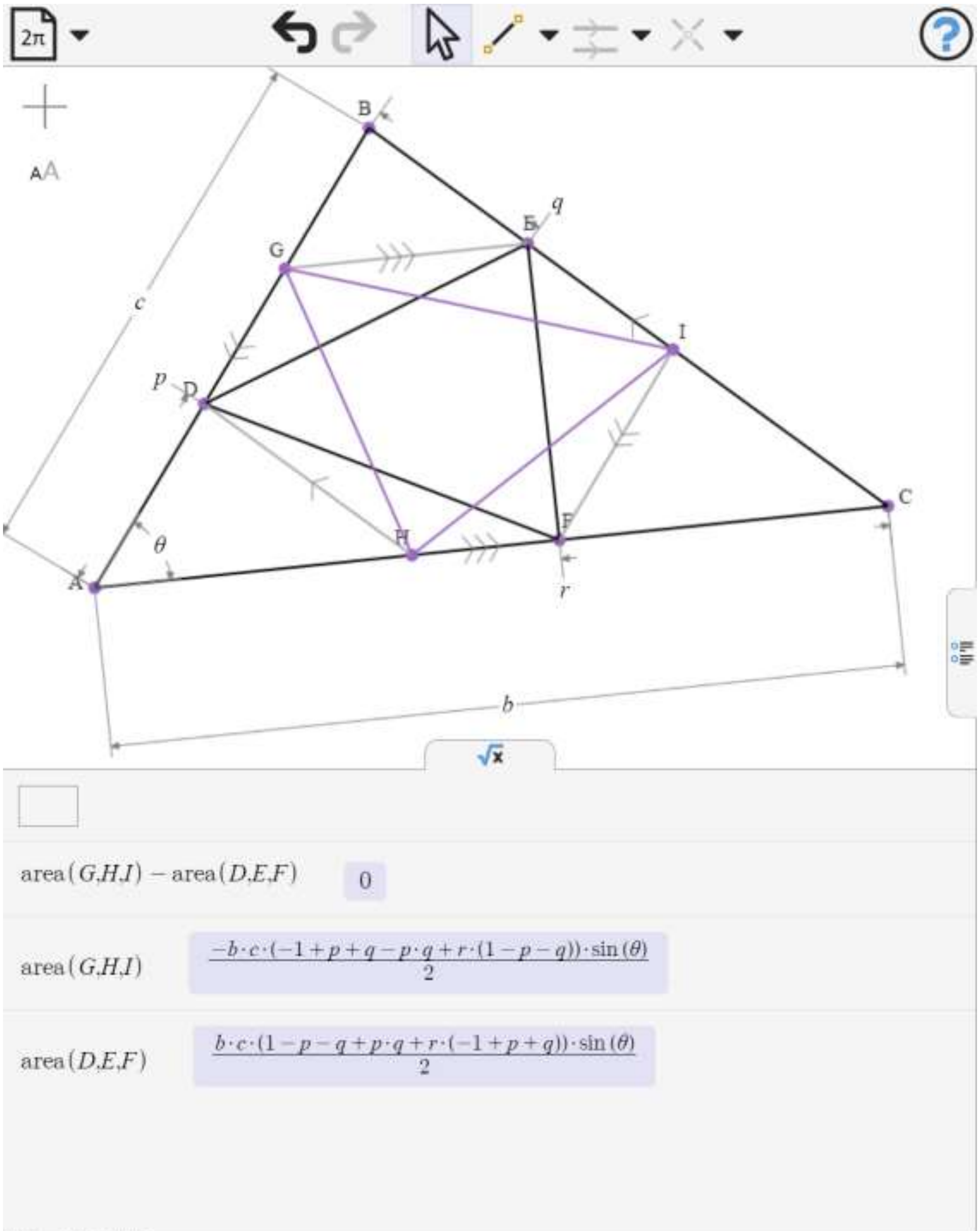
The proportional distances used by GXWeb represent the ratios $\frac{BD}{BC}, \frac{CE}{CA}, \frac{AF}{AB}$. Hence, we need to set

$\frac{BD}{BC}=\frac{p}{1+p}, \frac{CE}{CA}=\frac{q}{1+q}, \frac{AF}{AB}=\frac{r}{1+r}$.



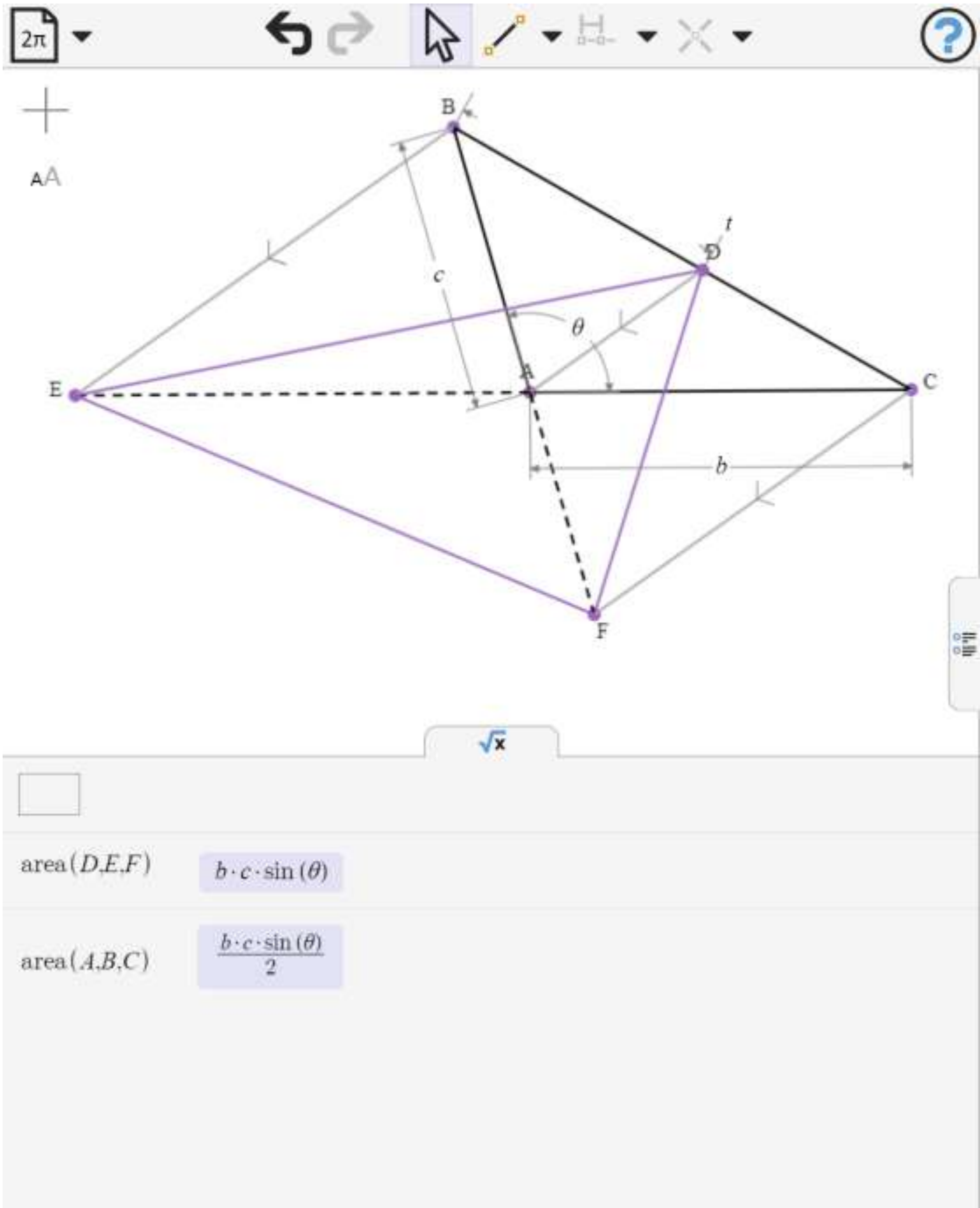
118. Two equiareal triangles

Let E,F,G be points on sides BC,CA,AB of a triangle ABC. Show that if G,H,I are points on sides AB, AC and BC such that EG is parallel to AC, DH is parallel to BC and FI is parallel to AB, then triangles EFG and GHI have the same areas.



119. A triangle defined by parallels

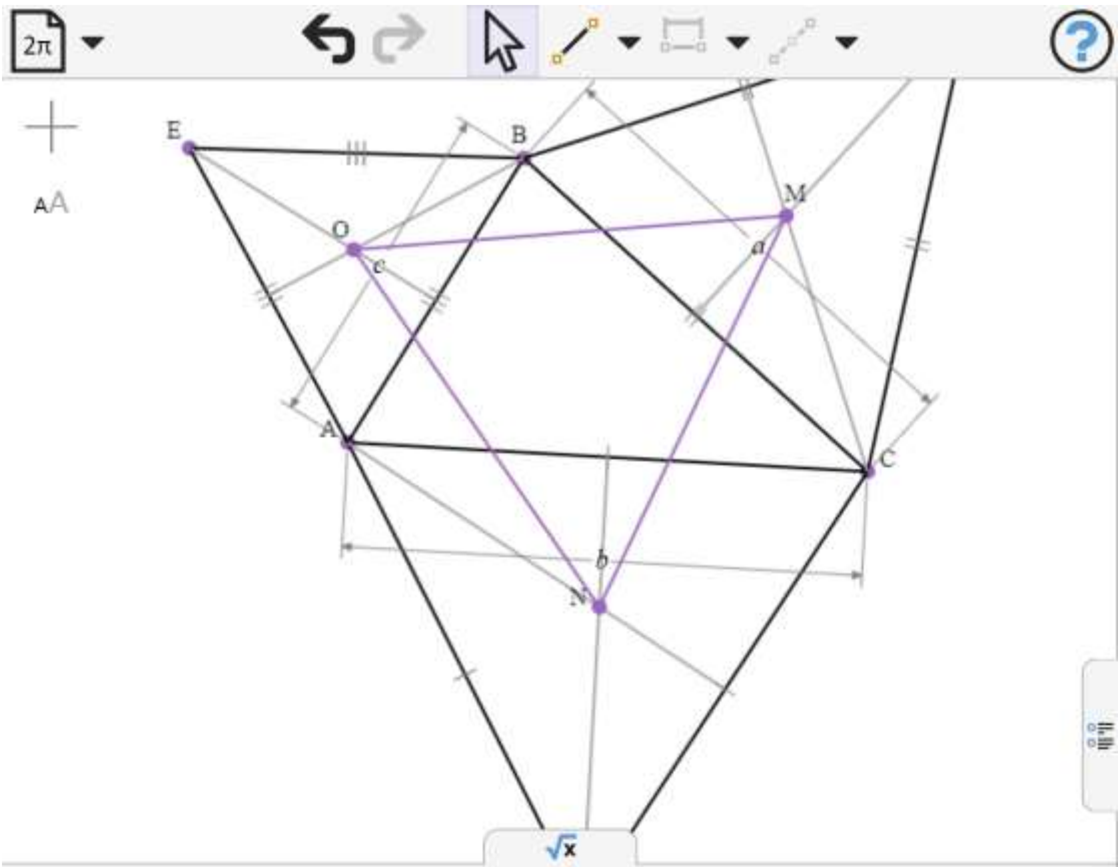
Three parallel lines drawn through the vertices of a triangle ABC meet the respectively opposite sides in the points E, F, G. Show that area EFG is twice area ABC.



