

141 theorems proved using GXWeb

GXWeb gives both numeric and symbolic measurements taken from your diagram. The symbolic measurements are exact, computed algebraically, and hence provide a form of automated proof of theorems. We give examples here.

1 Introduction

GXWeb is not designed primarily for geometry theorem proving, however, the fact that it can derive symbolic results from a geometry diagram means that it can be used as a theorem prover with a little creativity. In this document, we use the software to prove 141 different theorems, taken from the book “Machine Proofs in Geometry” by Chou, Gao and Zhang.

Geometry theorems frequently give results which are stated in terms of unexpected properties which hold. For example, while it is unremarkable that two points lie on a straight line, it may be the result of a theorem that three points are collinear. Similarly, a theorem might state that three lines are concurrent, or four points lie on a circle. GXWeb generates symbolic quantities for measurements (such as distance, angle, area) taken from the drawing. Here are some ideas for making symbolic measurements to establish theorem results.

3 Collinear points: To show that points A,B,C are collinear, you could draw the line segment between A and B, then ask GXWeb to measure the distance(C,AB). This will measure the perpendicular distance between C and the line AB. If the distance is 0, then the lines are collinear. Another approach is to evaluate area(A,B,C). This returns the area of the triangle ABC and will be 0 if the points are collinear.

Note the difference if you make the measurement in the numeric pane versus the symbolic pane. If you measure in the numeric pane it tells you that the points are collinear for this specific geometry, but if there are points which can be dragged, or parameters changed, it may not remain 0. If it is 0 in the symbolic pane, it is guaranteed to be 0 for any value of the parameters.

3 concurrent lines: To show that lines L0, L1, L2 are concurrent, first create the intersection point A between L0 and L1. Now evaluate distance(A,L2). If this is 0, then A lies on all 3 lines.

4 concyclic points: To show A,B,C,D lie on a circle, you can create the circumcircle of A,B,C, let's assume its center is called E. now you would need to show distance(D,E)=distance(A,E). In some cases you can do this by inspection. In other cases you might need some simplification done for you, so you can ask for

$$\frac{\text{distance}(D, E)}{\text{distance}(A, E)}$$

Which should evaluate to 1 if the distances are identical.

Alternatively,

$$\text{distance}(D, E) - \text{distance}(A, E)$$

should evaluate to 0.

Equilateral triangle: To check that triangle ABC is equilateral, you could ensure all the side lengths are equal, by examining

$$\frac{\text{distance}(A, B)}{\text{distance}(B, C)} = 1$$

and

$$\frac{\text{distance}(A, C)}{\text{distance}(B, C)} = 1$$

Alternatively, you could check the angles:

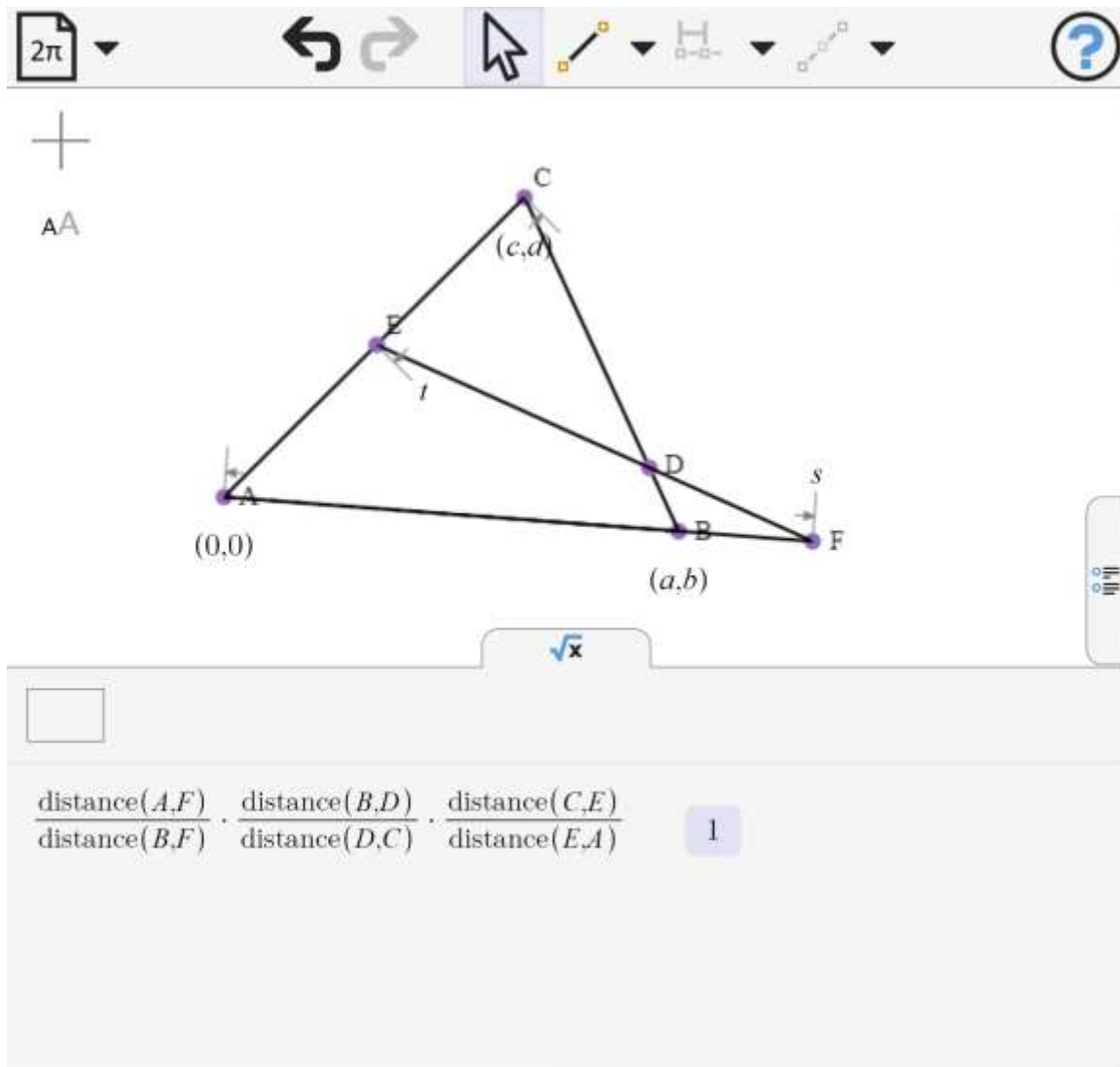
$$\text{angle}(A, B, C) = \frac{\pi}{3}$$

and

$$\text{angle}(B, C, A) = \frac{\pi}{3}$$

2 Geometry of Incidence

1. Menelaus Theorem



Without loss of generality, we make A be the origin.

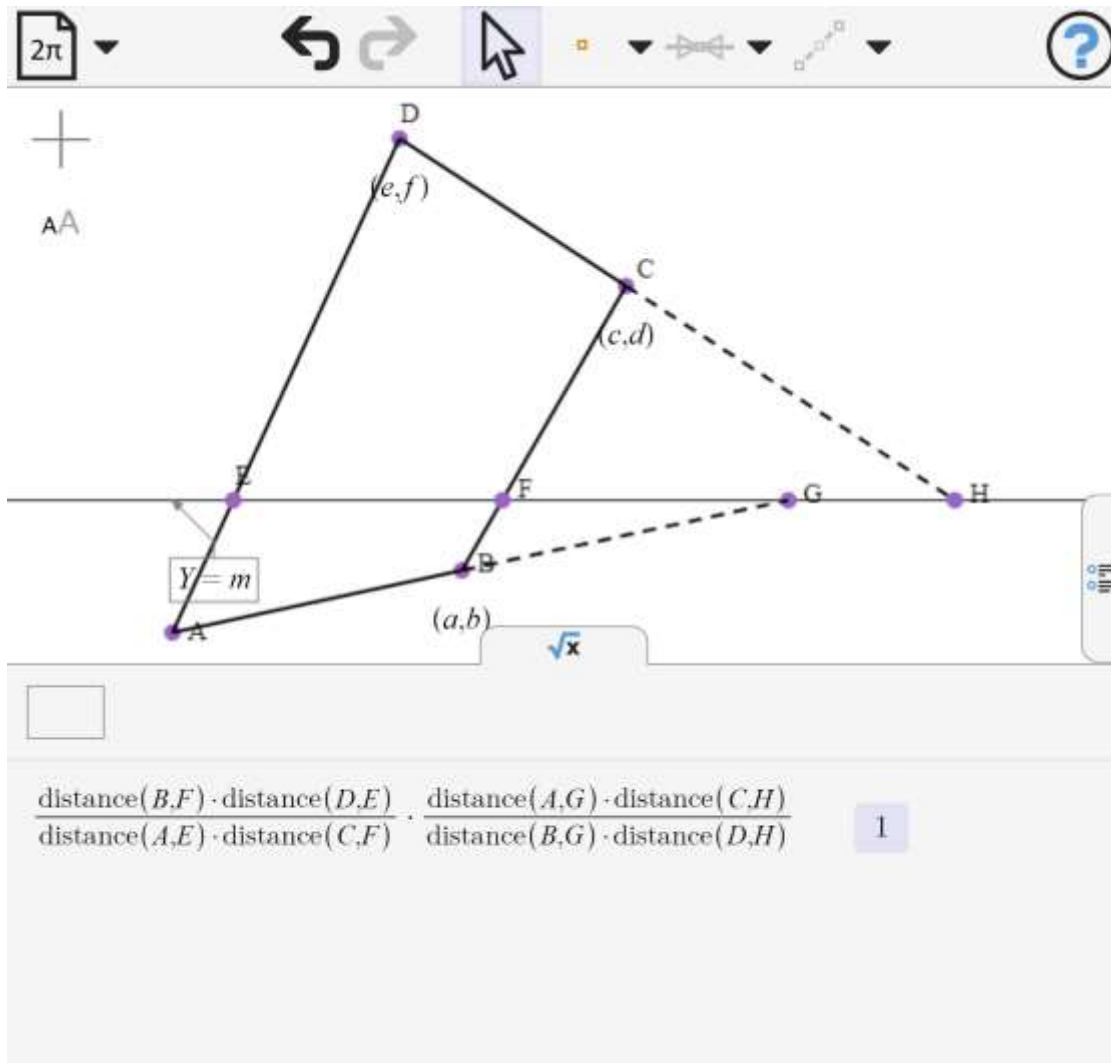
2. Converse of Menelaus Theorem

Ratio r_1 is defined to be BD/DC . GX point proportional constraint accepts the ratio BD/BC . So we enter $r_1/(1+r_1)$.

The diagram shows a triangle ABC with vertices $A(0,0)$, $B(a,b)$, and $C(c,d)$. A line passes through points D on BC , E on AC , and F on the extension of AB . The ratio $BD/DC = r$ is indicated, and the ratio $CE/EA = 1/r$ is also shown. The ratio BF/FA is shown to be $r \cdot s$.

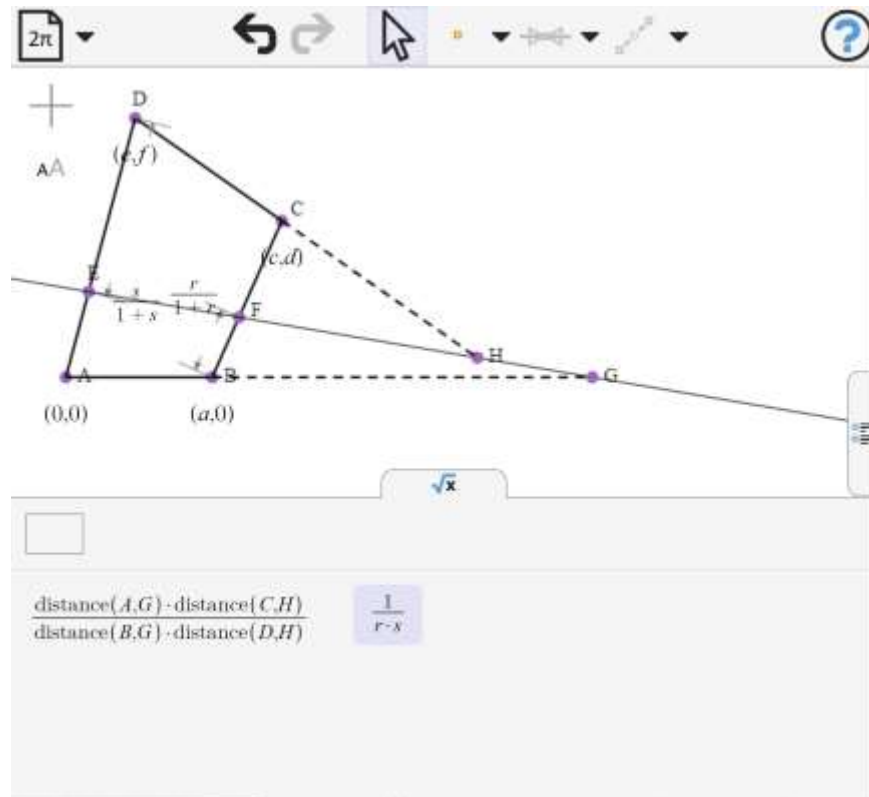
distance(B,F)
distance(A,F) $r \cdot s$

3. Menelaus Theorem for a Quadrilateral



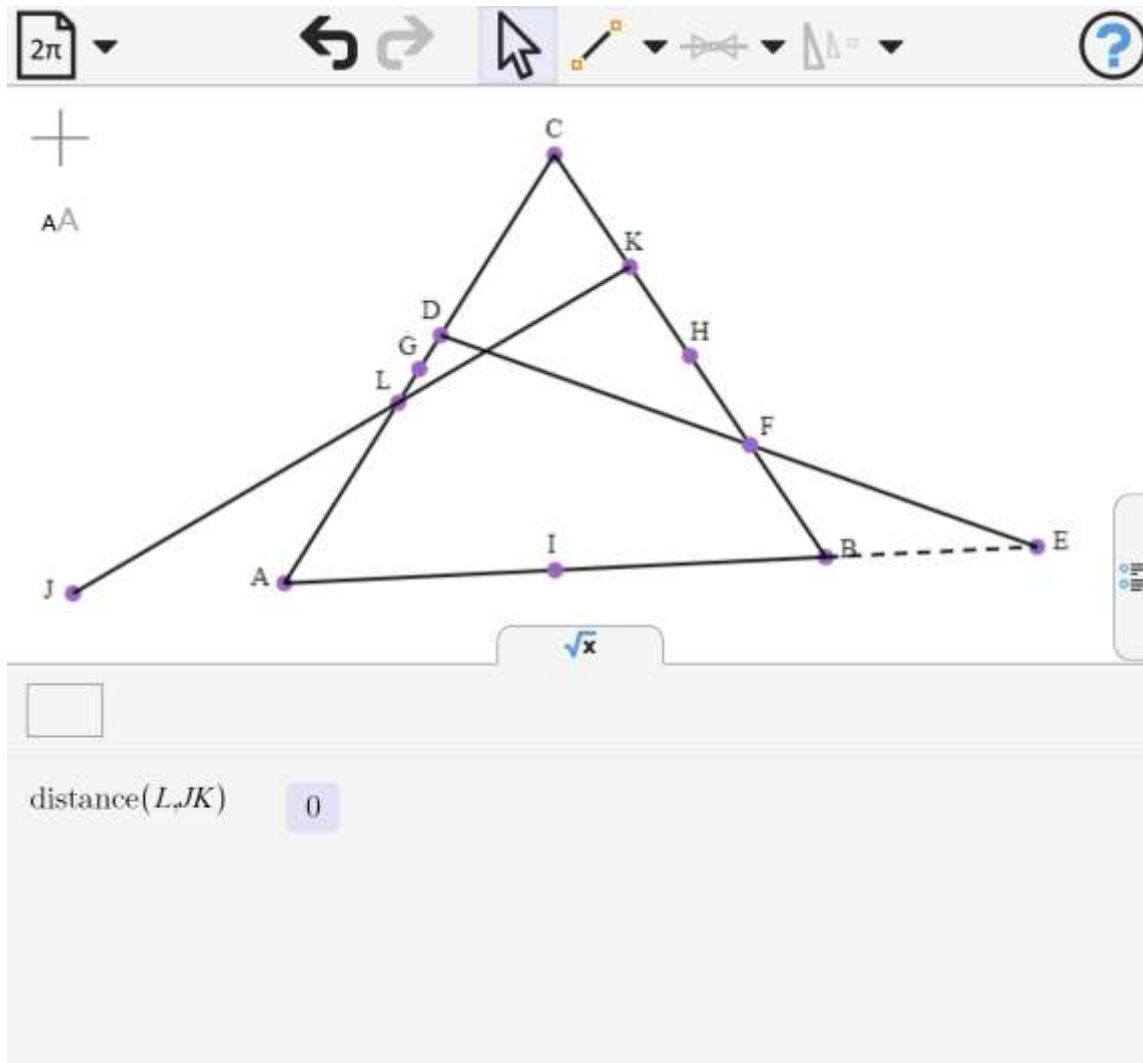
Here we have elected without loss of generality to make A the origin and the line $y=b$.