Equilateral triangles

120. The Napoleon Triangle

If equilateral triangles are erected externally (or internally) on the sides of any triangle, their centers form an equilateral triangle.



$$\sqrt{\frac{a^2}{6}+\frac{b^2}{6}+\frac{c^2}{6}+\frac{\sqrt{3}\cdot\sqrt{a+b+c}\cdot\sqrt{a+b-c}\cdot\sqrt{a-b+c}\cdot\sqrt{-a+b+c}}{6}}$$

distance(M,N)

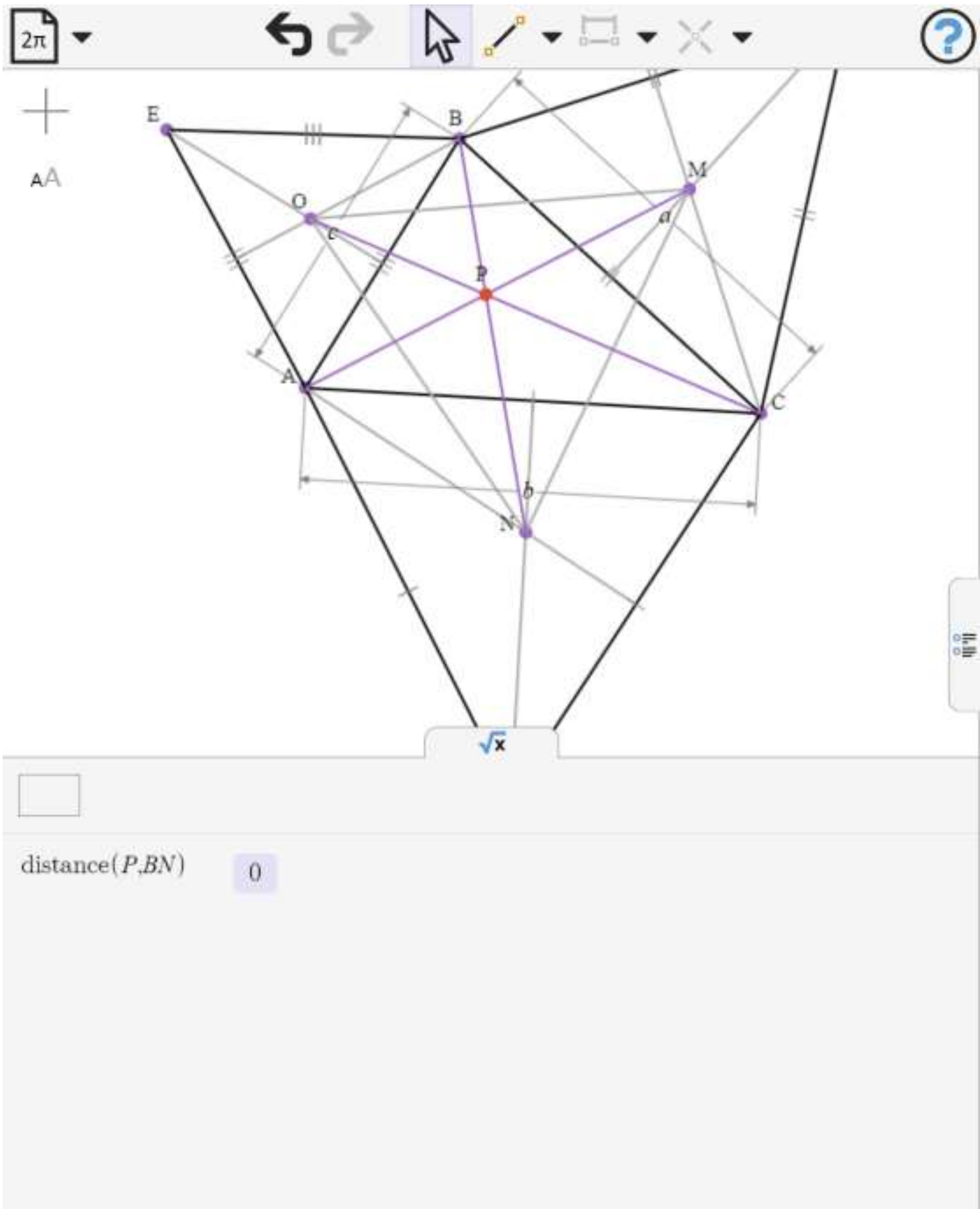
$$\sqrt{\frac{a^2}{6}+\frac{b^2}{6}+\frac{c^2}{6}+\frac{\sqrt{3}\cdot\sqrt{a+b+c}\cdot\sqrt{a+b-c}\cdot\sqrt{a-b+c}\cdot\sqrt{-a+b+c}}{6}}$$

distance(M,O)

$$\sqrt{\frac{a^2}{6}+\frac{b^2}{6}+\frac{c^2}{6}+\frac{\sqrt{3}\cdot\sqrt{a+b+c}\cdot\sqrt{a+b-c}\cdot\sqrt{a-b+c}\cdot\sqrt{-a+b+c}}{6}}$$

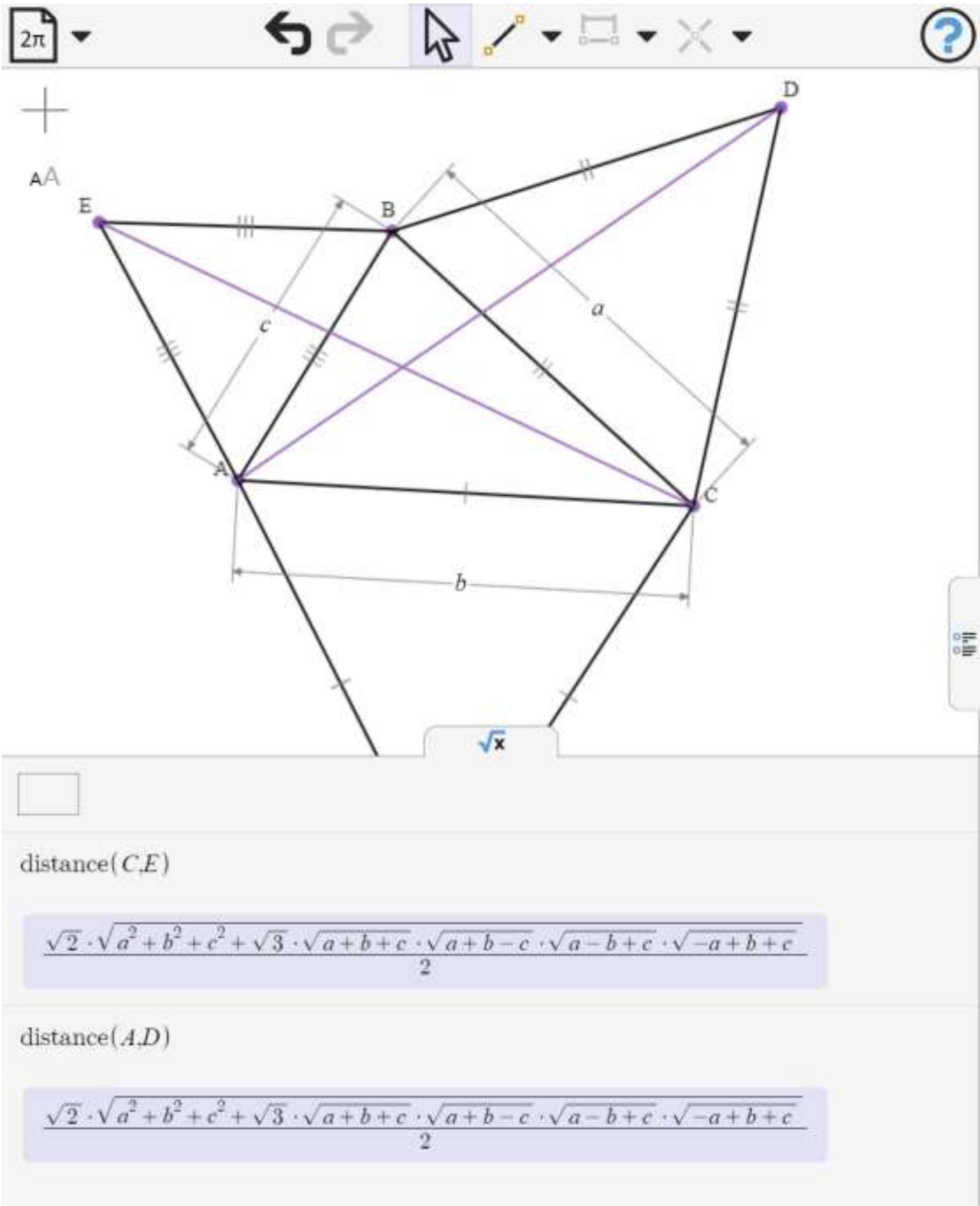
121. Lines joining vertices to centers of equilateral triangles

Continuing the above example, show that AM, BN and CO are concurrent



122. Fermat point

Let equilaterals BCD, ABF and ACE be erected externally on the sides of triangle ABC. Show that AD=CF=BE

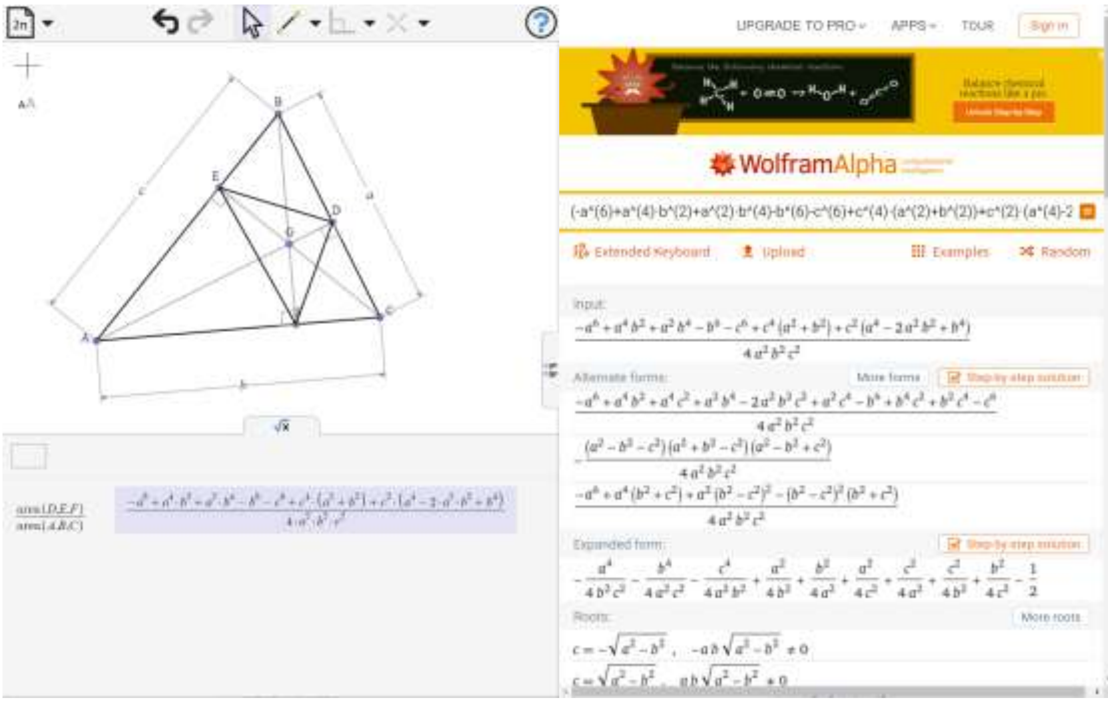


Pedal Triangles

From a point P three perpendicular lines are drawn to the sides of a triangle. The triangle whose vertices are the feet of these perpendiculars is called the pedal triangle of point P with respect to the given triangle

123. Pedal triangle of the orthocenter

Let K be the area of the pedal triangle of the orthocenter of ABC. What is the ratio of K to ABC?

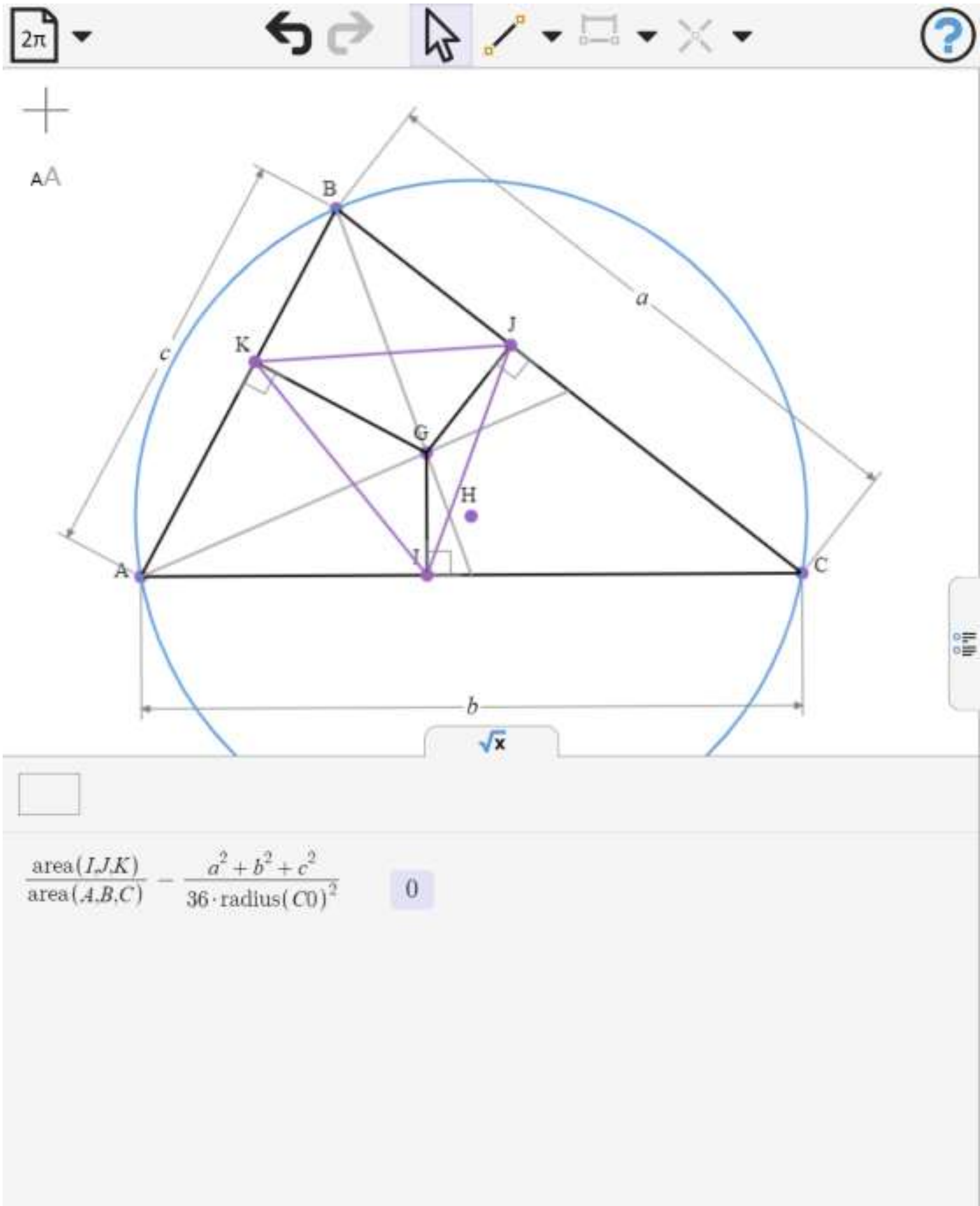


Wolfram Alpha lets us see his in a number of different forms, including a factored form

124. Pedal triangle of the centroid

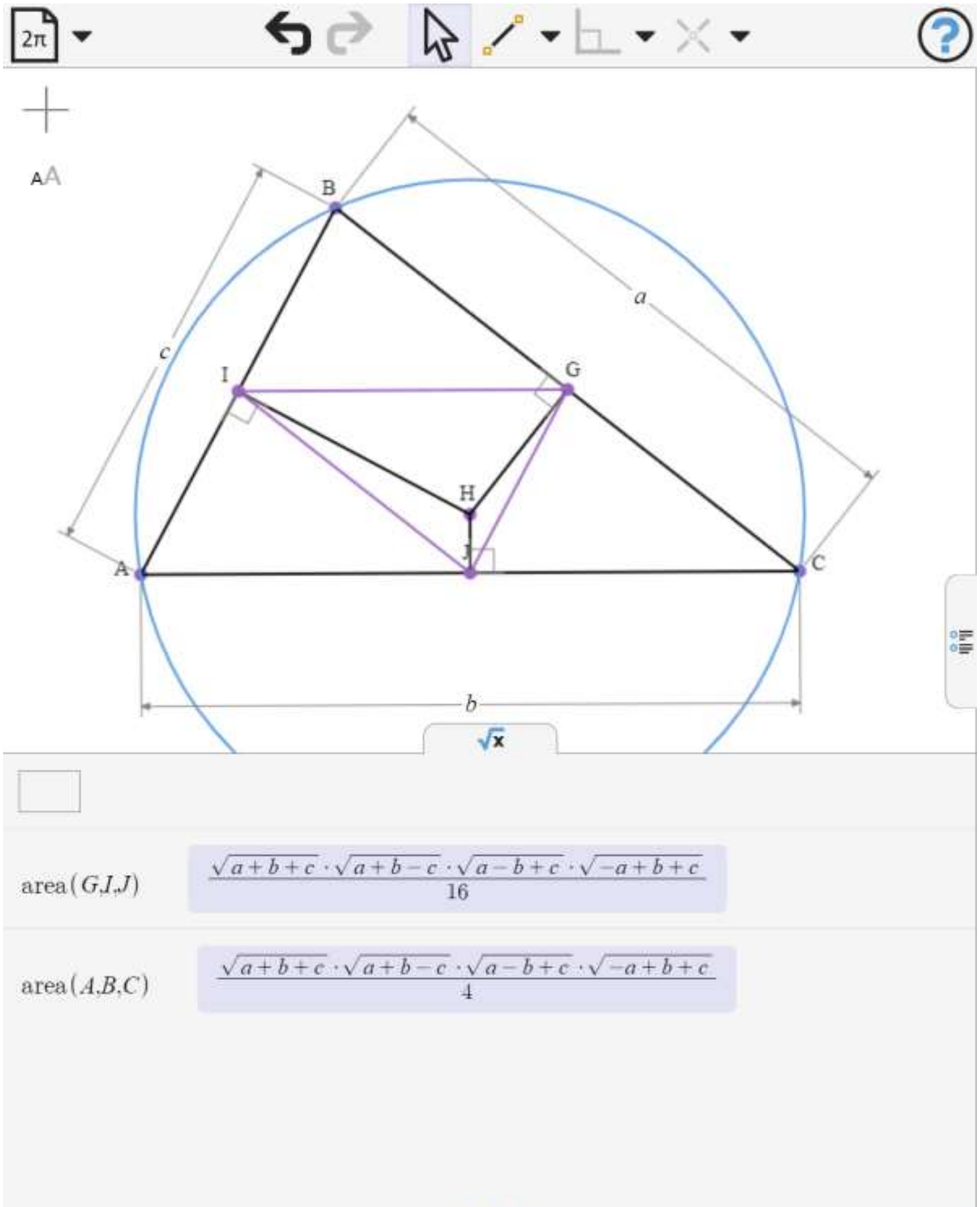
Let K be the area of the pedal triangle of the centroid of ABC and R the circumradius of ABC.

Show that $\frac{K}{ABC} = \frac{AB^2+BC^2+AC^2}{36R^2}$



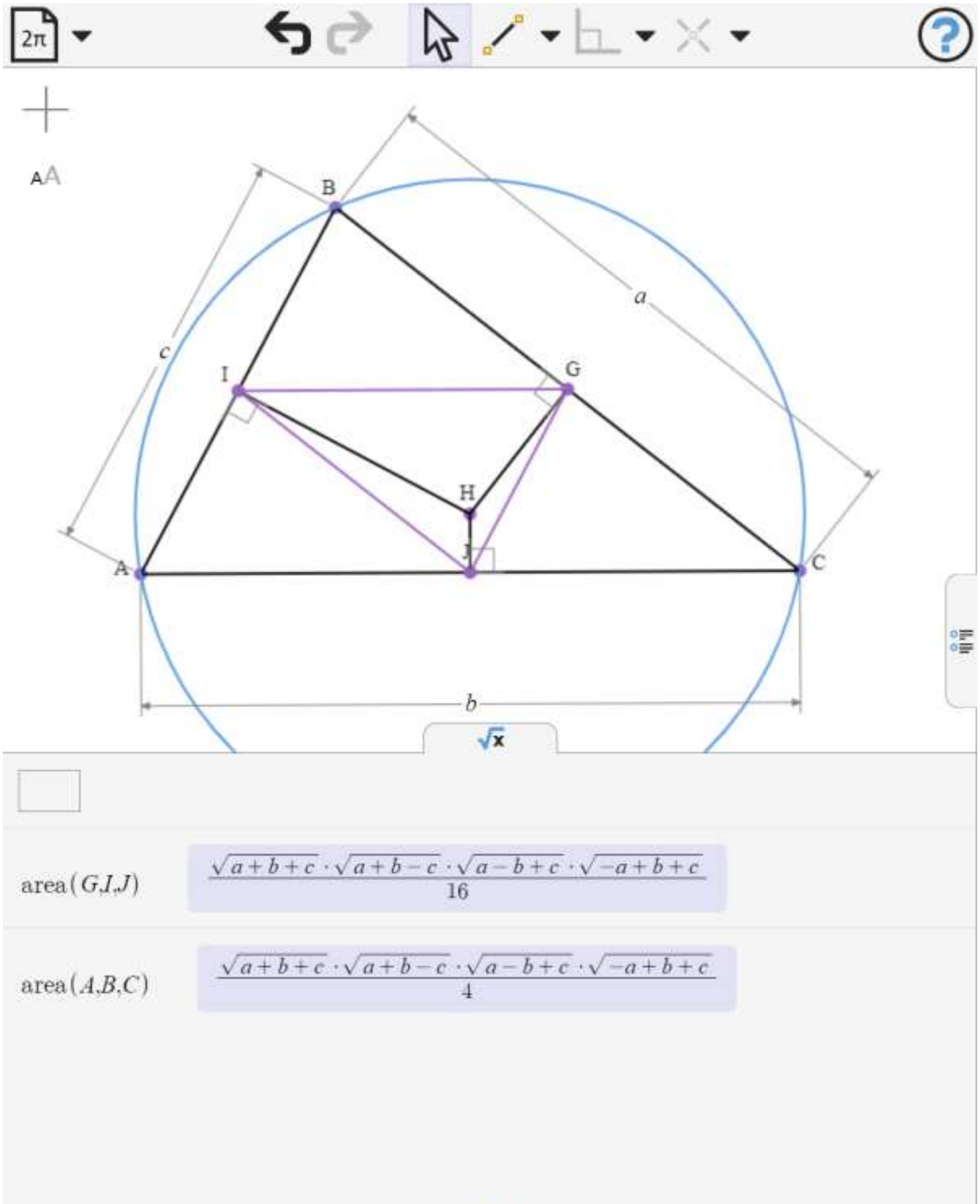
125. Pedal triangle of the circumcenter

Let K be the area of the pedal triangle of the circumcenter of ABC. Show that the area of ABC is 4K.