4. Menelaus Theorem for a Quadrilateral (done a little differently)



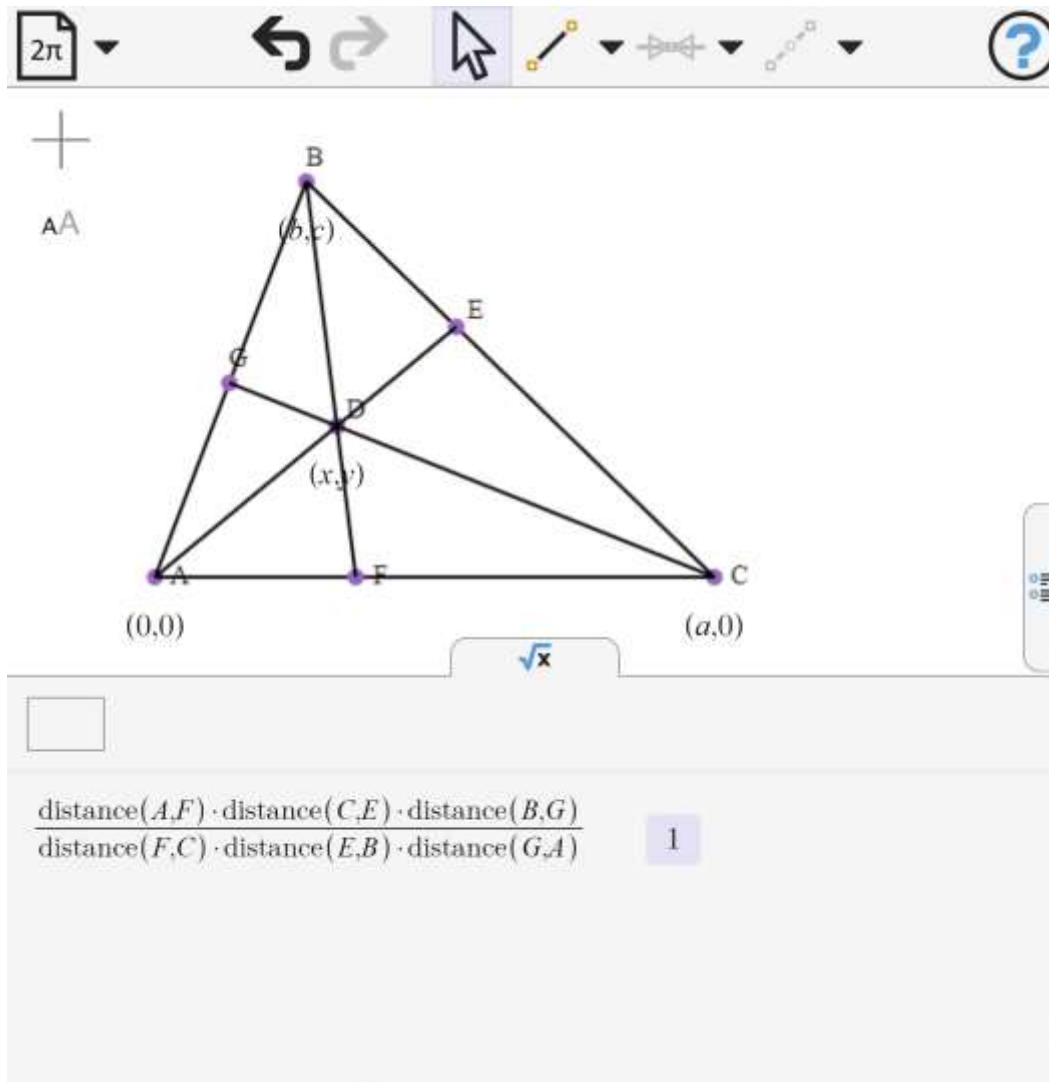
In this one, we specify the coordinates of the quad points and proportions for E and F, and measure the other proportions. A useful trick is to make one of the coordinates (0,0), and another (a,0) which cuts down on the complexity of the results.

5. The isotomic points of three collinear points are collinear



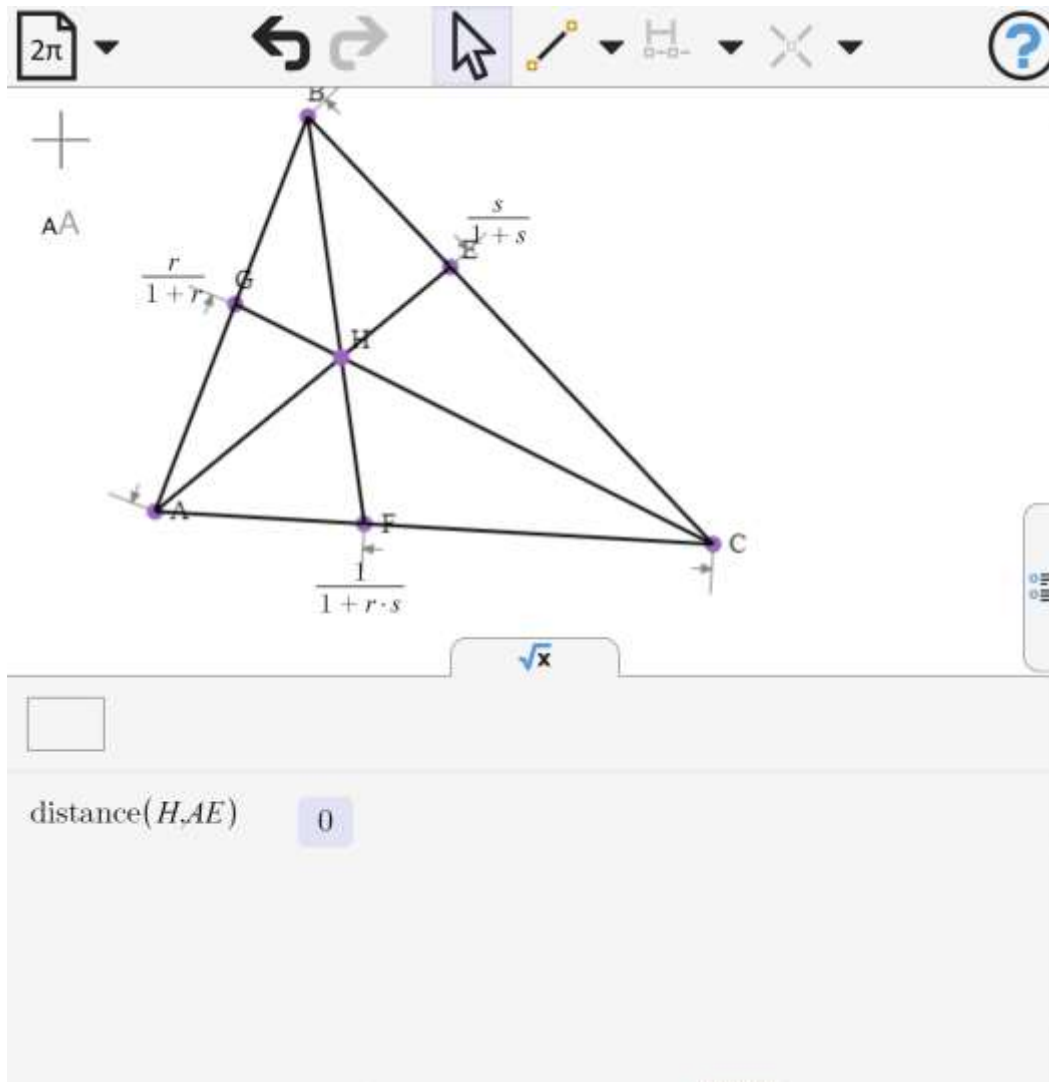
Isotomic points are created by reflecting points in segment midpoints. (reflection in a point is done in Geometry Expressions using dilation of scale -1).

6. Ceva's Theorem



Here is a direct proof of Ceva's theorem.

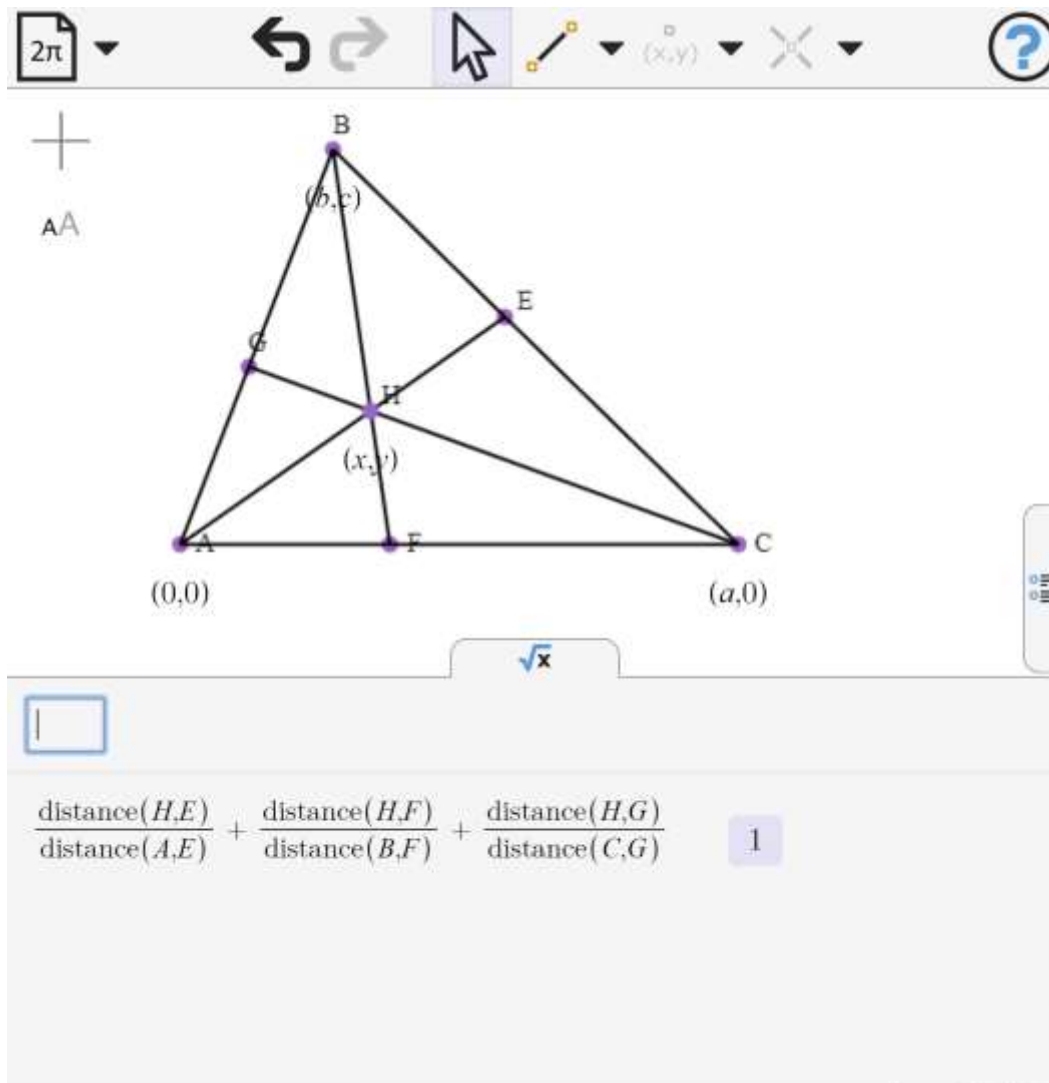
7. Converse of Ceva's Theorem



Here we have set H to be the intersection of lines AD and BE . We measure its distance to line CF

8. Ratios on the Cevian

If LMN is the Cevian Triangle of point S in triangle ABC, we have $\frac{SL}{AL} + \frac{SM}{BM} + \frac{SN}{CN} = 1$



SN and CN are hidden for clarity.

9. Sub triangle areas

If EFH is the Cevian Triangle of point S in triangle ABC , we have the following area relationship

The diagram shows a triangle ABC with vertices $A(0,0)$, $B(b,c)$, and $C(a,0)$. A point $H(x,y)$ is located inside the triangle. Lines AH , BH , and CH are drawn. Points E , F , and G are on sides BC , AC , and AB respectively, such that EFH is the cevian triangle of H .

The area relationship is given by:

$$\frac{\text{area}(B,E,F) \cdot \text{area}(A,G,E) \cdot \text{area}(C,F,G)}{\text{area}(B,F,G) \cdot \text{area}(A,E,F) \cdot \text{area}(C,G,E)} = 1$$

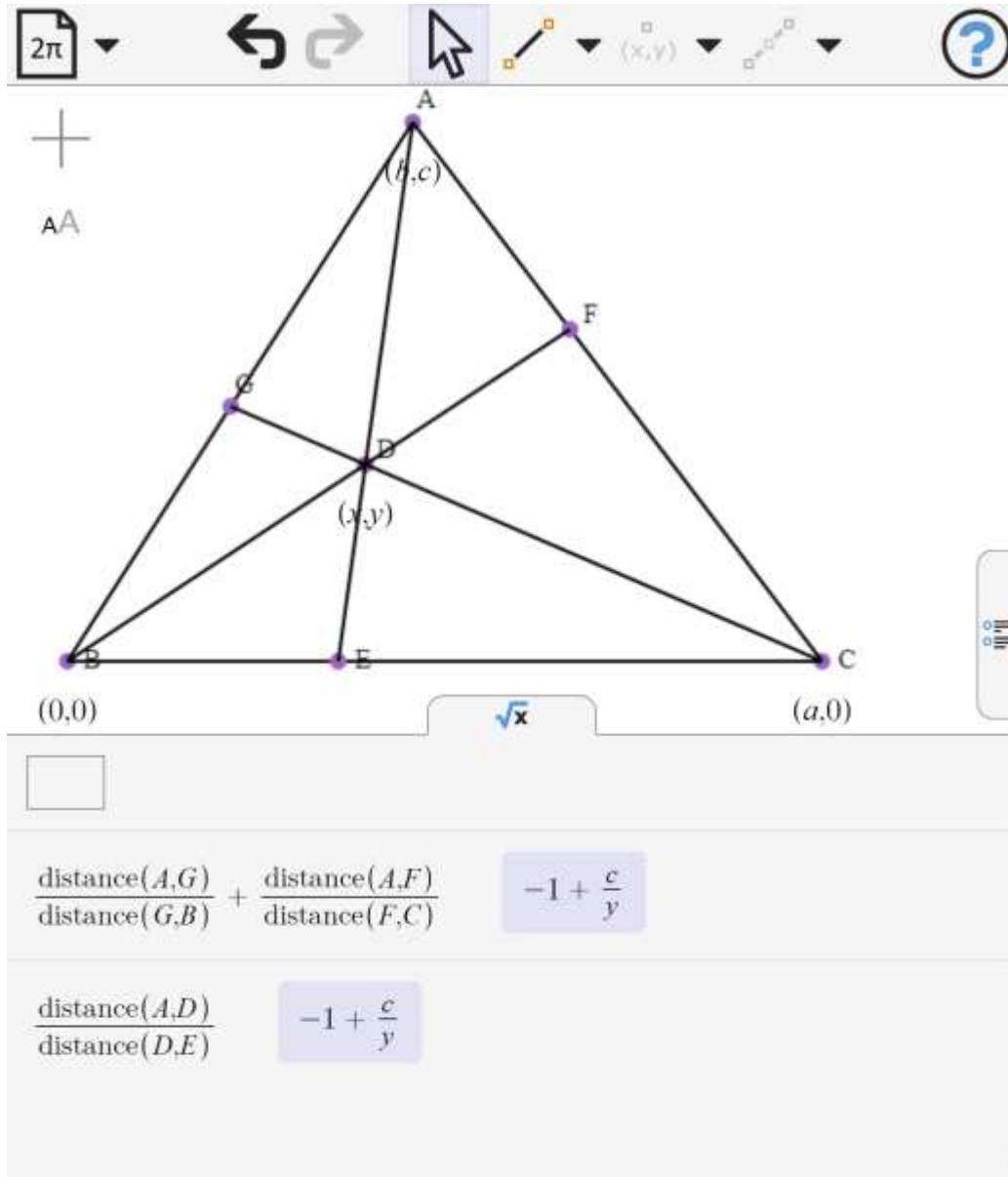
10. Ratios of parallels

$\frac{\text{distance}(G,E)}{\text{distance}(C,H)} + \frac{\text{distance}(G,F)}{\text{distance}(A,I)} + \frac{\text{distance}(G,D)}{\text{distance}(B,J)} = 1$

D,E,F are the feet of the altitudes of ABC. G is an arbitrary point. H lies on AB and CH is parallel to GE. I lies on BC and AI is parallel to GF. J lies on AC and BJ is parallel to GD. The sum of the ratios of the parallel sides is 1.