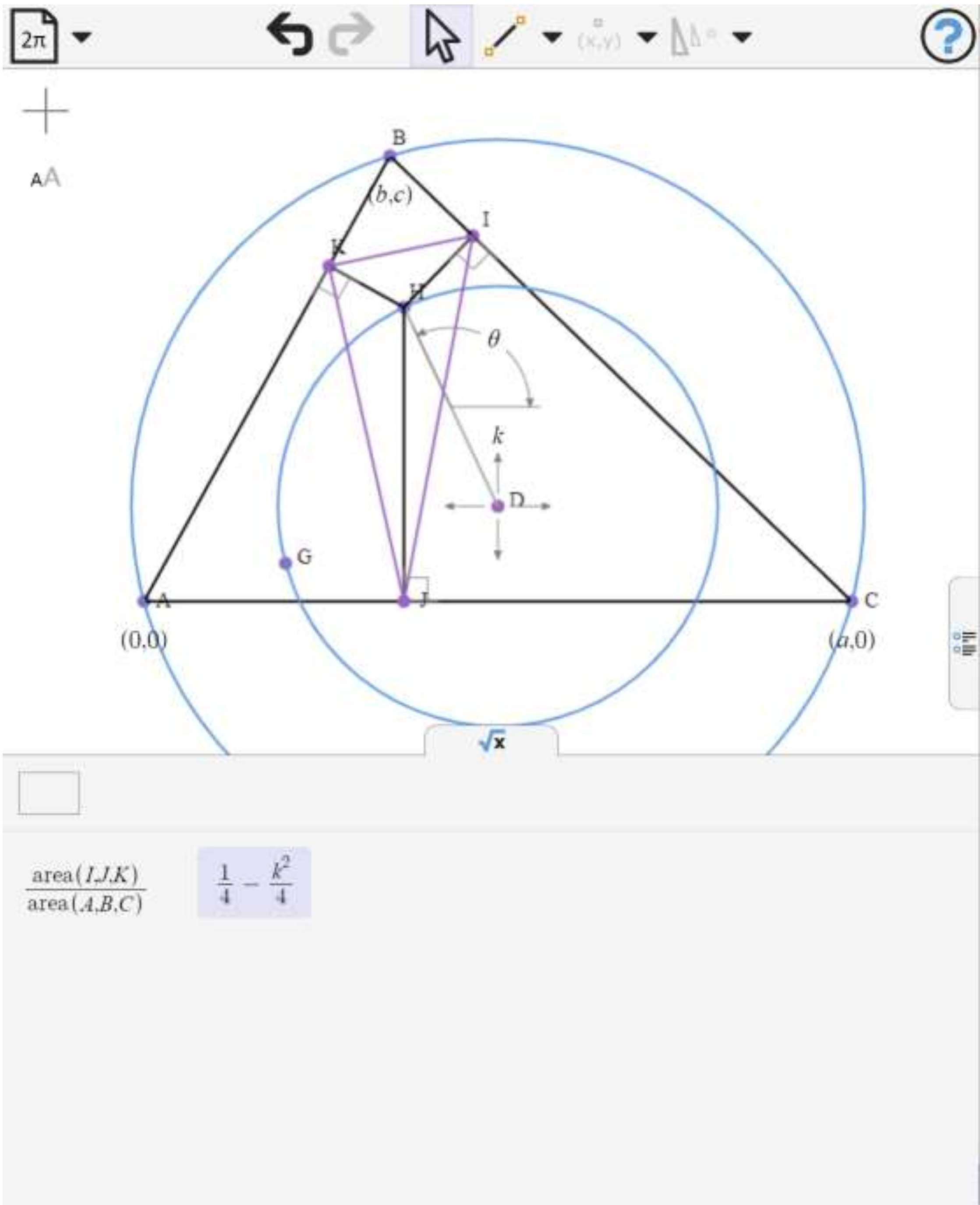
126. Pedal triangle of the incenter

Let K be the area of the pedal triangle of the incenter of ABC. Show that $\frac{K}{ABC} = \frac{r}{2R}$



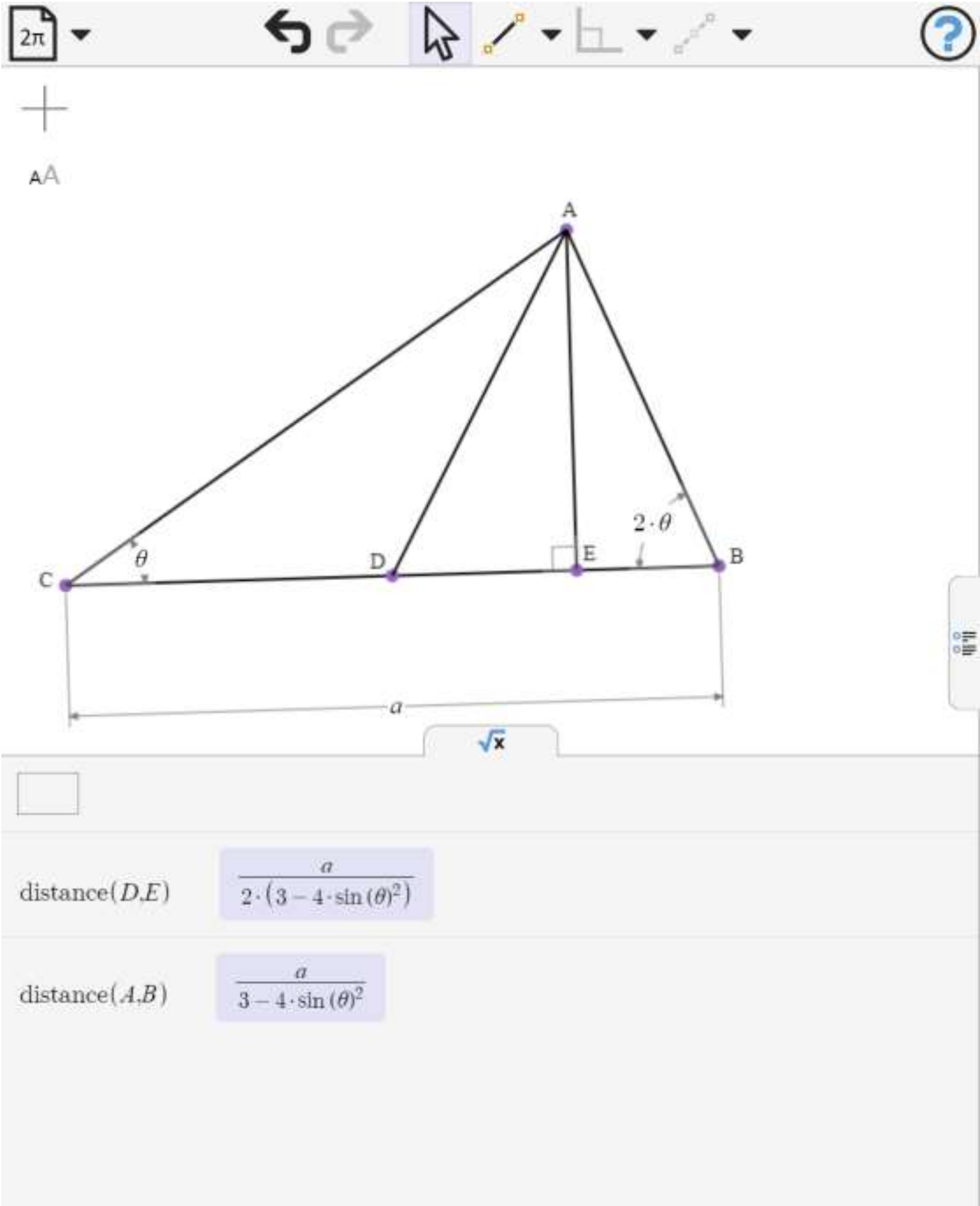
127. Pedal triangle of a point on a concentric circle to the circumcircle

Let ***C*** be a circle concentric to the circumcircle with radius k times the radius of the circumcircle. Let H be a point on ***C***. Let IJK be the pedal triangle of H. Show $\frac{IJK}{ABC} = \frac{1-k^2}{4}$



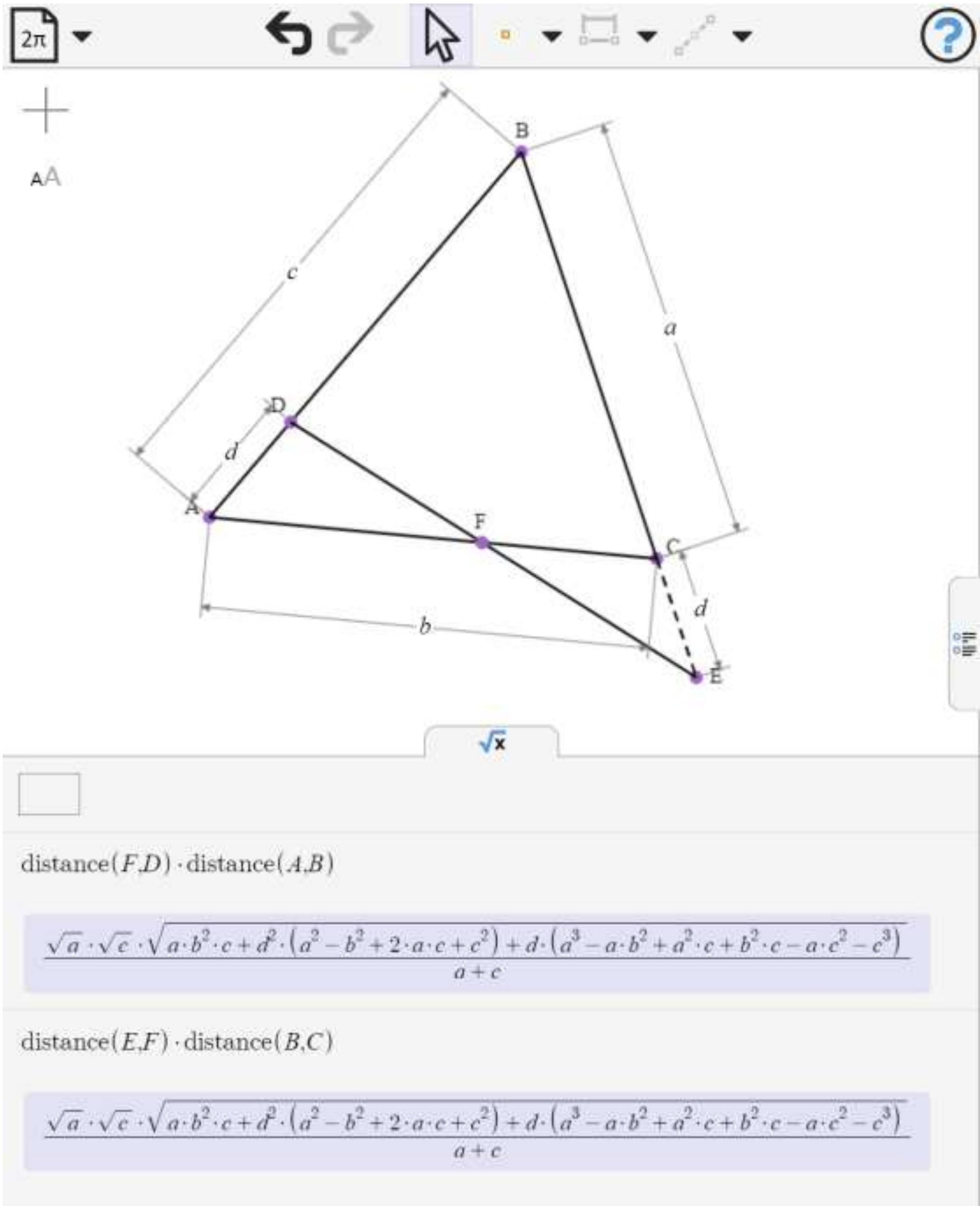
128. A triangle with one angle twice the other

Let angle ABC be twice ACB, and let D be the midpoint of side BA and E the foot of the altitude from A. Show that AB=2.DE



129. Points equal distance, but opposite directions along triangle sides from the base

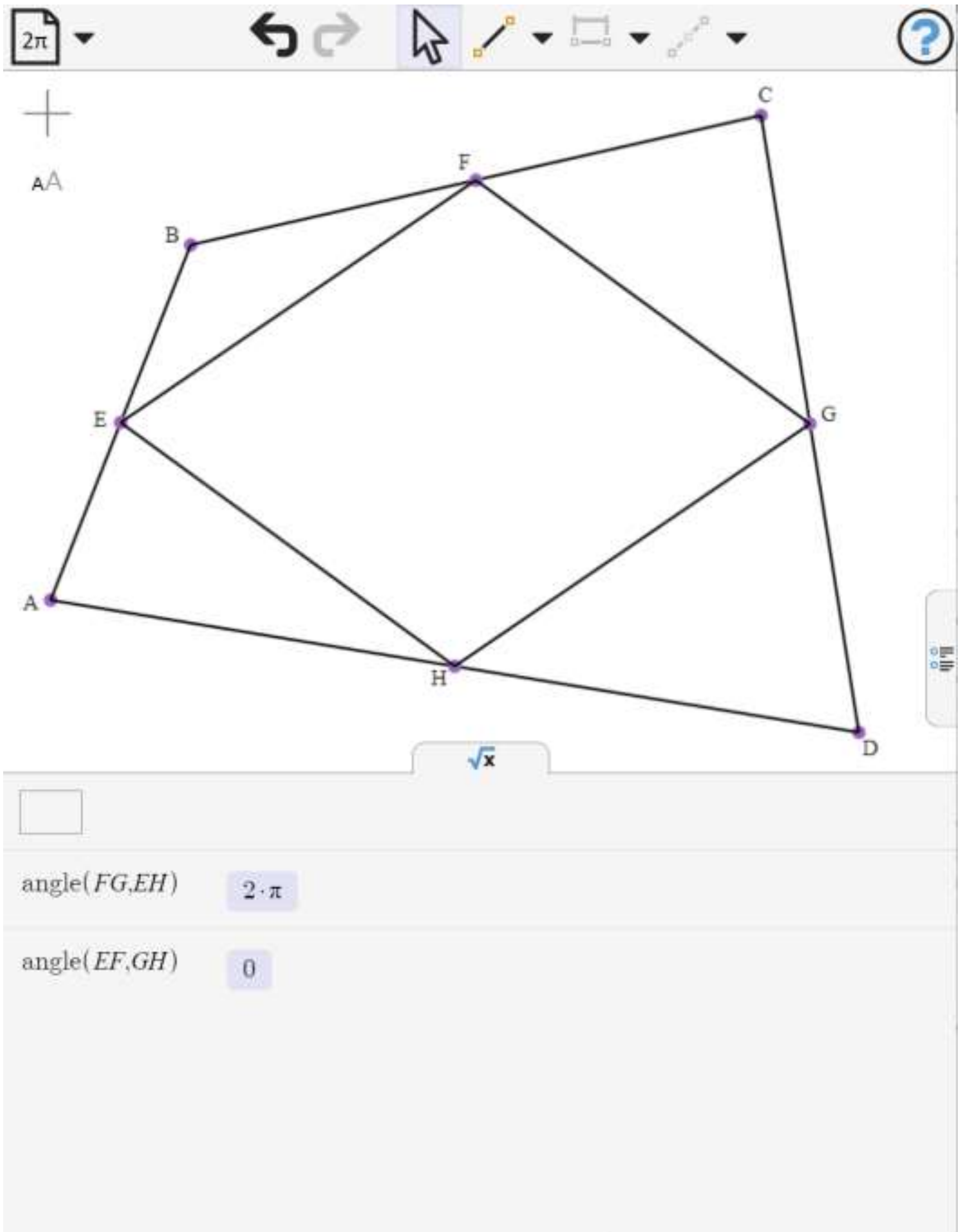
Let D and E be two points on two sides BA and BC of triangle ABC such that AD=CE. Let F be the intersection of AC and DE. Show that FD.AB=EF.BC



Quadrilaterals

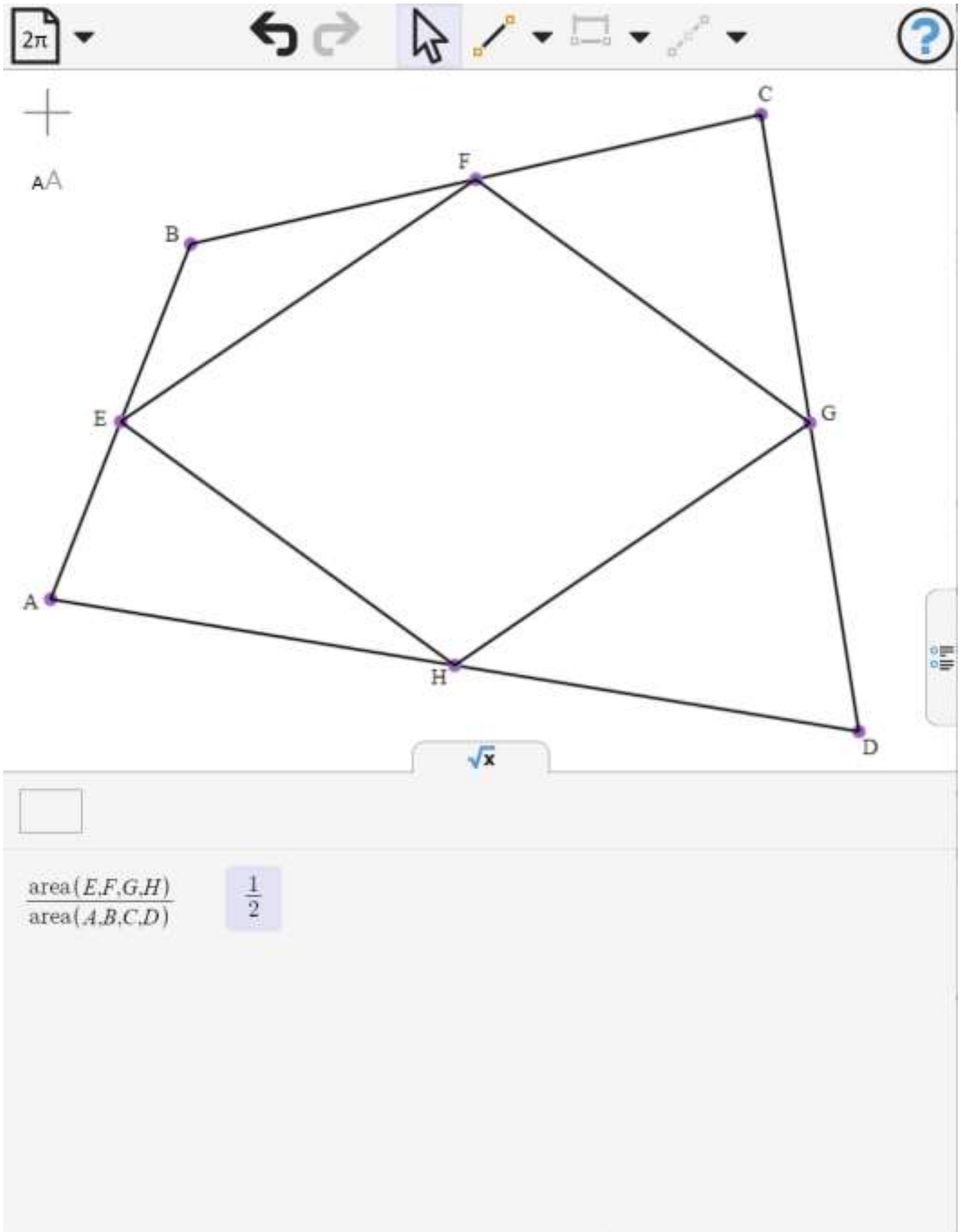
130. Midpoints of a quadrilateral

The figure formed when the midpoints of the sides of a quadrilateral are joined in order is a parallelogram