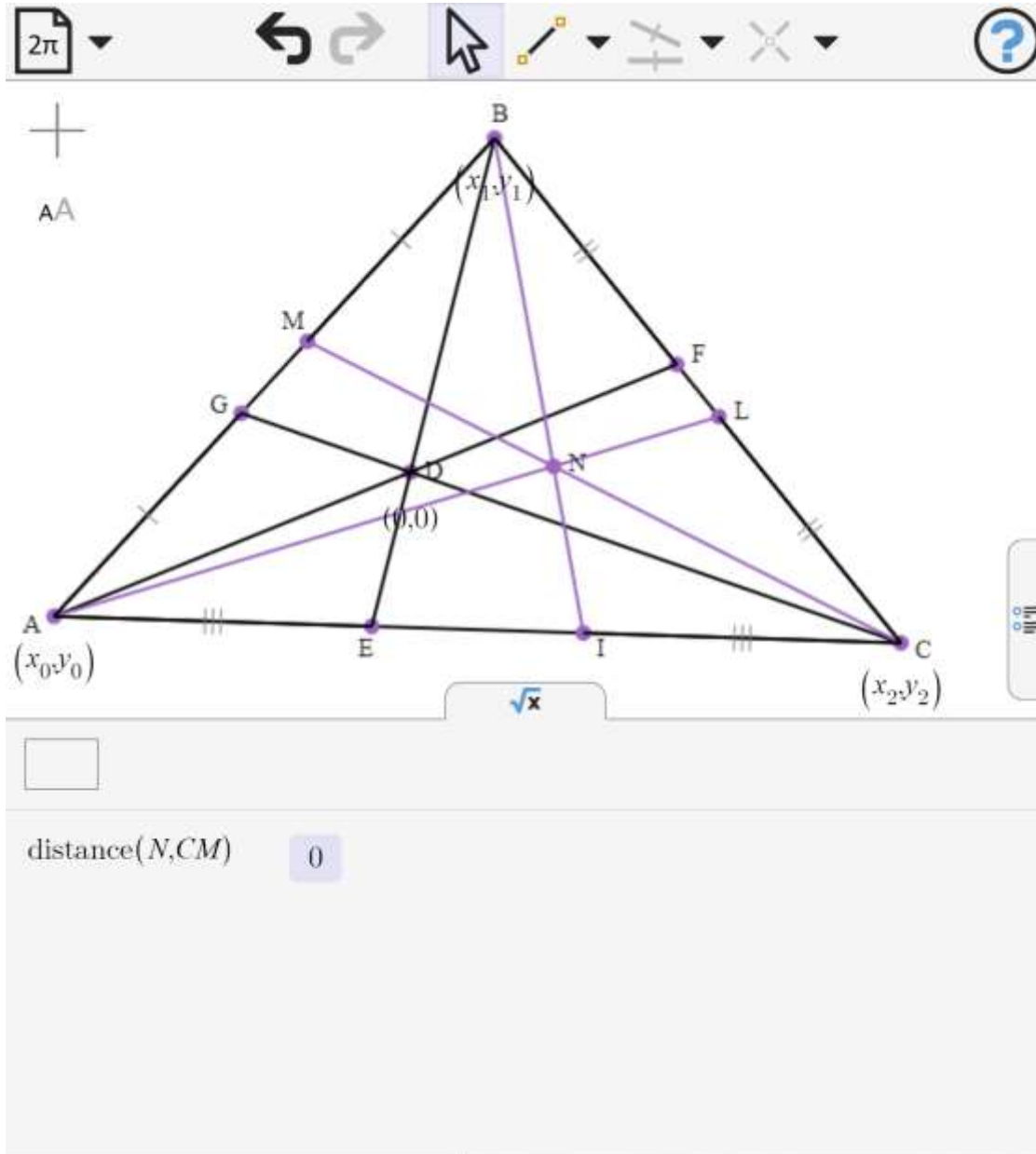
11. More ratios

If EFG is the Cevian triangle of the point D for the triangle ABC, we have: $\frac{AD}{DE} = \frac{AG}{GB} + \frac{AF}{FC}$



12. Isotomics of Cevian point

If the three lines joining three points marked on the sides of a triangle to the opposite vertices are concurrent, the same is true of the isotomics of the given points.



N is defined to be the intersection of AL and BI we measure its distance to CM

13. The cross ratio of four points on a line is unchanged by projection

Let A,B,C,D be four collinear points. The cross ratio denoted $(ABCD) = \frac{CA/CB}{DA/DB}$

$$\frac{\text{distance}(F,H) \cdot \text{distance}(G,I)}{\text{distance}(F,I) \cdot \text{distance}(G,H)} = \frac{t \cdot (1-s)}{s \cdot (1-t)}$$

$$\frac{\text{distance}(B,D) \cdot \text{distance}(E,C)}{\text{distance}(B,E) \cdot \text{distance}(D,C)} = \frac{t \cdot (1-s)}{s \cdot (1-t)}$$

Medians and Centroids

14. Line joining side midpoints

The line joining the midpoints of two sides of a triangle is parallel to the third side and is equal to one half the length.

The diagram shows a triangle with vertices A , B , and C . Side AB has length c , side BC has length a , and the base AC has length b . Points D and E are the midpoints of sides AB and BC respectively. A line segment DE connects these two midpoints. The length of DE is shown to be $\frac{b}{2}$. The angle between DE and AC is π , indicating they are parallel. The diagram is displayed in a software interface with a toolbar at the top and a data panel at the bottom.

angle(DE, AC) π

length(DE) $\frac{b}{2}$

15. Centroid Theorem

The three medians of a triangle meet in a point and each median is trisected by this point.

The diagram shows a triangle with vertices A, B, and C. Medians AD, BE, and CF are drawn, intersecting at the centroid G. The sides are labeled a, b, and c. The centroid G is the point where the medians intersect, and it is the point where each median is trisected.

$\frac{\text{distance}(G,E)}{\text{distance}(A,E)}$	$\frac{1}{3}$
$\text{distance}(G,BF)$	0

16. Median triangle

With the medians of a triangle a new triangle is constructed. The medians of the second triangle are equal to three quarters of the respective sides of the given triangle.

The diagram illustrates the construction of a second triangle from the medians of a first triangle. Triangle ABC has vertices A , B , and C . Medians AD , BE , and CF are drawn, intersecting at the centroid G . The midpoints of these medians are labeled I , E , and H . A second triangle IGH is formed by connecting these midpoints. The sides of triangle IGH are parallel to the medians of triangle ABC . The distance between points I and H is given by the formula:

$$\text{distance}(I,H) = \frac{3 \cdot c}{4}$$

17. Area of the median triangle

The area of the triangle having for sides the medians of a triangle is equal to three quarters the area of the given triangle.

The diagram illustrates a triangle ABC with medians AD , BE , and CF intersecting at the centroid G . The medians are extended to form a second triangle DEF . The sides of the original triangle are labeled a , b , and c . The diagram includes various geometric markers such as tick marks and arrows to indicate relationships between the lines.

$\frac{\text{area}(C,H,D)}{\text{area}(A,B,C)}$	$\frac{3}{4}$
$\text{distance}(I,H)$	$\frac{3 \cdot c}{4}$

18. Median midpoint

Show that the line joining the midpoint of a median to a vertex of the triangle trisects the side opposite the vertex considered.

The diagram shows a triangle ABC with vertices A , B , and C . The side lengths are labeled as a (opposite A), b (opposite B), and c (opposite C). Medians AD , BE , and CF are drawn, intersecting at the centroid E . Point F is the midpoint of side BC . A line segment AF is drawn, representing the line joining the midpoint of a median (E) to the vertex A . The length of BC is labeled as a . The diagram is shown in a software interface with various toolbars and a command input field.

distance(F,C) $\frac{a}{3}$

19. Parallel through the centroid

Show that a parallel to a side of a triangle through the centroid divides the area of the triangle into two parts, in the ratio of 4:5

The diagram shows a triangle with vertices A, B, and C. The base BC is labeled 'a'. The sides AB and AC are labeled 'c' and 'b' respectively. The medians AD, BE, and CF intersect at the centroid F. A line segment GH is drawn parallel to the base BC, passing through the centroid F. Point G is on side AB and point H is on side AC. The line segment GH is labeled 'x'.

Below the diagram, the ratio of the areas of the two parts is given as:

$$\frac{\text{area}(A,G,H)}{\text{area}(B,G,H,C)} = \frac{4}{5}$$

20. Distances from a median point to triangle sides

Show that the distances of a point on a median of a triangle from the sides including the median are inversely proportional to these sides.

distance(E,BC) $\frac{b}{a}$
 distance(E,AC)

21. A line through the centroid

Show that, if a line through the centroid G of the triangle ABC meets AB in M and AC in N we have $AN \cdot MB + AM \cdot NC = AM \cdot AN$

distance(A,F) · distance(G,C) + distance(F,B) · distance(A,G)

$$\frac{-b \cdot c \cdot (-1 + t)^2}{-2 + 3 \cdot t}$$

distance(A,F) · distance(A,G) $\frac{-b \cdot c \cdot (1 - t)^2}{-2 + 3 \cdot t}$

22. Median dividing a particular line

Two equal segments AD, AE are taken on the sides AB, AC of the triangle ABC. Show that the median issued from A divides DE in the ratio of the sides AC, AB

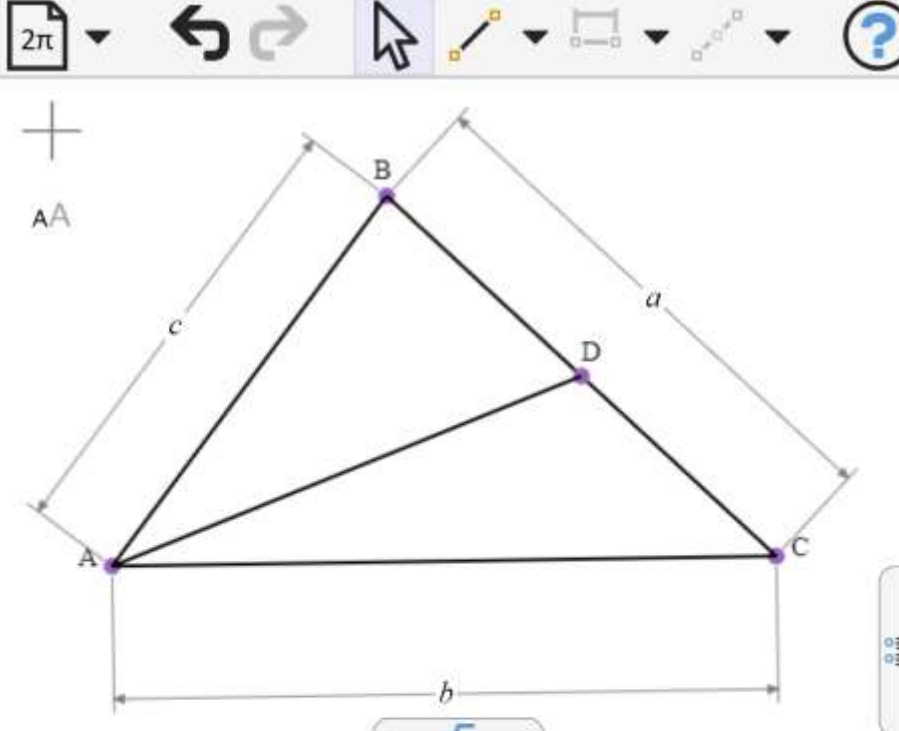
The diagram shows a triangle ABC with vertices A, B, and C. A median AF is drawn from vertex A to the midpoint F of side BC. Points D and E are marked on sides AB and AC respectively, such that AD = AE = d. A line segment DE is drawn. A line segment DG is drawn from D to G on BC such that AG is a median. The diagram illustrates that AF is parallel to DG and bisects DE at F. The side lengths are labeled: AC = b, AB = c, and AD = AE = d.

Below the diagram, a text input field contains the following expression:

$$\frac{\text{distance}(D,G)}{\text{distance}(E,G)} = \frac{b}{c}$$

23. Median length

Compute the square of the lengths of the medians

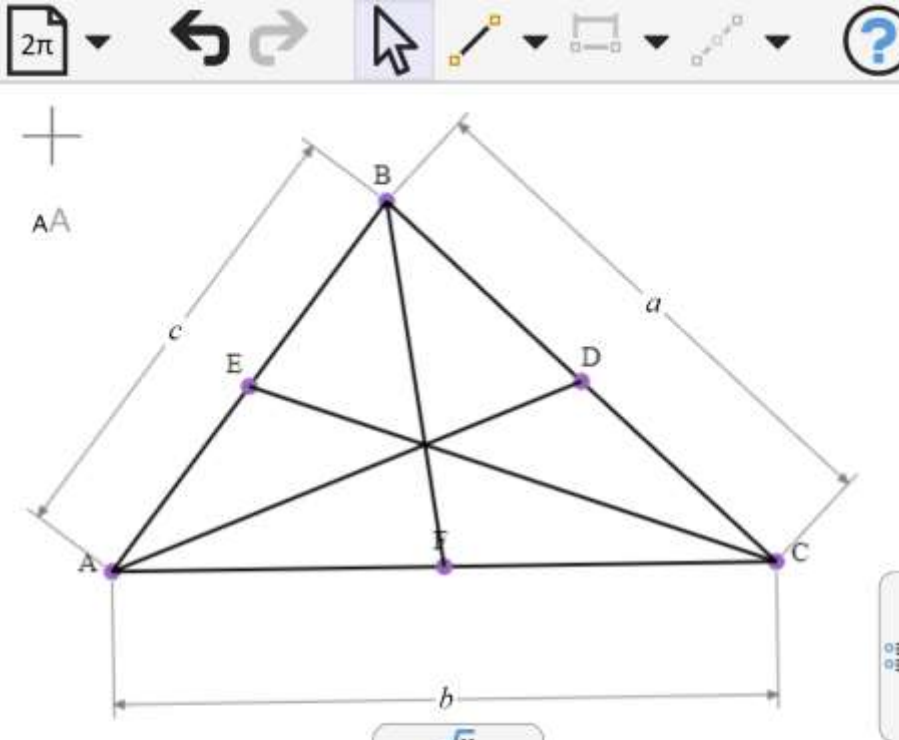


The diagram shows a triangle with vertices A, B, and C. Side BC is labeled 'a', side AC is labeled 'c', and side AB is labeled 'b'. A median AD is drawn from vertex A to the midpoint D of side BC. The diagram is part of a software interface with a toolbar at the top containing icons for a ruler, undo, redo, mouse, and other tools. Below the diagram is an input field with a square icon and a button labeled \sqrt{x} . The input field contains the expression $\text{length}(AD)^2$ followed by a highlighted formula: $\frac{-a^2}{4} + \frac{b^2}{2} + \frac{c^2}{2}$.

$\text{length}(AD)^2$ $\frac{-a^2}{4} + \frac{b^2}{2} + \frac{c^2}{2}$

24. Sum of squares of the medians

The sum of the squares of the medians is equal to $\frac{3}{4}$ the sum of squares of the sides of the original triangle



length(AD)² + length(BF)² + length(CE)²

$$\frac{3 \cdot (a^2 + b^2 + c^2)}{4}$$

length(AD)² $\frac{-a^2}{4} + \frac{b^2}{2} + \frac{c^2}{2}$

25. Sum of squares of the distance of a point from the vertices in terms of its distance from the centroid

If two points are equidistant from the centroid of a triangle, the sums of the squares of their distances from the vertices of the triangle are equal.

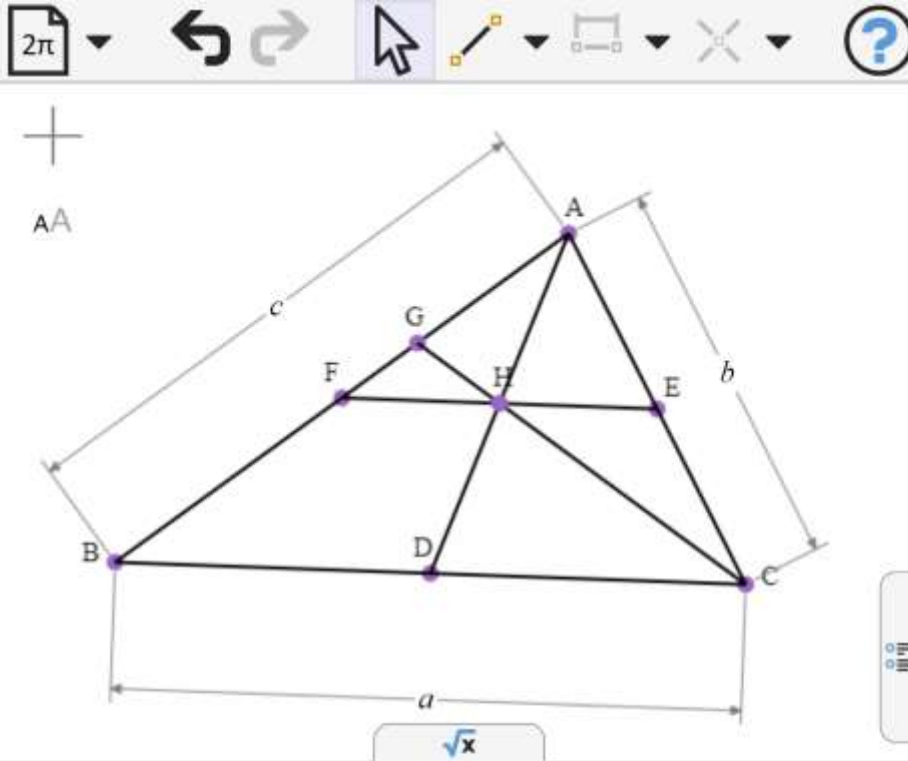
distance(H,A)² + distance(H,B)² + distance(H,C)²

$$\frac{a^2 + b^2 + c^2 + 9 \cdot d^2}{3}$$

The sum of squares is independent of θ , hence the result.

26. A median meets a side of the medial triangle

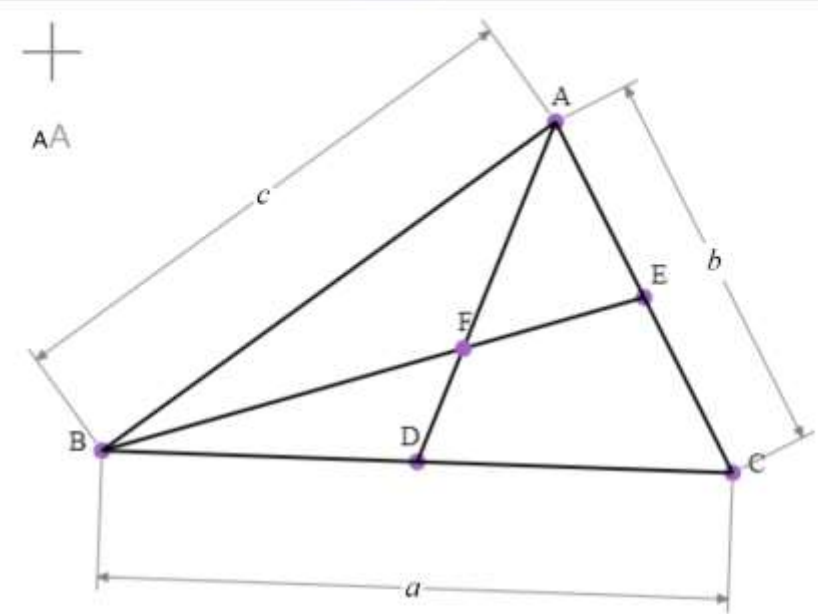
Let D, E, F be the midpoints of BC, CA, AB of triangle ABC . The median AD meets FE in H , and CH meets AB in G . Show that $AB=3AG$.



distance(A, G) $\frac{c}{3}$

27. Sum of squares of the distances from the vertices to the centroid

The sum of squares of the distances of the centroid of a triangle from the vertices is equal to one third the sum of squares of the sides.



The diagram shows a triangle with vertices A, B, and C. Medians AD, BE, and CF are drawn, intersecting at the centroid F. The side lengths are labeled as follows: side BC is labeled 'a', side AC is labeled 'b', and side AB is labeled 'c'. The centroid F is the intersection point of the medians.

distance(A,F)² + distance(B,F)² + distance(C,F)²

$$\frac{a^2 + b^2 + c^2}{3}$$

28. Sum of squares of the distances of a point from the vertices of a triangle

If G is any point in the plane of the triangle ABC and F is the centroid of ABC, we have $GA^2+GB^2+GC^2=FA^2+FB^2+FC^2+3FG^2$

distance(F,A)² + distance(F,B)² + distance(F,C)² + 3 · distance(F,G)²

$$a^2 + b^2 + c^2 - 2 \cdot a \cdot x - 2 \cdot b \cdot x + 3 \cdot x^2 - 2 \cdot c \cdot y + 3 \cdot y^2$$

distance(G,A)² + distance(G,B)² + distance(G,C)²

$$a^2 + b^2 + c^2 - 2 \cdot a \cdot x - 2 \cdot b \cdot x + 3 \cdot x^2 - 2 \cdot c \cdot y + 3 \cdot y^2$$

29. Exscribed triangle parallel to the medians

Show that the parallels through the vertices A, B, C of the triangle ABC to the medians of this triangle issued from the vertices B, C, A respectively form a triangle whose area is three times the area of the original triangle

The diagram illustrates the construction of an outer triangle GHI from triangle ABC. Medians AD, BE, and CF are drawn from vertices A, B, and C respectively, intersecting at centroid F. Lines through vertices A, B, and C are drawn parallel to the medians from the opposite vertices: line through A parallel to BE, line through B parallel to CF, and line through C parallel to AD. These three lines intersect at vertices G, H, and I of the outer triangle GHI. The diagram uses various markings to indicate parallelism and midpoints, such as tick marks on segments and dashed lines for auxiliary constructions.

Below the diagram, a software interface shows the calculation of the area ratio:

$$\frac{\text{area}(G,H,I)}{\text{area}(A,B,C)} = 3$$

30. Parallels to triangle sides through a point

The parallels to the sides of a triangle ABC through the same point D meet the respective medians in the points H, I, J. Prove that we have $KJ/KA = KJ/KB - KH/KC$

AA

distance(K,J) / distance(K,B) - distance(K,H) / distance(K,C) $1 - \frac{3 \cdot x}{2 \cdot a} - \frac{3 \cdot y}{2 \cdot c} + \frac{3 \cdot b \cdot y}{2 \cdot a \cdot c}$

distance(K,J) / distance(K,A) $1 - \frac{3 \cdot x}{2 \cdot a} - \frac{3 \cdot y}{2 \cdot c} + \frac{3 \cdot b \cdot y}{2 \cdot a \cdot c}$

31. Altitude segments divided by orthocenter

In a given triangle, the three products of the segments into which the orthocenter divides the altitudes are equal.

The diagram shows a triangle ABC with vertices A , B , and C . The altitudes are AD , BE , and CF , where D , E , and F are the feet of the altitudes on the opposite sides. The orthocenter is the point F . The altitudes are divided into segments AF , FE , BF , FD , and CF , CD . The sides of the triangle are labeled a , b , and c . The software interface includes a toolbar at the top with icons for undo, redo, mouse, and other tools. Below the diagram, a text input field contains the expression $\text{distance}(A,F) \cdot \text{distance}(F,E) - \text{distance}(B,F) \cdot \text{distance}(F,D)$ and a button with the number 0 .

32. Side segments divided by foot of altitude

The product of the segments into which the side of a triangle is divided by the foot of the altitude is equal to this altitude multiplied by the distance of the side from the orthocenter.

The diagram shows a triangle with vertices A, B, and C. Altitudes are drawn from each vertex to the opposite side, intersecting at the orthocenter F. The foot of the altitude from A is D, from B is E, and from C is F. The side BC is labeled 'a', AC is 'c', and AB is 'b'. The orthocenter is labeled 'F'. The diagram illustrates the relationship between the segments of the side BC and the altitude AD.

Below the diagram, a text input field contains the following expression:

$$\frac{\text{distance}(B,E) \cdot \text{distance}(E,C)}{\text{distance}(E,F) \cdot \text{distance}(A,E)}$$

The result of the expression is displayed as 1.

33. Distances of a point from the sides of a triangle related to altitude lengths

If p, q, r are the distances of a point inside a triangle ABC from the sides of the triangle, show that

$$\frac{p}{h_a} + \frac{q}{h_b} + \frac{r}{h_c} = 1.$$

distance(G,BC) / distance(A,BC) + distance(G,AC) / distance(B,AC) + distance(G,AB) / distance(C,AB) = 1

Specifying the triangle by coordinates allows us to give the location of O by coordinate.

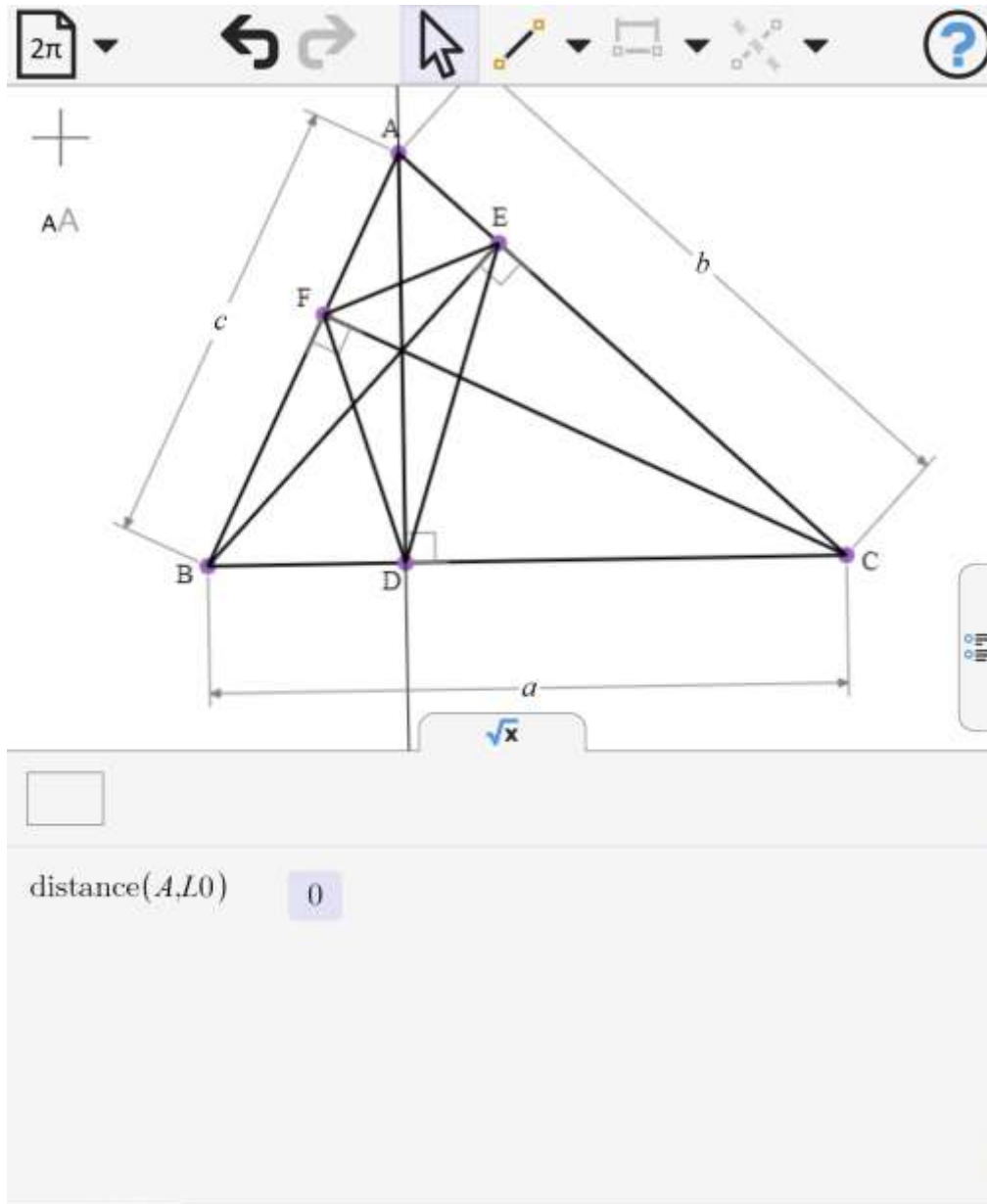
34. distances of a point on the base of an equilateral triangle to the other sides

Show that the sum of the distances of a point on the base of an isosceles triangle to its two sides is equal to the altitude on that side.

distance(E,BC) + distance(E,AC) - distance(A,D) 0

35. Altitudes are angle bisectors of the orthic triangle

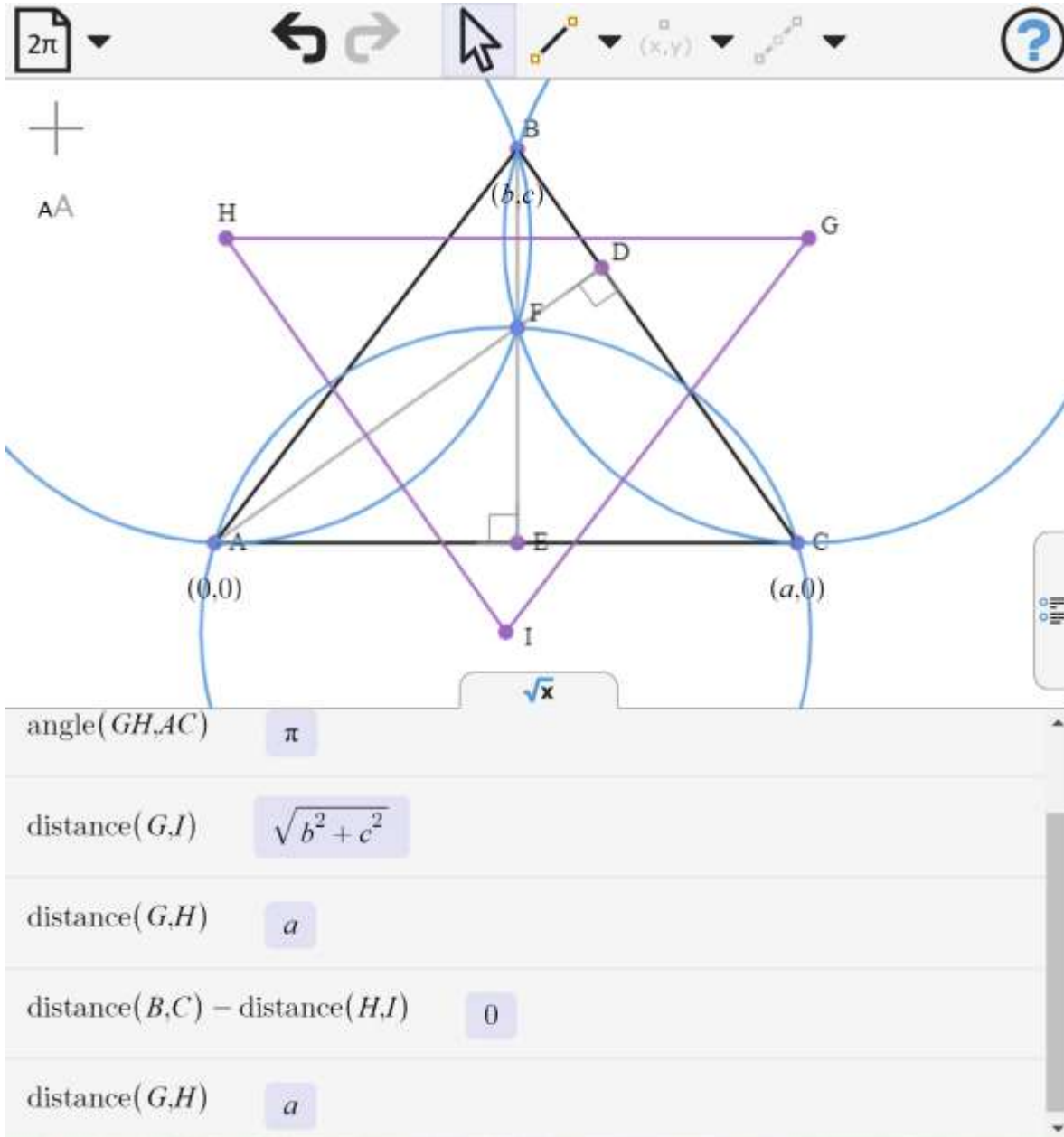
The altitudes of a triangle bisect the internal angles of its orthic triangle.



We can show that point A lies on the bisector of angle FDE , hence the altitude is the bisector.