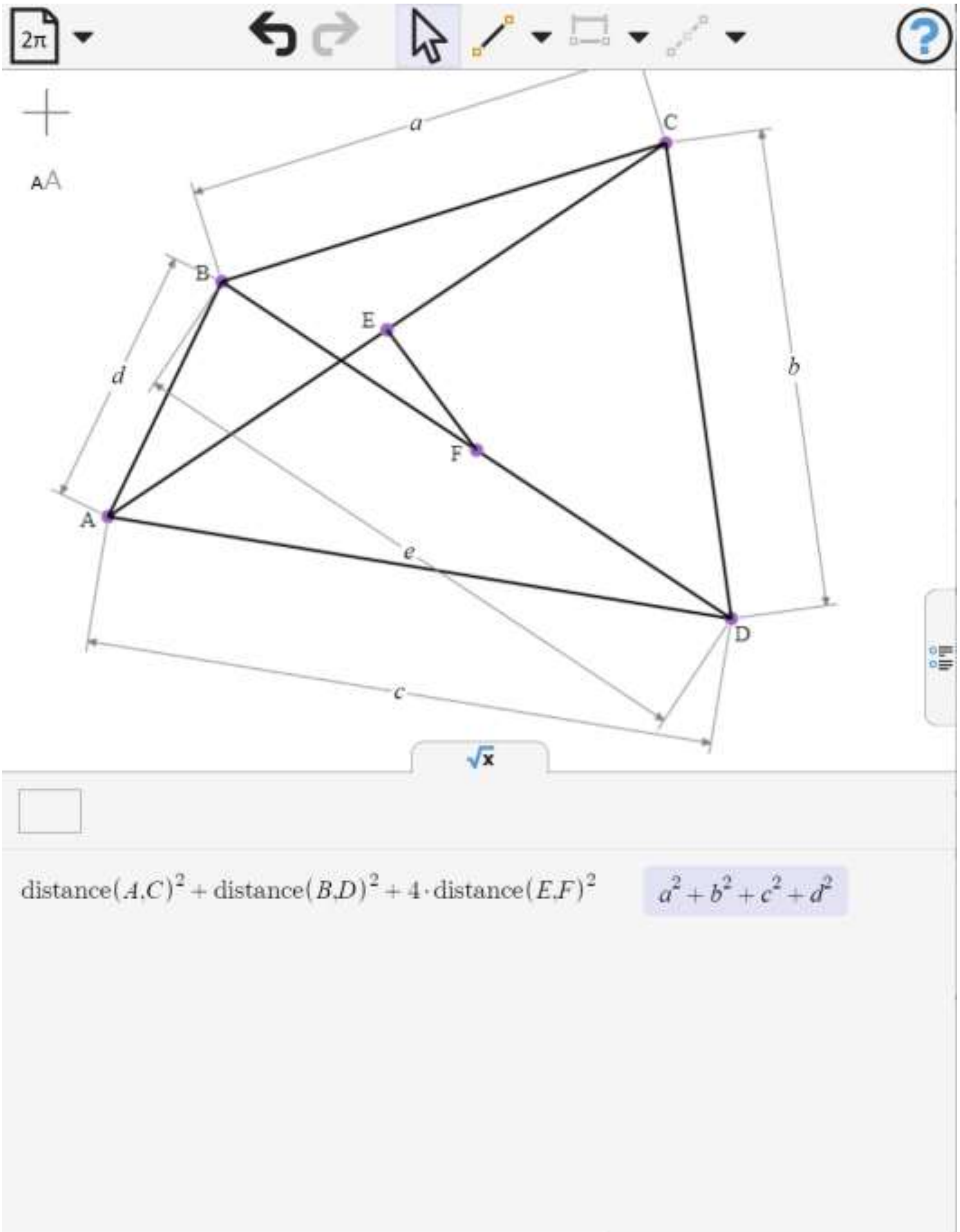
131. Area of the midpoint parallelogram

The area of the parallelogram whose vertices are the midpoints of the sides of a quadrilateral is equal to half the area of the quadrilateral.



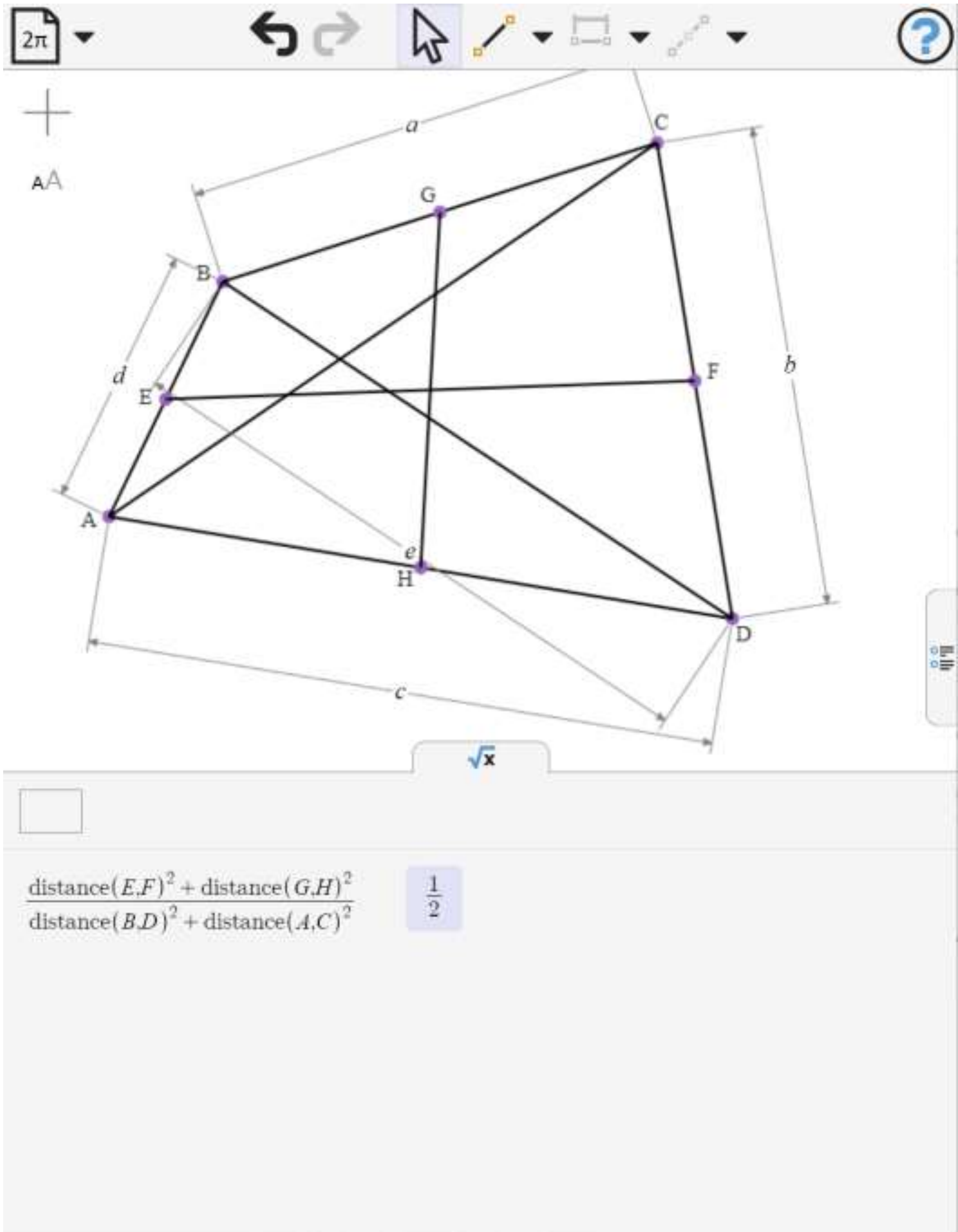
132. Sum of squares of the diagonals related to side lengths

The sum of squares of the sides of a quadrilateral is equal to the sum of squares of the diagonals increased by four times the square of the segment joining the midpoints of the diagonals.



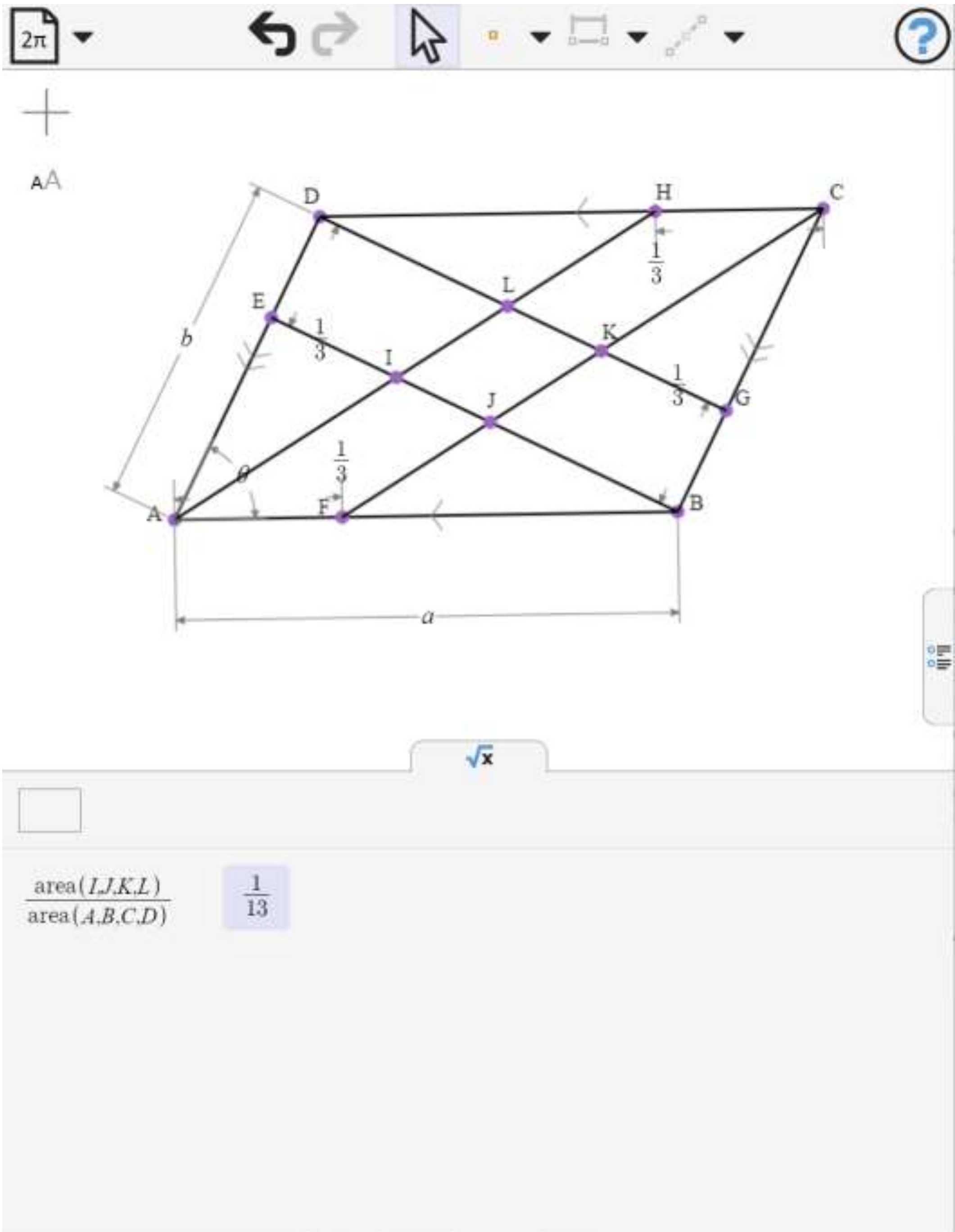
133. Diagonals related to lines joining midpoints

The sum of squares of the diagonals of a quadrilateral is equal to twice the sum of squares of the lines joining the midpoints of the opposite sides of the quadrilateral.