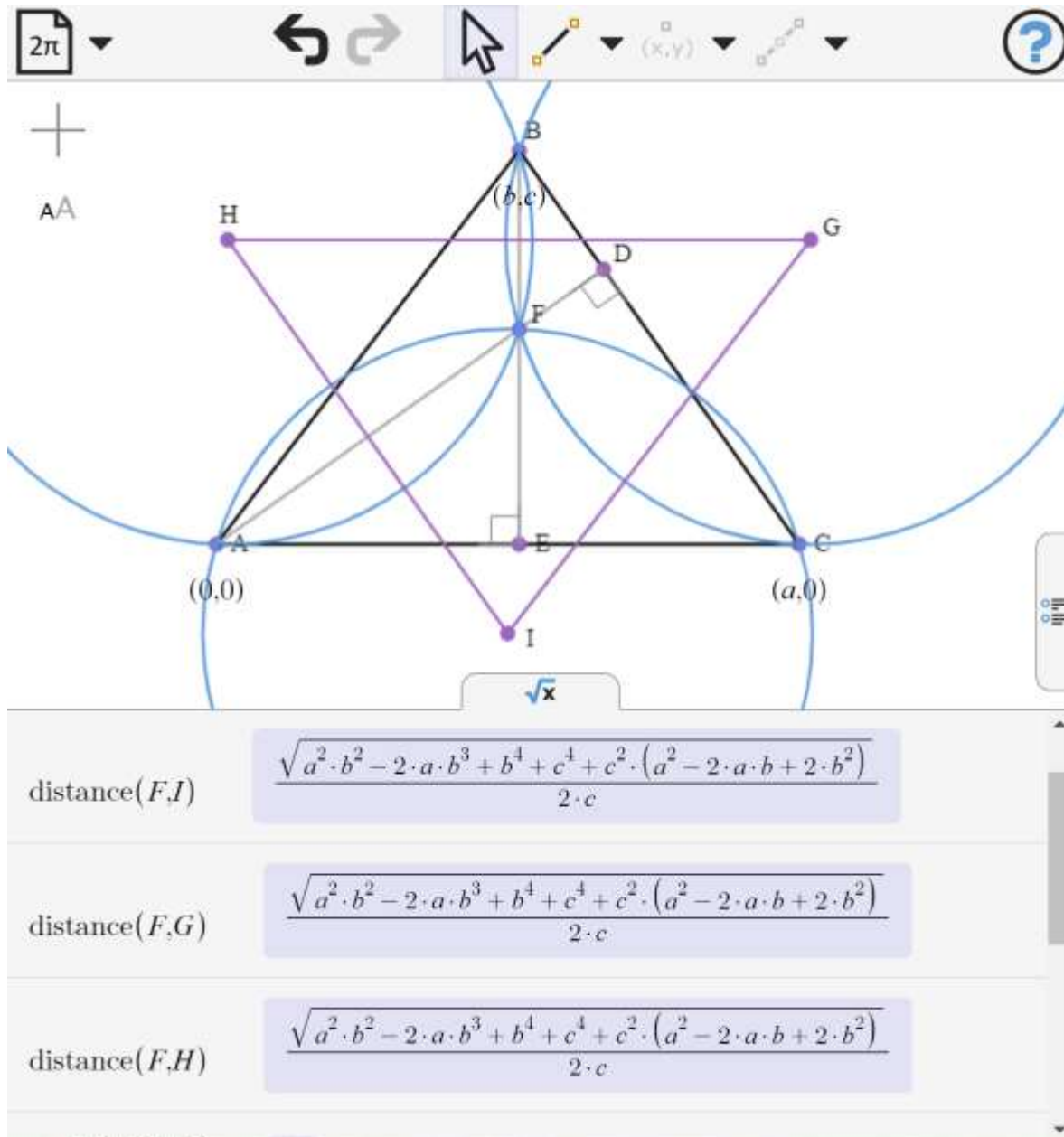
36. Circumcircles of two vertices and the orthocenter

Let F be the orthocenter of triangle ABC . Then the circumcenters of the four triangles ABF , ACF , FBC form a triangle congruent to ABC . The sides are parallel.



37. More on the above

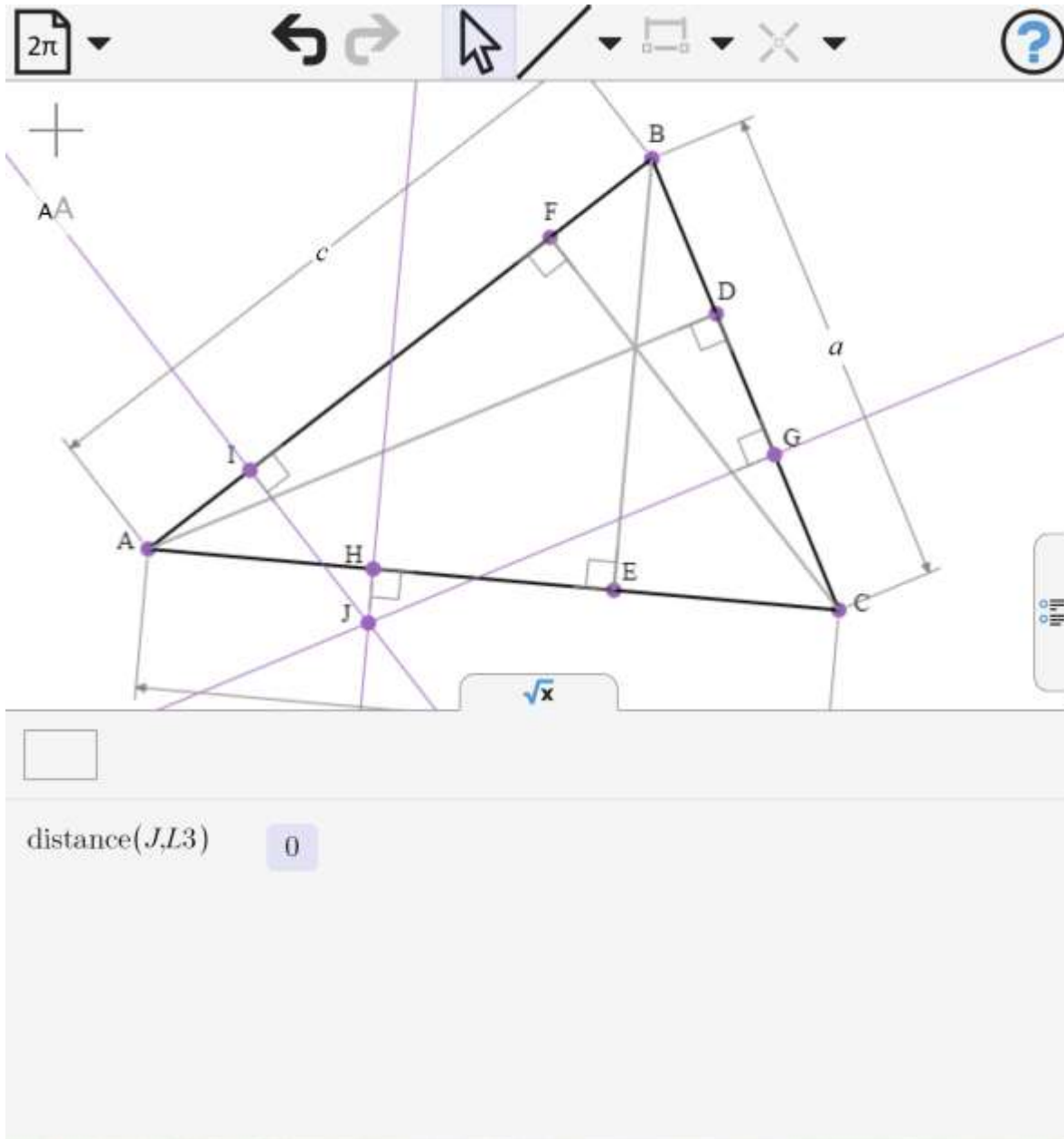
Continuing from Example 6.70, show that F is the circumcenter of GHI.



By inspection, the three lengths HI, HJ, HK are the same.

38. Perpendiculars at isotomic points to altitude feet

Show that the three perpendiculars to the sides of a triangle at the points isotomic to the foot of the respective altitudes are concurrent.



Note: Isotomic points are constructed by reflection in the perpendicular bisector. For clarity, these lines, though present, have been hidden in the diagram.

39. Reflections of altitude feet in triangle sides

Show that the symmetries of the foot of the altitude to the base of the triangle with respect to the other two sides lie on the side of the orthic triangle relative to the base.

The diagram shows a triangle ABC with vertices A , B , and C . The altitudes from A , B , and C are AD , BE , and CF respectively, where D , E , and F are the feet of the altitudes. The orthocenter is H . The orthic triangle is DEF . Points H and I are reflections of E across lines AB and BC respectively. The diagram shows that H , I , and D are collinear on the line DF .

distance(I,DF) 0

distance(H,DF) 0

H, I are the images of E under reflection in AB and BC . We show their distance from the line DF is zero.

40. Products of segments involving the orthic triangle

Show that the product of the segments into which a side of a triangle is divided by the corresponding vertex of the orthic triangle is equal to the product of the sides of the orthic triangle passing through the vertex considered.

The diagram shows a triangle ABC with orthic triangle DEF . Side BC is labeled a , side AC is labeled c , and side AB is labeled b . The orthic triangle DEF has vertices D on BC , E on AC , and F on AB . Right angle symbols are shown at D , E , and F . A toolbar at the top includes a 2π icon, undo/redo arrows, a mouse cursor, a selection tool, a zoom tool, and a help icon. A status bar at the bottom shows a square root symbol with an x .

$\frac{\text{distance}(A,E) \cdot \text{distance}(E,C)}{\text{distance}(E,F) \cdot \text{distance}(D,E)}$	1
$\text{distance}(E,F) \cdot \text{distance}(D,E)$	$\frac{-a^4 + b^4 + 2 \cdot a^2 \cdot c^2 - c^4}{4 \cdot b^2}$
$\text{distance}(A,E) \cdot \text{distance}(E,C)$	$\frac{(a^2 + b^2 - c^2) \cdot (-a^2 + b^2 + c^2)}{4 \cdot b^2}$

41. Perpendiculars to the sides of the orthic triangle

If P, Q are the feet of the perpendiculars from the vertices B, C of the triangle ABC on the sides DF, DE respectively, of the orthic triangle DEF, show that EQ=FP.

The diagram shows a triangle ABC with vertices A , B , and C . The orthic triangle DEF is formed by the feet of the altitudes AD , BE , and CF from A , B , and C respectively. The orthocenter is H . The orthic triangle DEF is inscribed within ABC . The diagram is part of a software interface with various tool icons at the top and a calculation panel at the bottom.

distance(D,I) $\frac{-a^4 + b^4 + 2 \cdot a^2 \cdot c^2 - c^4}{4 \cdot a \cdot b \cdot c}$

distance(F,H) $\frac{-a^4 + b^4 + 2 \cdot a^2 \cdot c^2 - c^4}{4 \cdot a \cdot b \cdot c}$

$\frac{\text{distance}(A,E) \cdot \text{distance}(E,C)}{\text{distance}(E,F) \cdot \text{distance}(D,E)}$ 1

42. Projections of altitude feet onto sides and altitudes are collinear

The four projections of the foot of the altitude on a side of a triangle upon the other two sides and the other two altitudes are collinear.

The diagram shows a triangle ABC with vertices A , B , and C . The altitudes from A , B , and C are AD , BE , and CF respectively, intersecting at the orthocenter H . The feet of the altitudes are D , E , and F . The projections of H onto the sides are I on BC , J on AC , and K on AB . A purple line segment connects I , J , K , and H , demonstrating that these four points are collinear. The diagram is part of a software interface with a toolbar at the top and a command window at the bottom.

distance(K, HI) 0

distance(J, HI) 0

43. Projections of altitude foot onto triangle sides cyclic with ends of base

DE, DF are perpendiculars from the foot D of the altitude BD of the triangle ABC on the sides AB, BC. Prove that the points A, C, E, F are cyclic.

distance(F,G) - distance(E,G) 0

distance(F,G)

$$\frac{\sqrt{a^8 - 6 \cdot a^4 \cdot b^4 + 8 \cdot a^2 \cdot b^6 - 3 \cdot b^8 - 4 \cdot a^2 \cdot c^6 + c^8 + c^4 \cdot (6 \cdot a^4 - 6 \cdot b^4) + c^2 \cdot (-4 \cdot a^6 + 12 \cdot a^2)}}{4 \cdot b \cdot \sqrt{a+b+c} \cdot \sqrt{(a+b-c) \cdot (a-b+c) \cdot (-a+b+c)}}$$

We put a circle through A,E,C then show that F is the same distance from the center of this circle as E.

44. Perpendiculars to the other sides from an altitude base meet perpendiculars to the base from its end points

The perpendiculars DP , DQ dropped from the foot D of the altitude AD of the triangle ABC upon the sides AB , AC meet the perpendiculars BP , CQ erected to BC at B , C in the points P , Q respectively. Prove that the line PQ passes through the orthocenter H of ABC

The diagram shows a triangle ABC with vertices A (top), B (bottom left), and C (bottom right). The orthocenter is H . The altitude from A to BC is AD , where D is on BC . Perpendiculars BP and CQ are drawn from B and C to BC . Perpendiculars DP and DQ are dropped from D to AB and AC respectively. The line PQ is shown in purple, passing through H . Other points F , E , G , and I are also marked on the diagram. The software interface includes a toolbar at the top with icons for undo, redo, selection, and other tools. Below the diagram, a text input field shows the command `distance(G,HI)` with a value of `0`.

45. Perpendicular from side midpoint bisects orthic triangle side

In triangle ABC, let D be the midpoint of the side BC, E and F the feet of the altitudes on AB, AC respectively. DG is perpendicular to EF at G. Show that G is the midpoint of DE.

distance(F,G) $\frac{a \cdot (-a^2 + b^2 + c^2)}{4 \cdot b \cdot c}$

distance(E,G) $\frac{a \cdot (-a^2 + b^2 + c^2)}{4 \cdot b \cdot c}$

Angle Examples

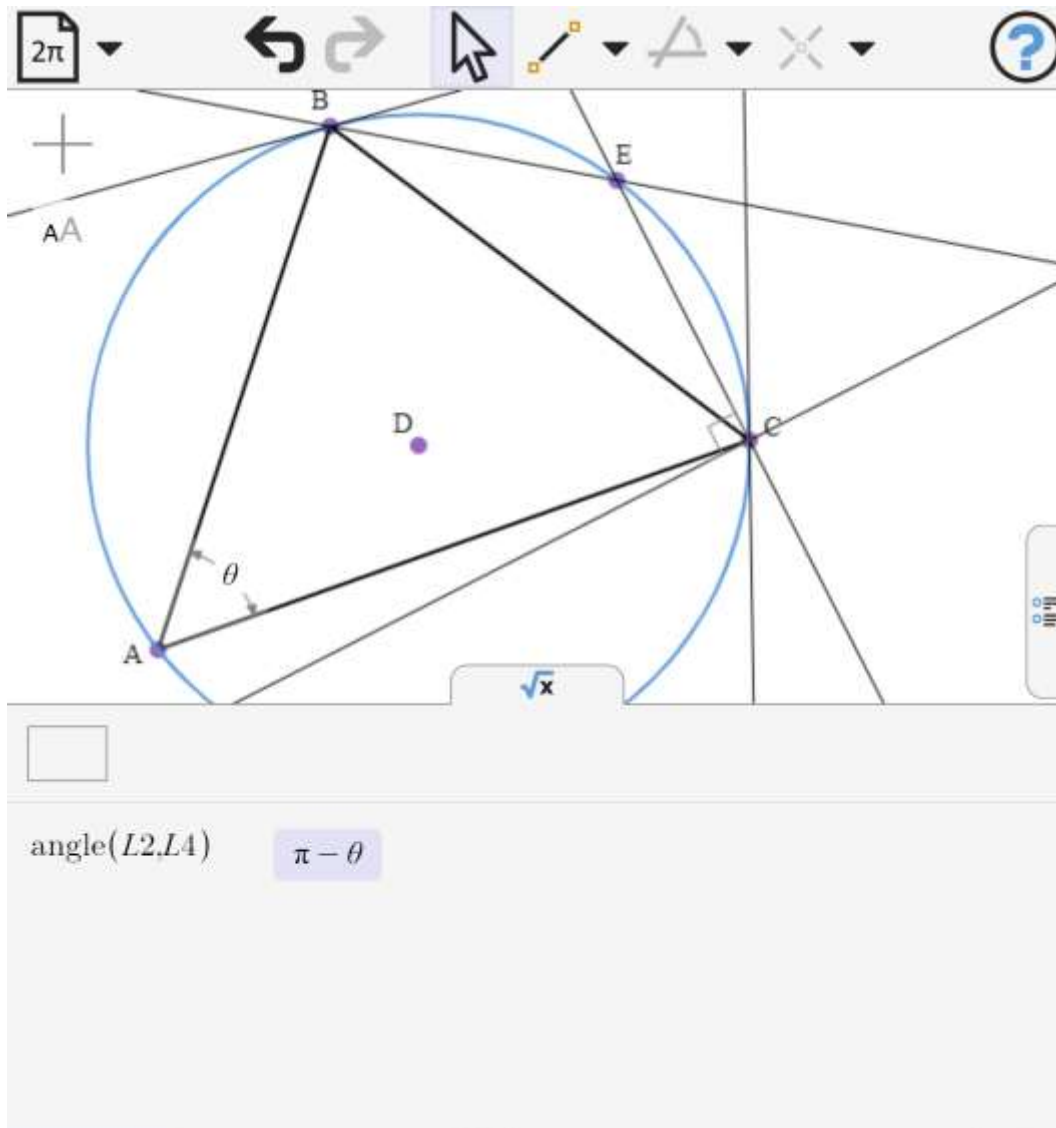
46. Problem 6.294 from Zhang et al

Let $ABCD$ be a cyclic quadrilateral and E the intersection point of its diagonals. Let F be the center of the circumcircle of AED . Then EF is perpendicular to BC

The diagram illustrates a geometric proof. A cyclic quadrilateral $ABCD$ is shown with vertices A , B , C , and D . The diagonals AC and BD intersect at point E . A circle is constructed passing through points A , E , and D , with its center at point F . A line segment EF is drawn, and it is shown to be perpendicular to the side BC . A point G is also marked on the circle AED . The diagram is displayed in a software interface with various tool icons at the top and a command line at the bottom.

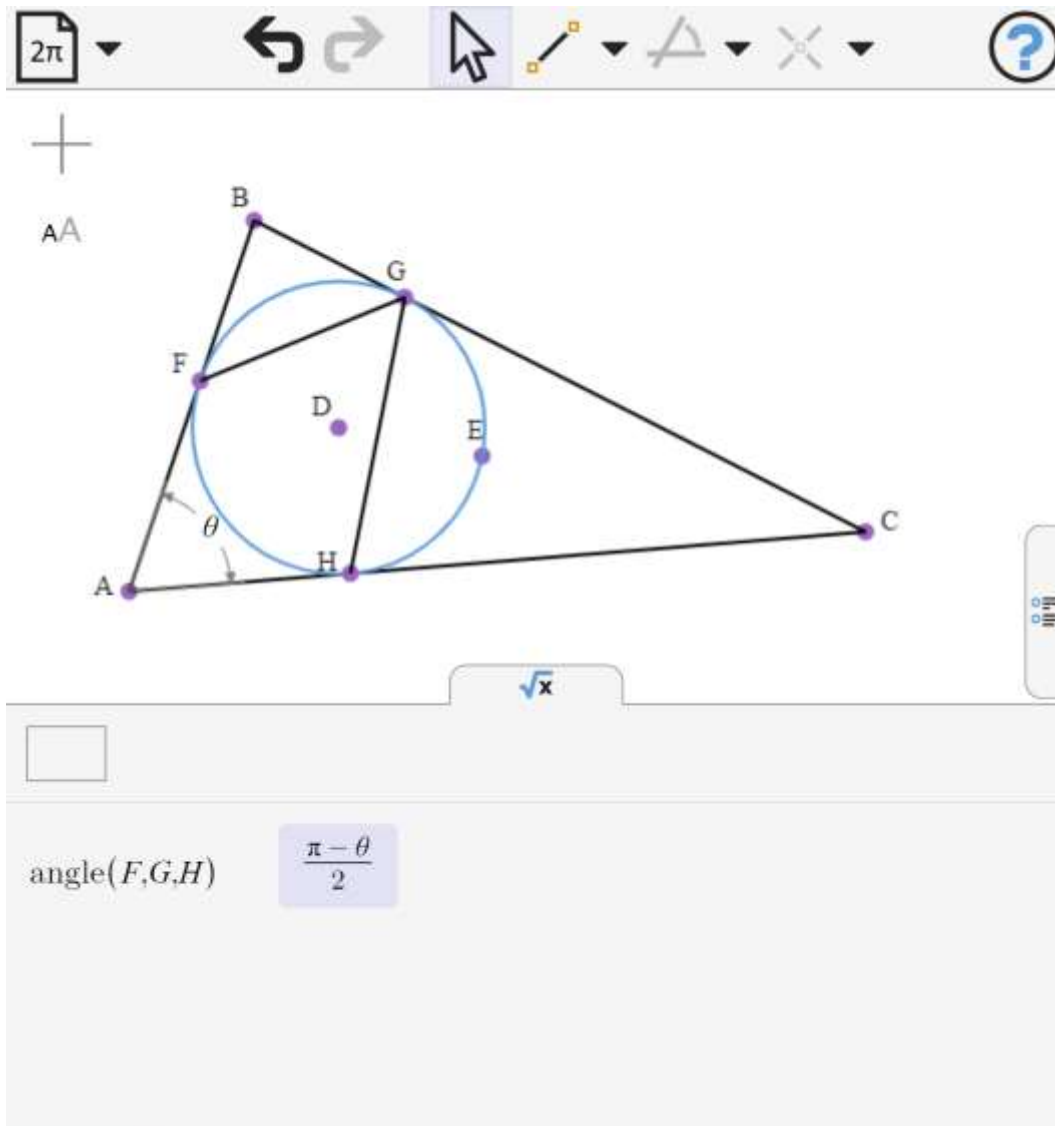
angle(BC,EF) $\frac{\pi}{2}$

47. Angle bisectors of chord and tangents meet on the circle



If chord BC subtends an angle θ on the circumference of a circle, the angle bisectors between the chord and the circle tangents at B and C are at angle $\pi - \theta$. We can infer that they meet on the circumference of the circle

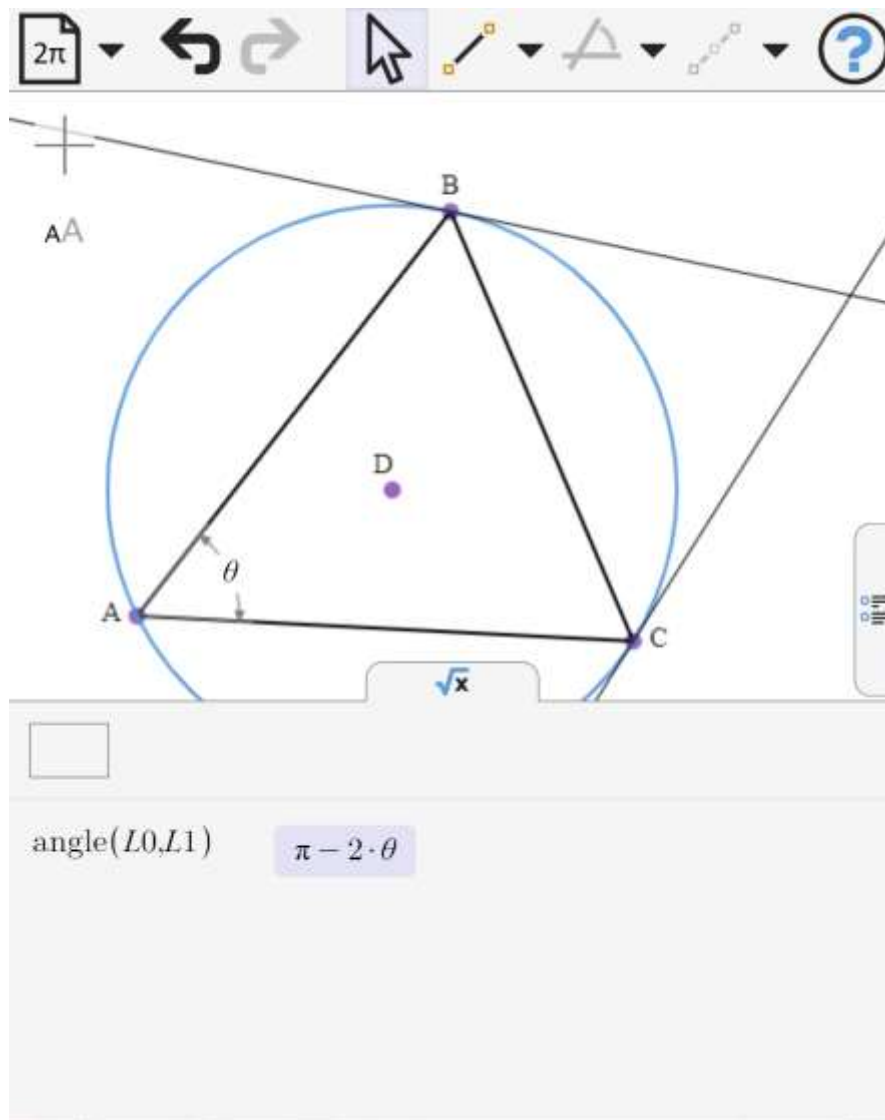
48. Points of contact with the incircle



angle(F,G,H) $\frac{\pi - \theta}{2}$

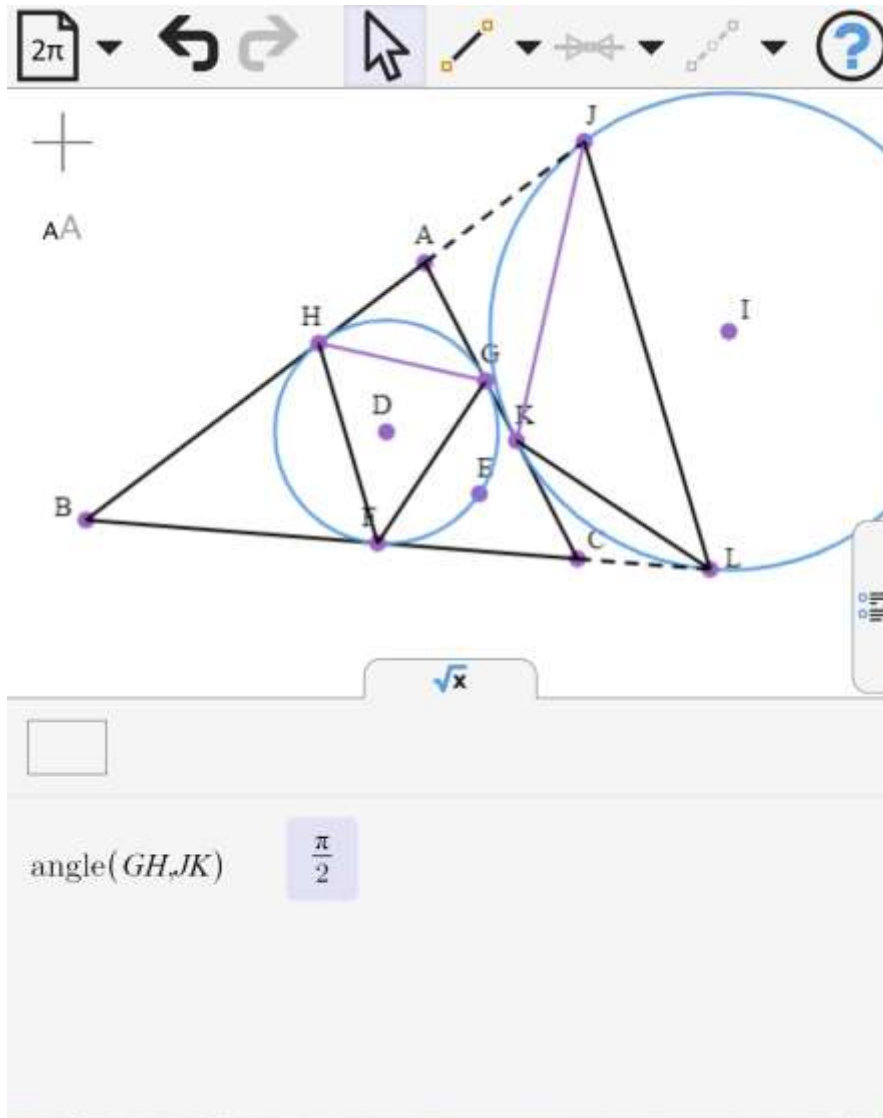
Given triangle ABC , let FGH be the points of contact between the incircle and sides AB , BC , CD respectively. If angle BAC is θ , then angle FGH is $\frac{\pi - \theta}{2}$

49. Tangent triangle



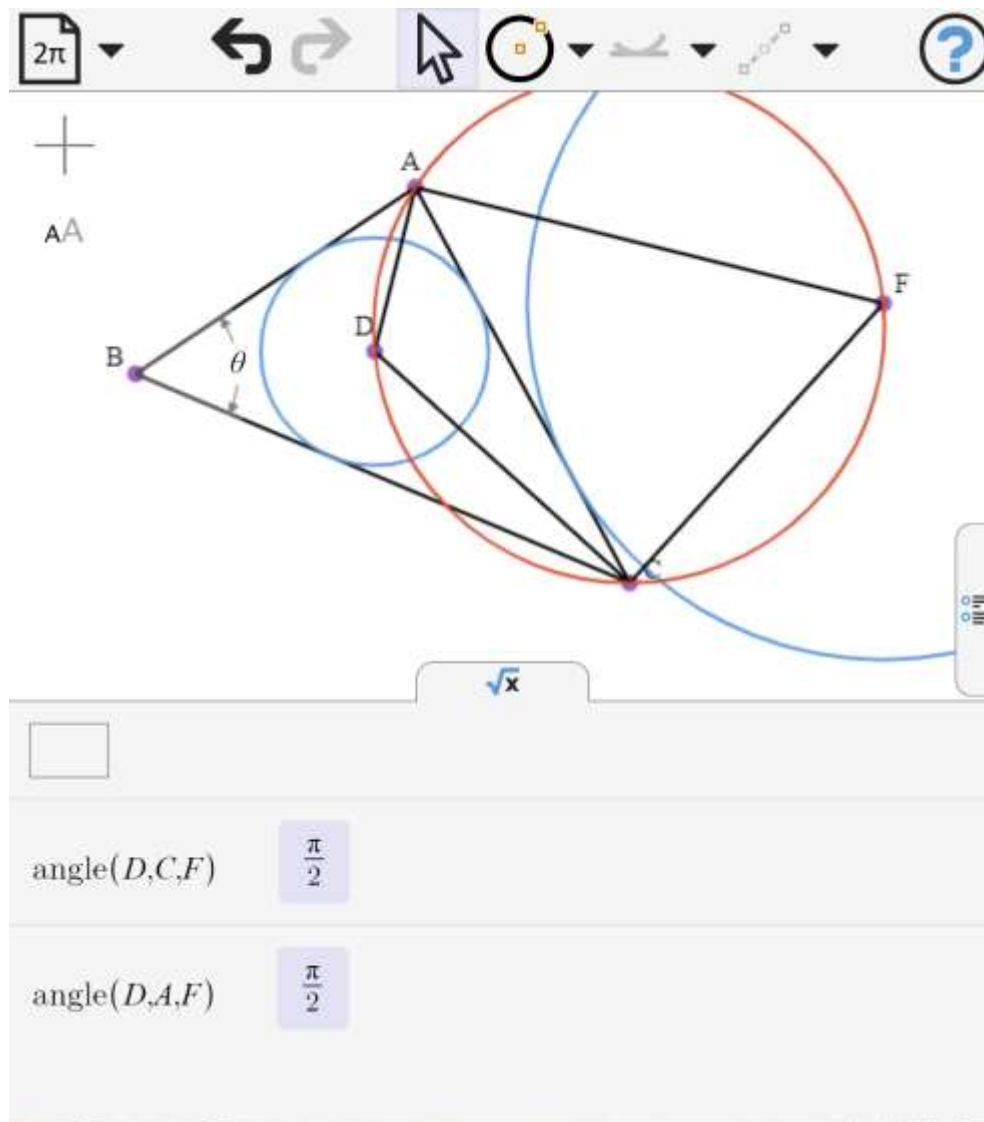
Conversely, the angle between the tangents to the circumcircle at B and C is $\pi - 2\theta$

50. Incircle and excircle



Let F, G, H be the points of contact between the incircle and sides BC, AC, AB of triangle ABC . Let C be the excircle external to AC . Let J, K, L be the points of contact between C and AB, AC, BC . Then GH is perpendicular to JK .

51. A cyclic quadrilateral in the incircle excircle diagram



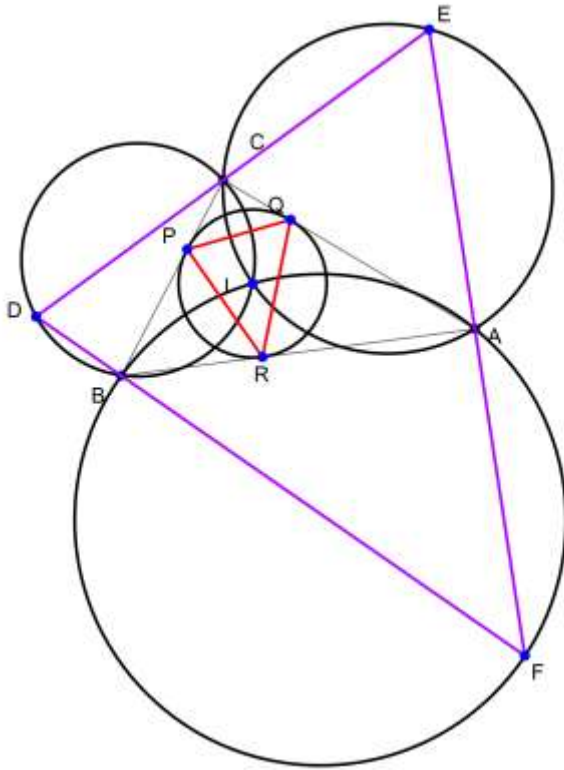
Let D be the incenter of ABC and F the center of the excircle external to AC . The quadrilateral $AFCD$ is cyclic, with DF as a diameter.

52. Circumcircle of Incenter and two vertices

The diagram shows a triangle ABC with vertices A , B , and C . The incenter is I . A circle is drawn with I as the center, passing through B and C . Point G is on this circle. The angle BAC is labeled θ . The software interface includes a toolbar with icons for 2π , undo, redo, mouse, line, triangle, and dashed line. Below the diagram is a command input field with a \sqrt{x} button and the text `angle(B,G,C)` followed by the value $\frac{\pi - \theta}{2}$.