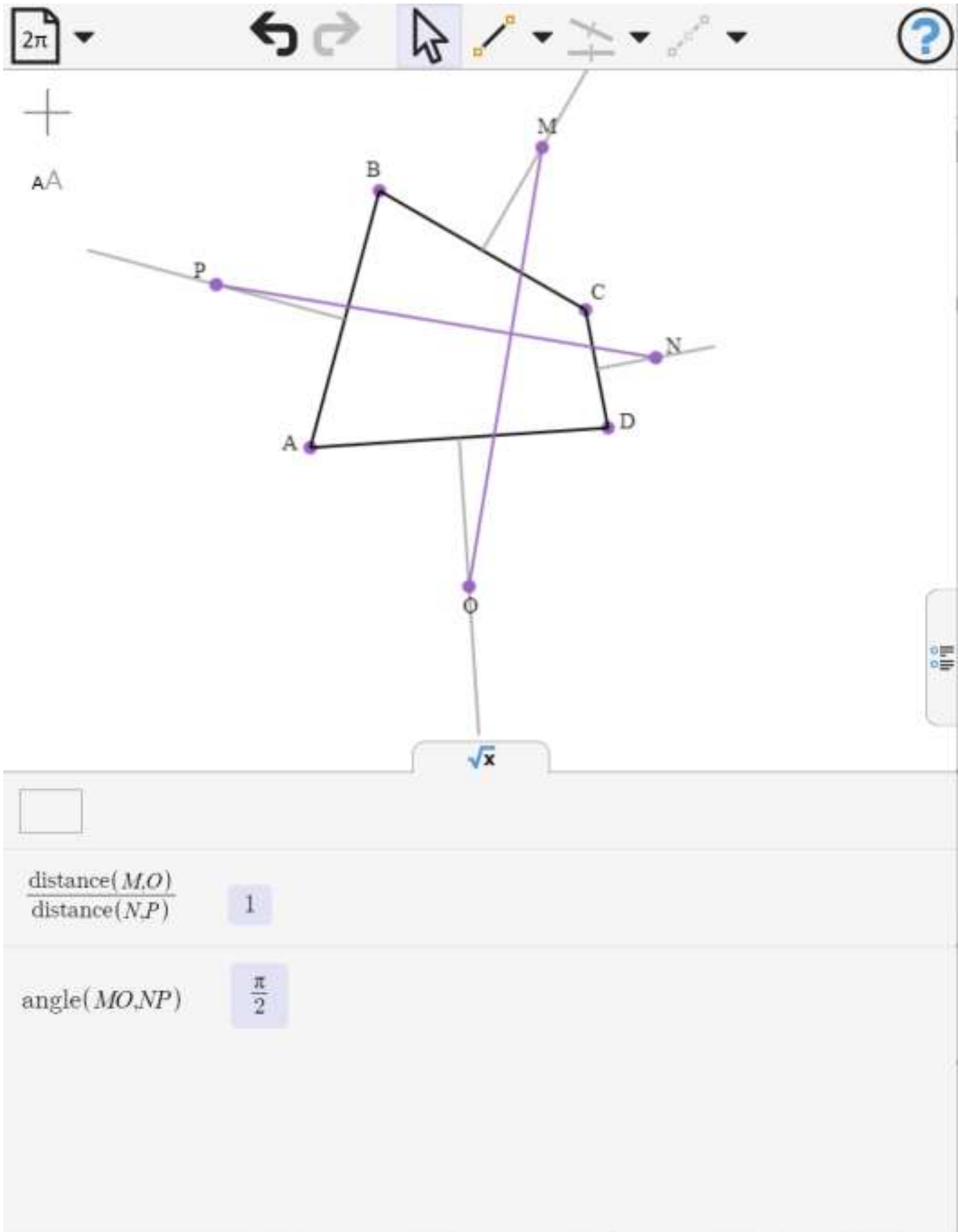
134. Area of an intersection parallelogram

Let points E,F,G,H be 1/3 of the way along segments DA, AB, BC, CD of parallelogram ABCD. Let I be the intersection of AH and BE, let J be the intersection of CF and BE, let K be the intersection of DG and CF, and let L be the intersection of AH and DG. Then the ratio of areas of ABCD to IJKL is 1:13.



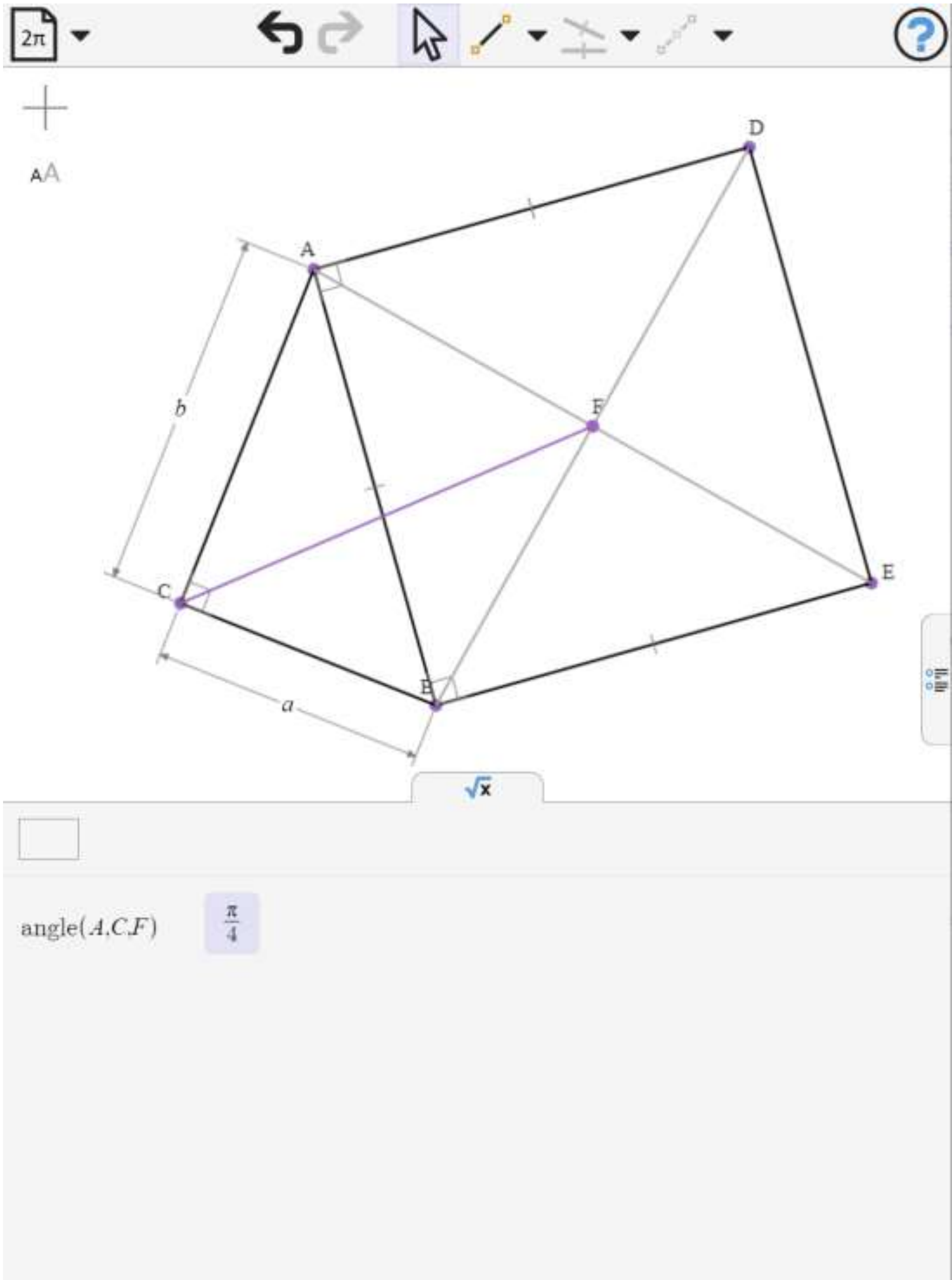
135. Von Abuel's Theorem

Squares are drawn externally on the sides of a quadrilateral, Show that the segments joining the centers of the opposite squares are equal and perpendicular.



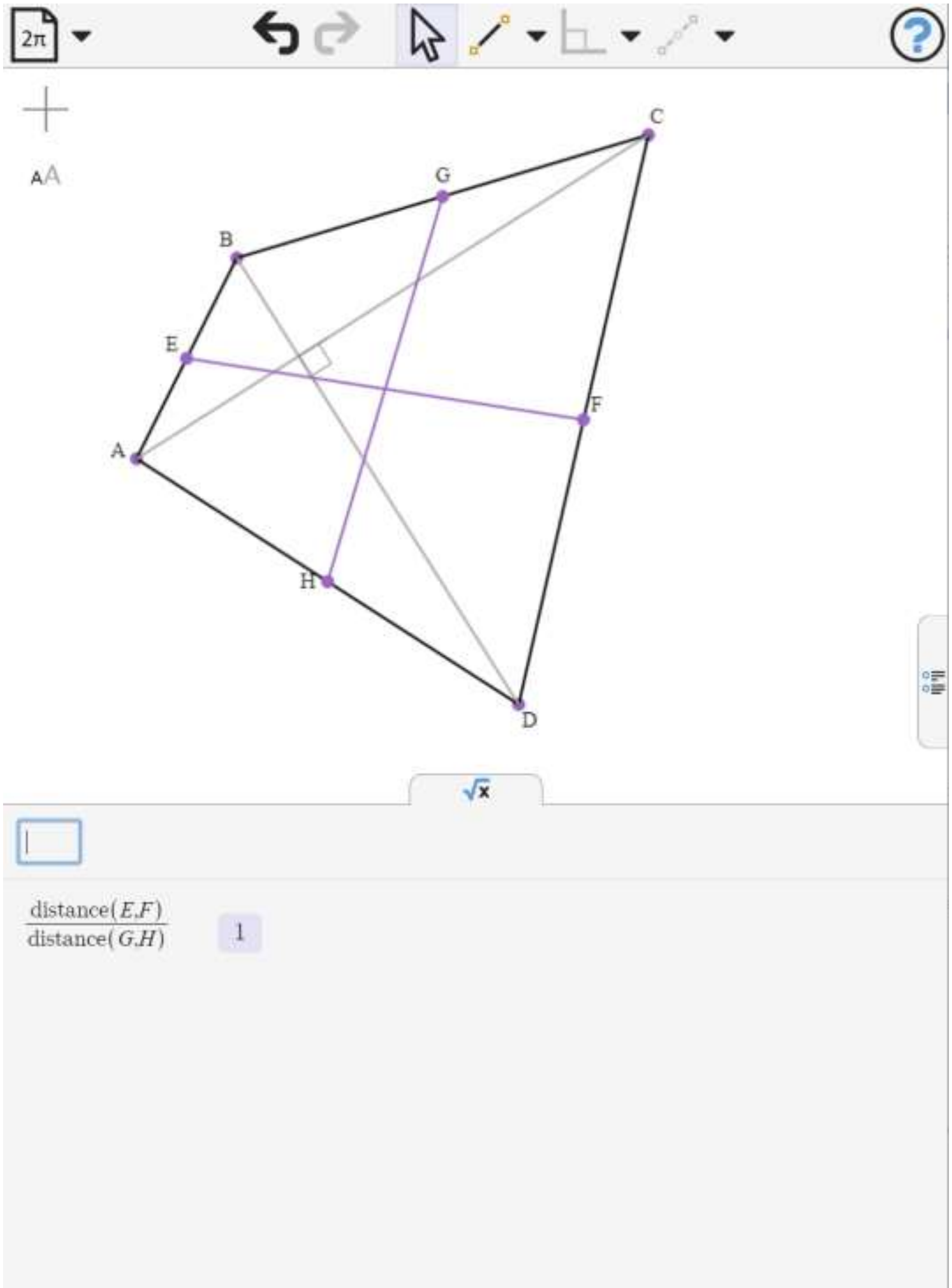
136. The square on the hypotenuse

On the hypotenuse AB of a right angled triangle ABC, a square is constructed. Let F be the intersection of its diagonals. Angle ACF = angle FCB.