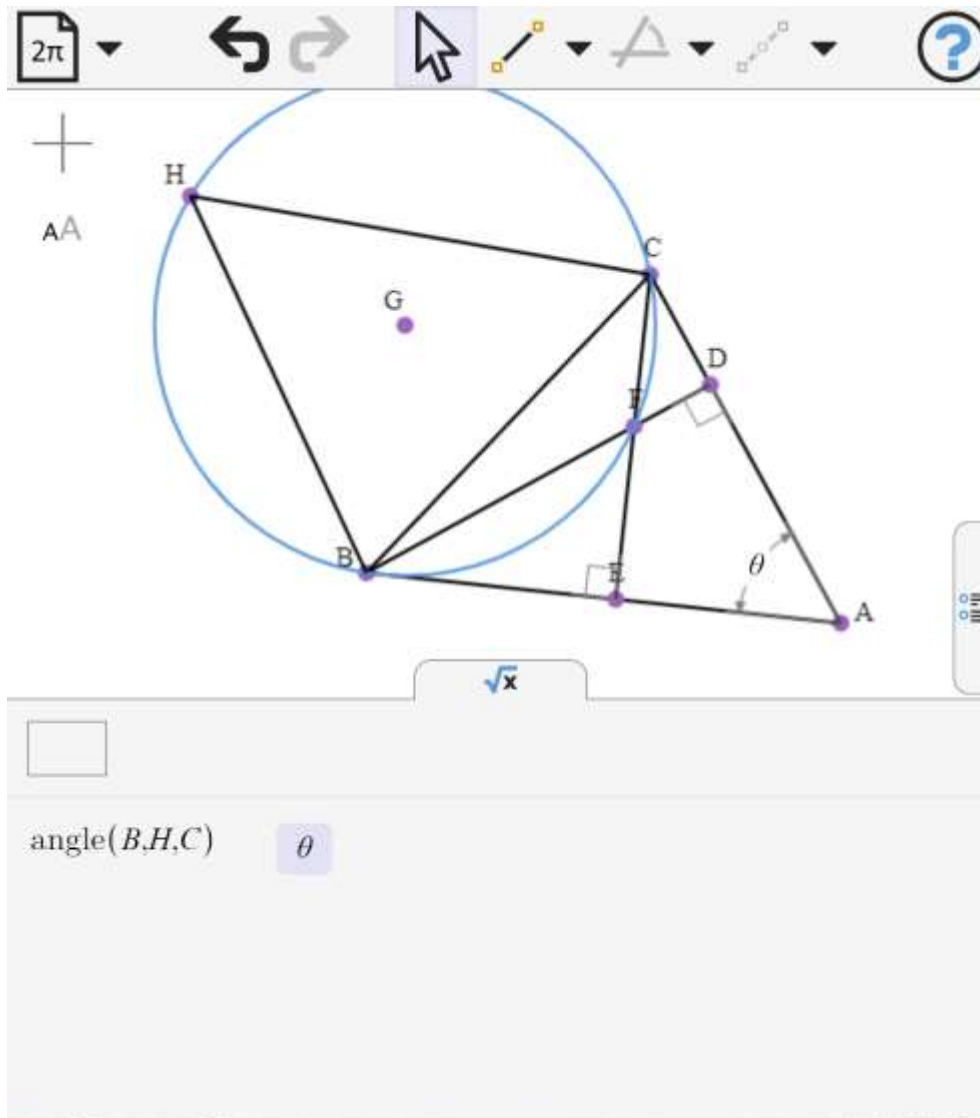
Let I be the incenter of triangle ABC , and let D be a point on the circumcircle of IAB . If angle BAC is θ , then angle BDC is $\frac{\pi - \theta}{2}$

53. Three such circumcircles



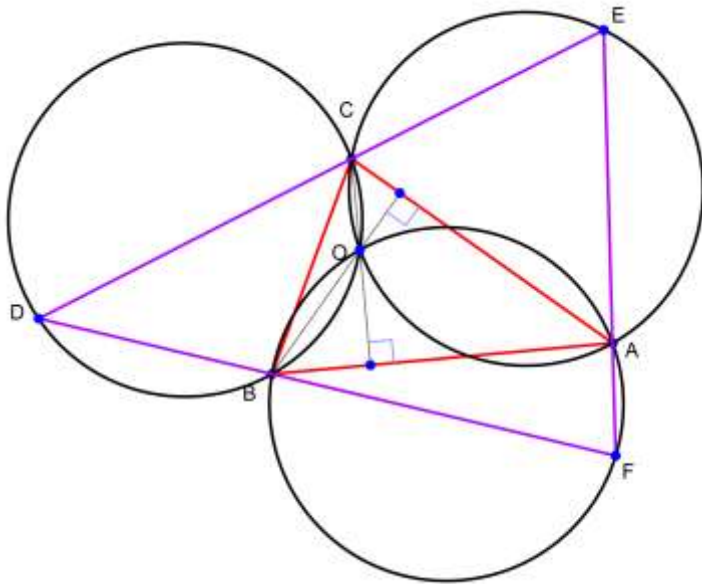
Let D lie externally on the circumcircle of BIC , let E be the intersection of DC extended and the circumcircle of CIA and let F be the intersection of EA extended and the circumcircle of AIB . Let PQR be the points of contact between the incircle and the sides BC, CA, AB . Then F, B and D are collinear and the triangle DEF is similar to the triangle PQR . This follows from the above results.

54. Circumcircle of two vertices and the orthocenter



Let F be the orthocenter of triangle ABC , and let H be a point on the circumcircle of FAB . Angle BAC is equal to angle BHC .

55. Three such circumcircles



Let D lie externally on the circumcircle of BOC , let E be the intersection of DC extended and the circumcircle of COA and let F be the intersection of EA extended and the circumcircle of AOB . Then F , B and D are collinear and the triangle DEF is similar to the triangle ABC . This follows from the above result.

56. Altitude side angle bisectors

The diagram shows a triangle with vertices A, B, and C. Vertex A is at the bottom left, B is at the top, and C is at the bottom right. Two altitudes are drawn: one from B to side AC at point D, and one from C to side AB at point E. Right-angle symbols are shown at D and E. Two purple lines are drawn: one bisecting angle A, and another bisecting angle C. The angle between these two bisectors is labeled $\theta + \phi$. The angle between the bisector of angle A and the altitude from C is labeled θ . The angle between the bisector of angle C and the altitude from B is labeled ϕ . The diagram is part of a software interface with a toolbar at the top containing icons for undo, redo, select, and other geometric tools. A status bar at the bottom left shows the command `angle(L1,L0)` and the result $\frac{\theta + \phi}{2}$.

angle(L1,L0) $\frac{\theta + \phi}{2}$

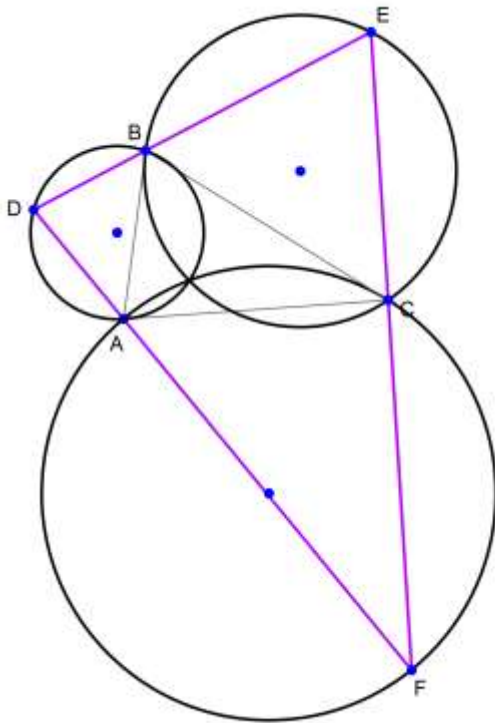
57. Mix-Linear circle

The diagram shows a circle passing through vertices A and B of a triangle ABC. The circle is tangent to the side AC at point A. Point E is another point on the circle. Point D is an external point. The angle ADB is labeled as θ .

Below the diagram, a software interface shows a command input field with the text `angle(A,E,B)` and a button labeled θ .

Let C be the circle through vertices A and B tangent to side AC of triangle ABC. Let D be an external point on this circle. Then $ADB = BAC$.

58. Three such circles



Let C be the circle through vertices A and B tangent to side AC . Let D be the circle through vertices A and B tangent to side AC . Let E be the circle through vertices A and B tangent to side AC . Let D lie externally on circle C , let E be the intersection of DB extended and circle D and let F be the intersection of EC extended and circle E . Then F , A and D are collinear and the triangle DEF is similar to the triangle ABC . This follows from the above result.

59. Angle subtended in the circumcircle of two vertices and the circumcenter

angle(A,D,C) $2 \cdot (\pi - \theta)$

If angle ABC is θ , let D be the circumcenter of ABC . Then angle $ADC=2(\pi-\theta)$.

60. Reflection of parallel rays

The diagram shows a triangle ABC with vertices A , B , and C . Two parallel rays are shown, one passing through A and the other through C . These rays are reflected across the sides BA and BC respectively. The reflected rays intersect at point D . The angle $\angle ABC$ is labeled θ . The angle $\angle ADC$ is labeled $2 \cdot (\pi - \theta)$. The diagram is part of a software interface with a toolbar at the top and a command input area at the bottom.

angle(A,D,C) $2 \cdot (\pi - \theta)$

Given triangle ABC with angle $ABC = \theta$, the images of two parallel rays under reflection in BA and BC meet at angle $2(\pi - \theta)$

From this along with the previous result, we can deduce that the intersection of the reflected images lies on the circumcircle of B, C and the circumcenter of ABC .

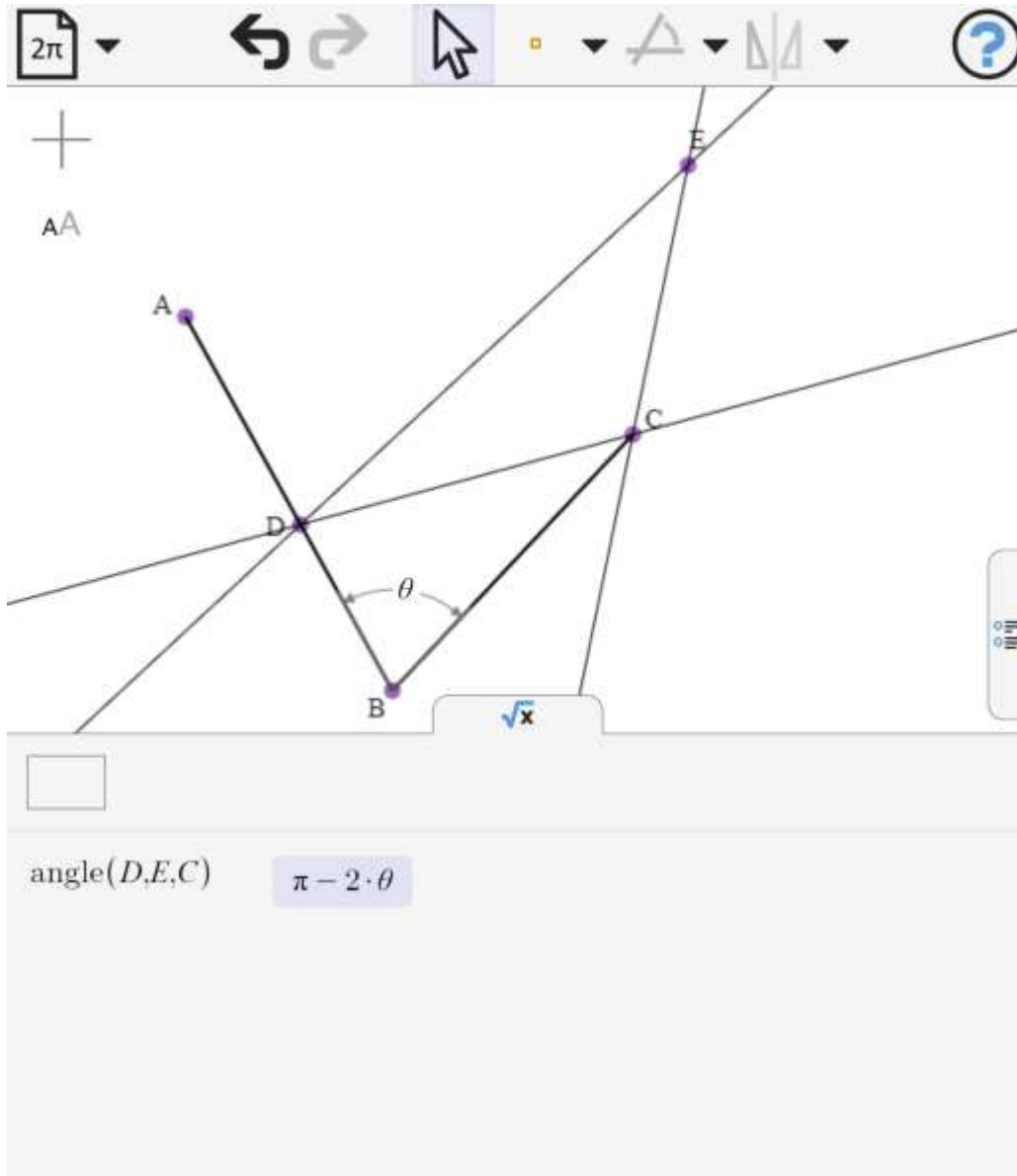
61. Reflections of angled rays

The diagram illustrates the reflection of a ray across two intersecting lines. The resulting path is a closed polygon ABCD. The angle at vertex B is labeled θ , and the angle at vertex C is labeled ϕ . The angle at vertex D is labeled $2 \cdot \pi - 2 \cdot \theta - \phi$. The diagram is shown in a software interface with a toolbar at the top and a command input area at the bottom.

angle(A,D,C) $2 \cdot \pi - 2 \cdot \theta - \phi$

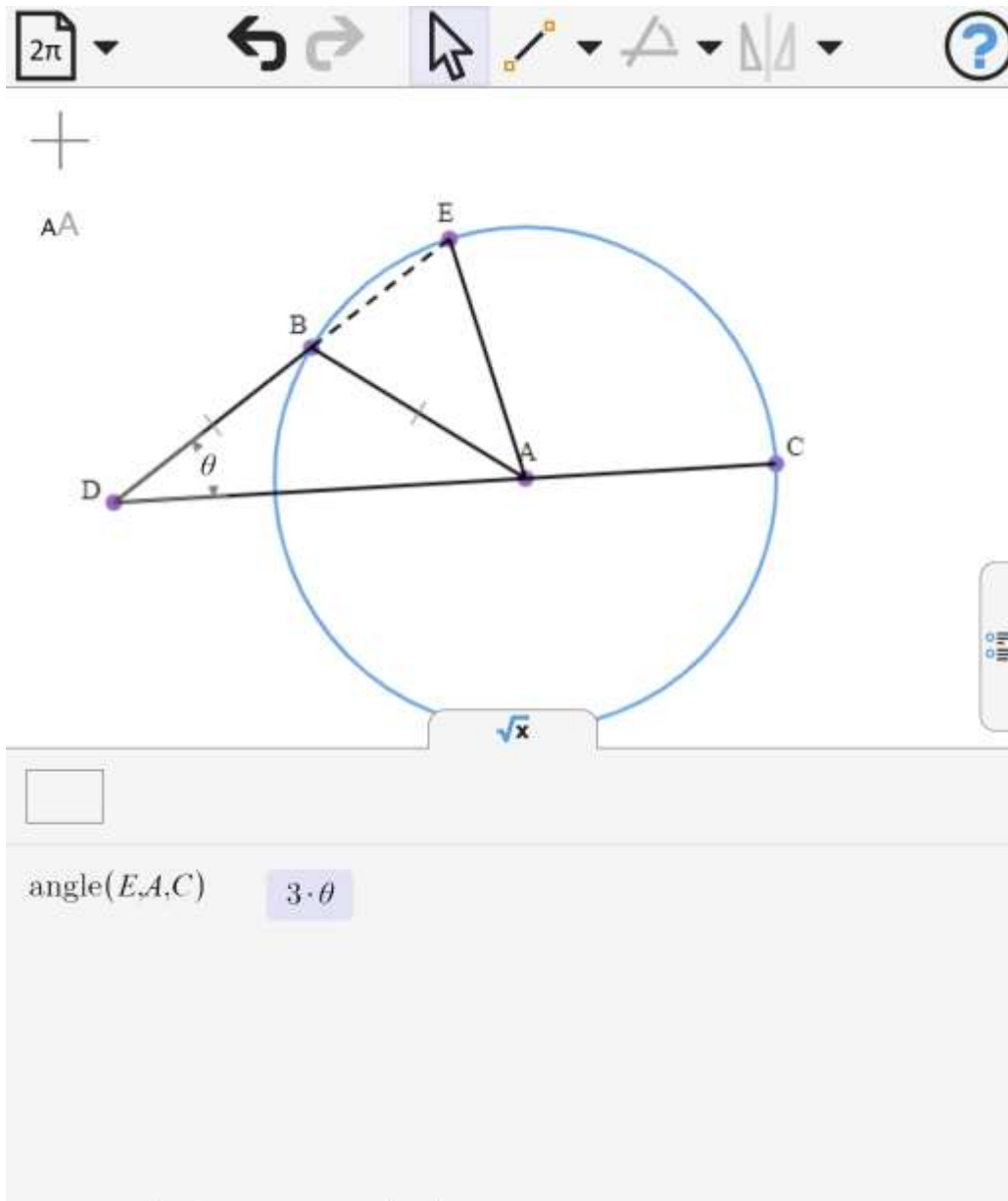
In the above, if the incident rays are at angle ϕ , then the reflected rays are at angle $2\pi - 2\theta - \phi$.