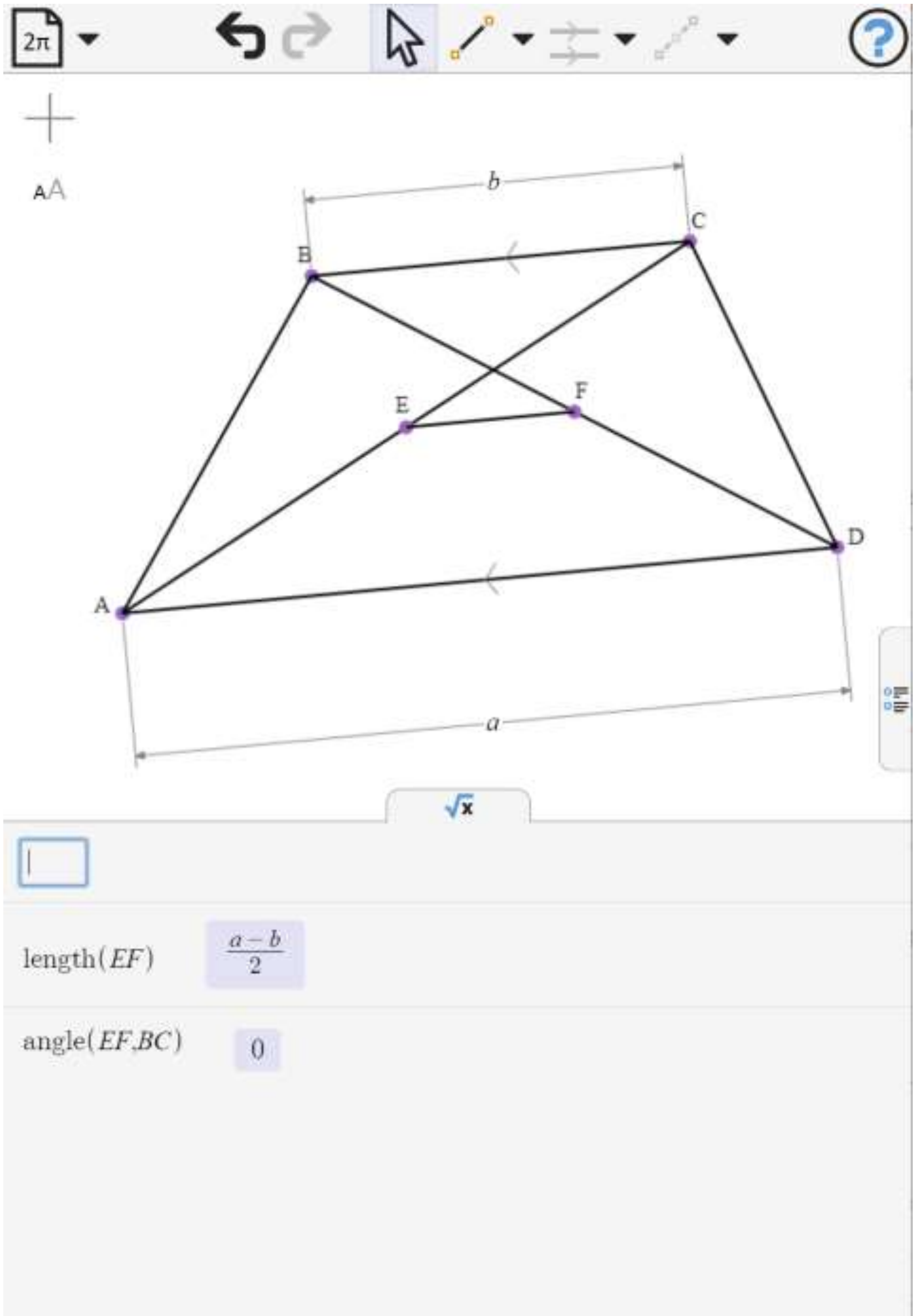
137. An orthodiagonal quadrilateral

In an orthodiagonal quadrilateral (quadrilateral with perpendicular diagonals) the lines joining the centers of opposite sides are equal.



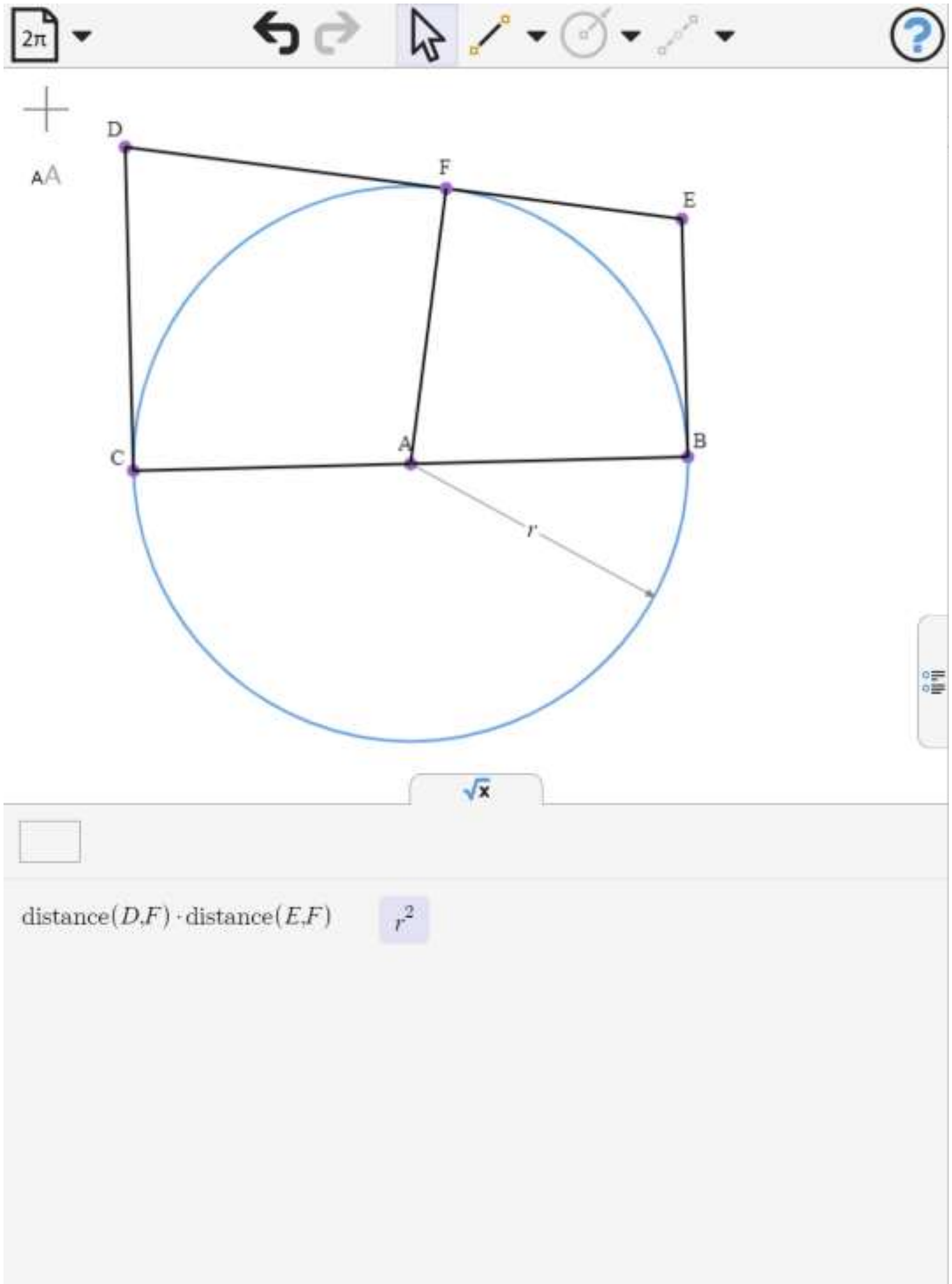
138. Midpoints of trapezoid diagonals

Let E,F be the midpoints of the diagonals of a trapezoid. Then EF is parallel to the two parallel sides of the trapezoid and its length is half the difference between the lengths of the parallel sides.



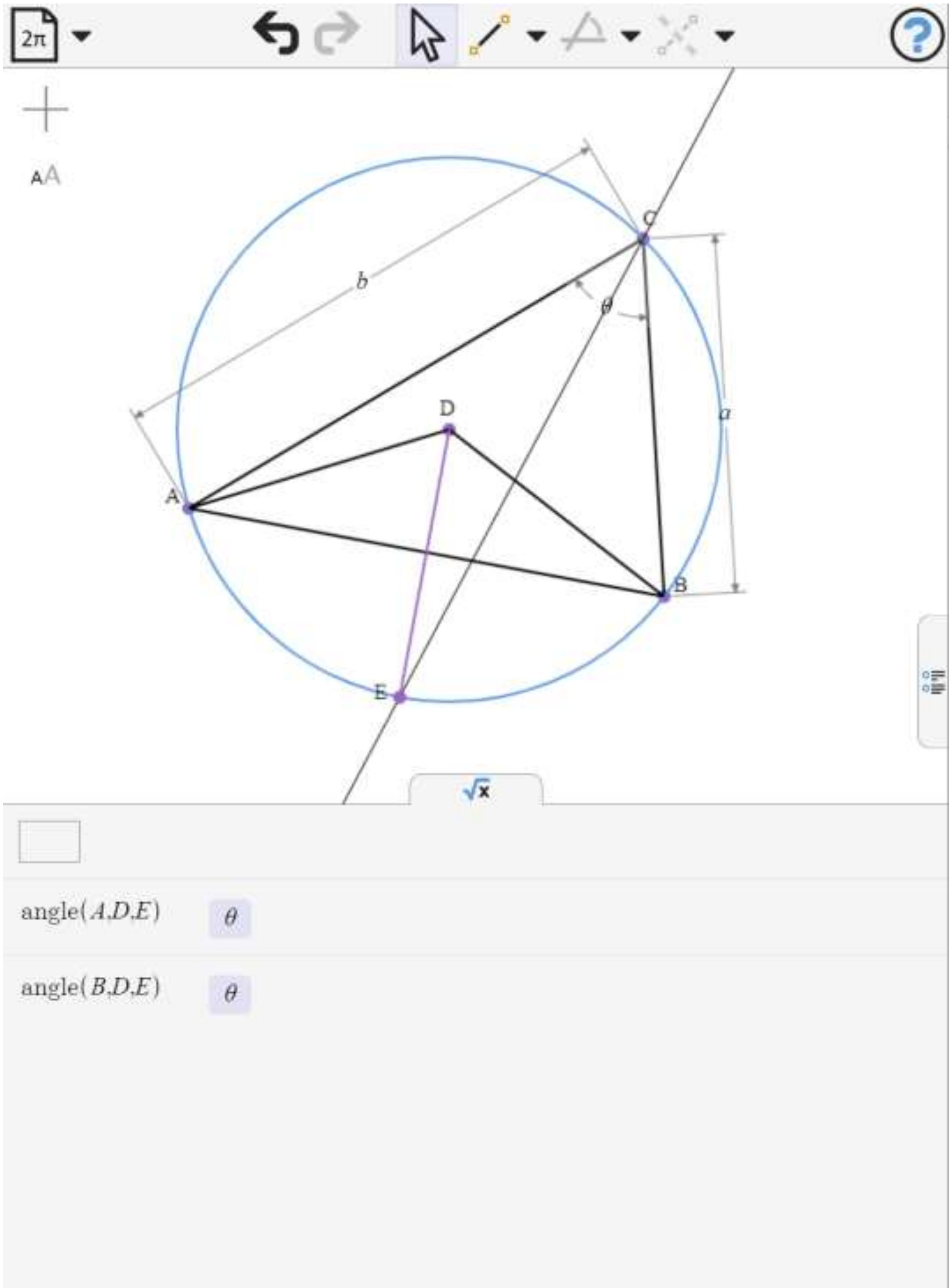
139. Three tangents and a diameter

Through point F on the circle with diameter BC, a tangent to the circle is drawn meeting the tangents at B and C in points D and E. Show that DF.EF=AB^2



140. Angle bisector and circumcircle

The angle bisector at C bisects the arc AB of the circumcircle of triangle ABC.



141. A circle through the midpoint of the hypotenuse

Let D be the midpoint of the hypotenuse of the right angled triangle ABC. A circle passing through A and D meets AB in F. G is the point on the circle such that FG is parallel to BC. Show that BC = 2FG.