62. Reflection in a corner



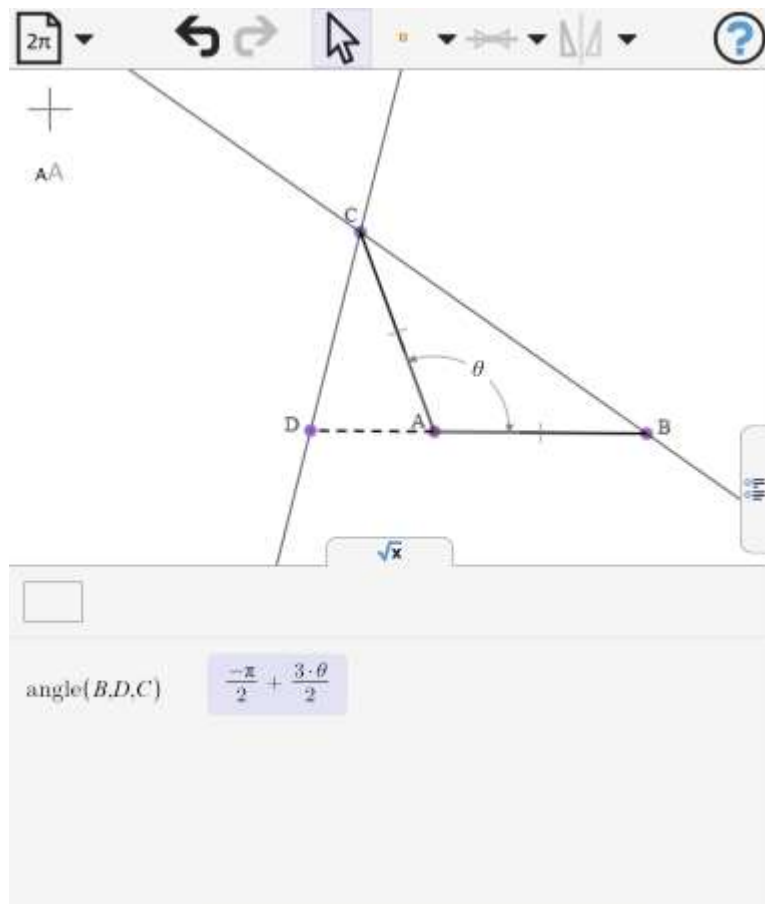
Let angle ABC be θ . A ray incident at C is reflected in BC, then the reflected ray is itself reflected in AB. The angle between the resulting ray and the original ray is $\pi - 2\theta$.

63. Archimedes Angle Trisector



This figure, constructed forward from D triples an angle. Ran backwards, it requires a 'neusis'.. that is a construction not performed with straightedge and compass, but does perform an angle trisection.

64. Reflections in a box lid



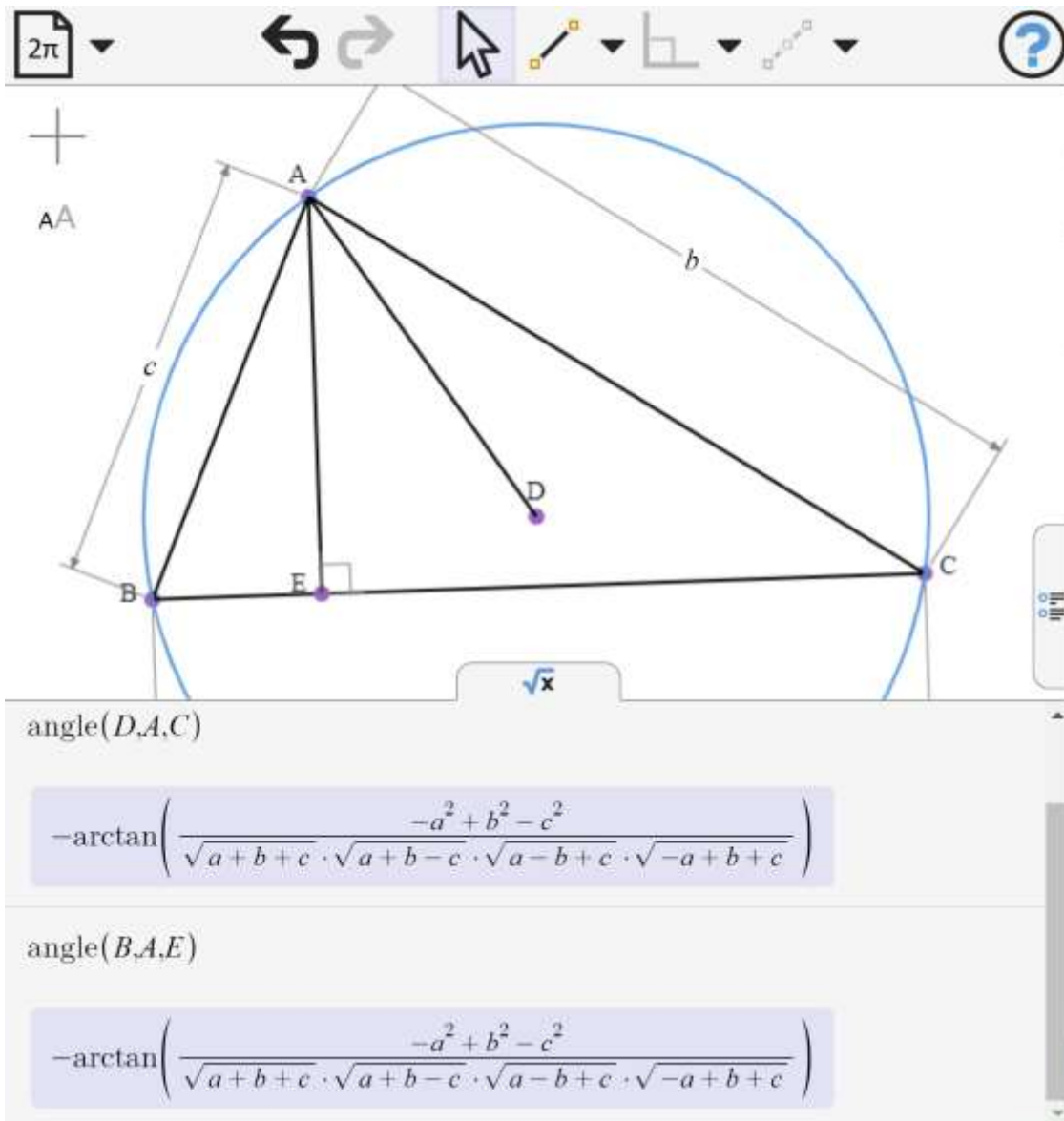
Let ABC be an isosceles triangle with AB and BC equal. Reflect the line BC in AC . The angle between the reflected image and AB is $\frac{3\theta}{2} - \frac{\pi}{2}$

The Circumcircle

65. Angle between circumdiameter and radius

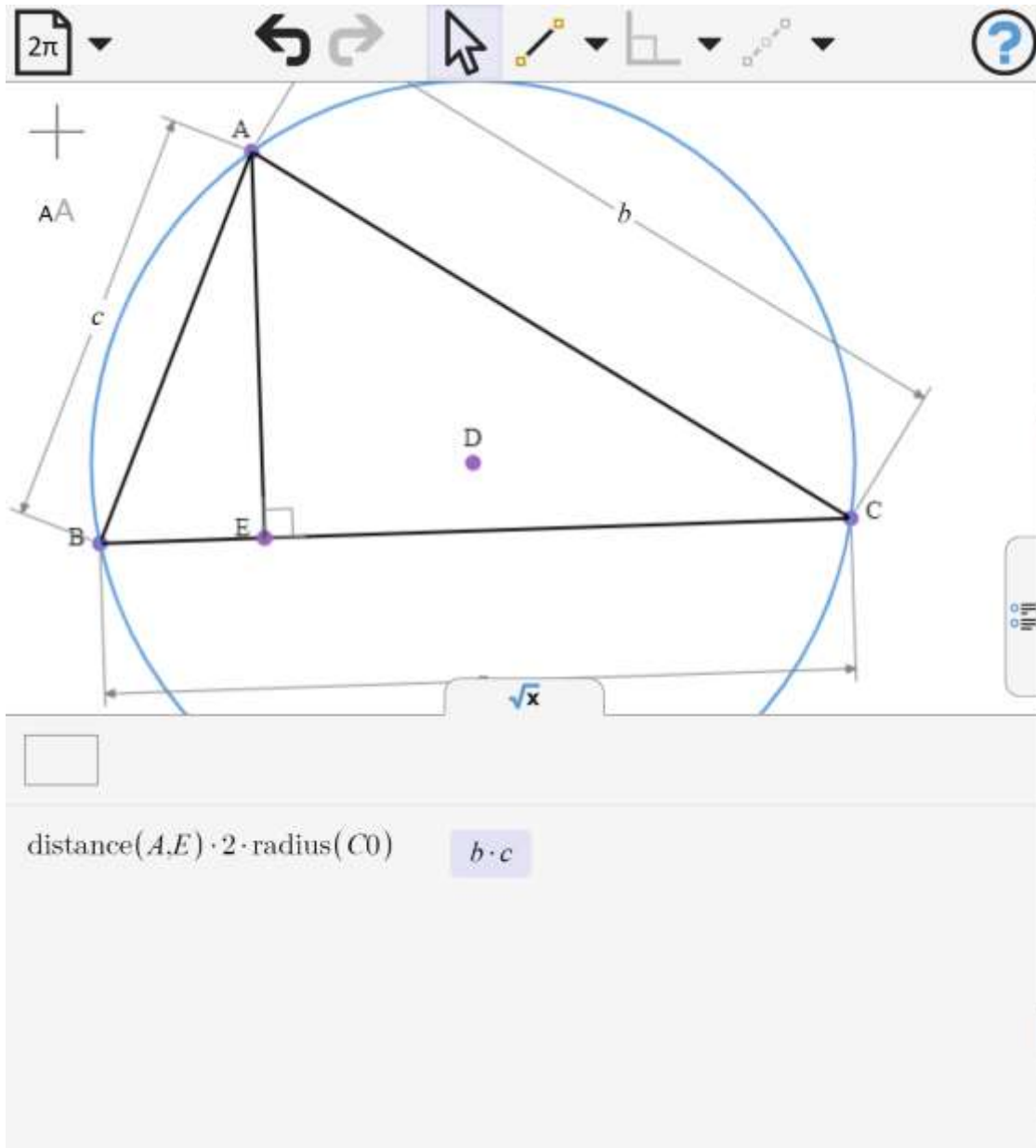
The angle between the circumdiameter and the altitude issued from the same vertex of a triangle is bisected by the bisector of the angle of the triangle at the vertex considered.

We show that angle BAE equals DAC in the diagram, which gives the result.



66. Triangle side lengths related to altitude and circumcircle diameter

The product of two sides of a triangle is equal to the altitude to the third side multiplied by the circumdiameter.



67. Circumcenter is orthocenter of medial triangle

Prove that the circumcenter of a triangle is the orthocenter of its medial triangle

The diagram illustrates a triangle ABC inscribed in a circle. The medial triangle DEF is formed by connecting the midpoints of the sides of ABC . The orthocenter G of the medial triangle DEF is shown, and it is the same point as the circumcenter of the original triangle ABC .

angle(EG,DF) $\frac{\pi}{2}$

angle(FG,DE) $\frac{\pi}{2}$

angle(DG,EF) $\frac{\pi}{2}$

68. Triangle area related to circumcircle radius

The area of a triangle is equal to the product of its three sides divided by the double circumdiameter of the triangle

The diagram shows a triangle with vertices A, B, and C inscribed in a circle. The center of the circle is marked as G. The side lengths are labeled: side BC is 'a', side AC is 'b', and side AB is 'c'. The software interface includes a toolbar at the top with icons for file operations, navigation, and drawing tools. At the bottom, there are two input fields for mathematical expressions.

Input field 1: $\frac{a \cdot b \cdot c}{4 \cdot \text{radius}(C0)}$

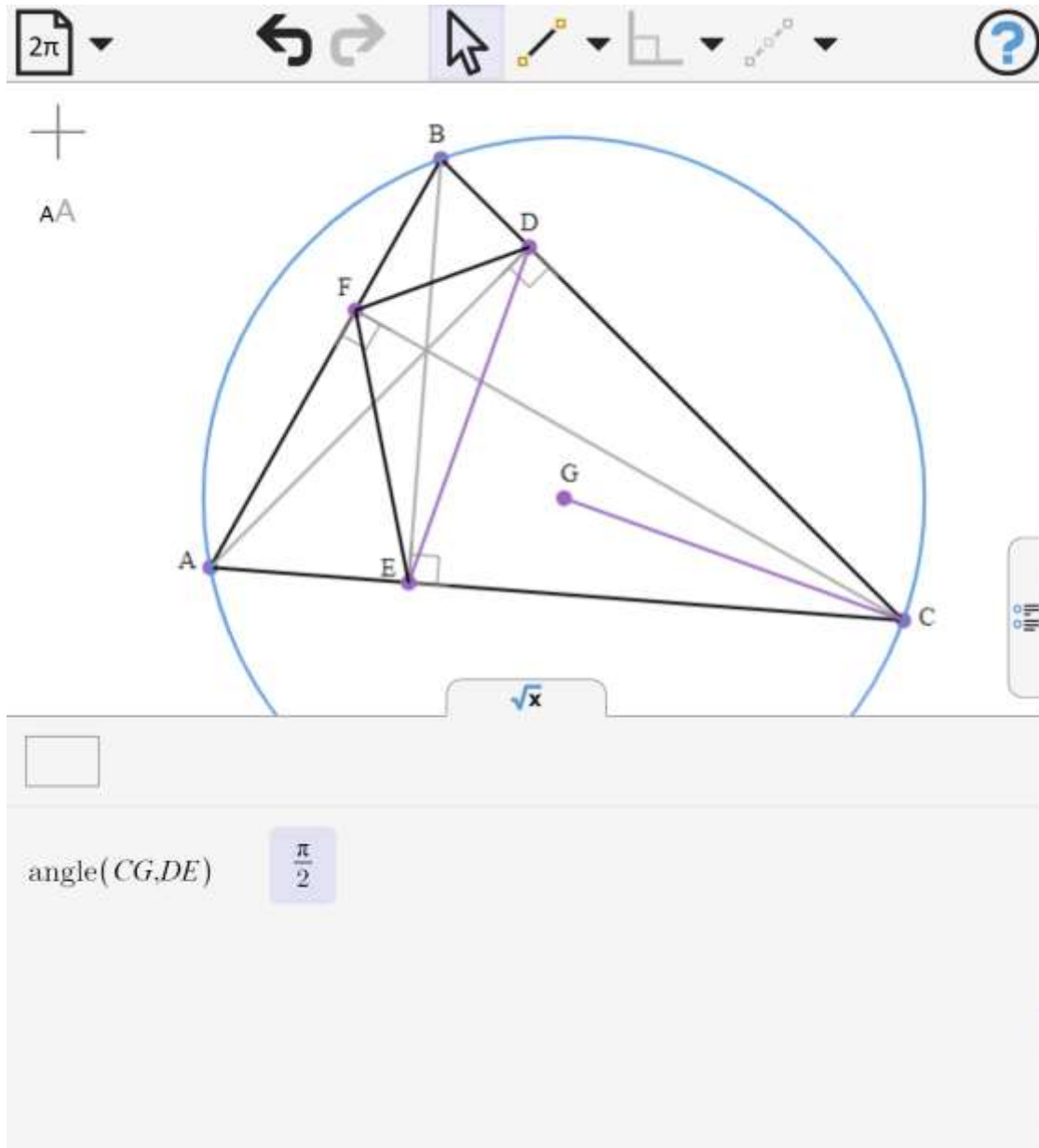
Input field 2: $\frac{\sqrt{a+b+c} \cdot \sqrt{a+b-c} \cdot \sqrt{a-b+c} \cdot \sqrt{-a+b+c}}{4}$

Input field 3: $\text{area}(A,B,C)$

Input field 4: $\frac{\sqrt{a+b+c} \cdot \sqrt{a+b-c} \cdot \sqrt{a-b+c} \cdot \sqrt{-a+b+c}}{4}$

69. Circumcircle and orthic triangle

The radii of the circumcircle passing through the vertices of a triangle are perpendicular to the corresponding sides of the orthic triangle



70. An isosceles triangle relationship

Let ABC be a triangle with $AC=AB$. E is a point on BC . Line AE meets the circumcircle of ABC at F . Show that $AB^2=AE \cdot AF$

distance(A,E) · distance(A,F) b^2

71. Circumcircle of an isosceles triangle

Let C be the midpoint of the arc AB of circle (D) . E is a point on the circle. F is the intersection of AB and CE . Show that $CA^2 = CE \cdot CF$.

distance(C,F) · distance(C,E) a^2

72. Distance to circumcenter and orthocenter

The distance of a side of a triangle from the circumcenter is equal to half the distance of the opposite vertex from the orthocenter.

AA

distance(D,BC) $\frac{1}{2}$

distance(A,G)

73. A relationship between the length of a side in the original triangle to distance between the feet of the altitudes

The ratio of a side of a triangle to the corresponding side of the orthic triangle is equal to the ratio of the circumradius to the distance of the side considered from the circumcenter.

AA

$\frac{\text{distance}(A,B)}{\text{distance}(E,D)} = \frac{2 \cdot a \cdot b}{a^2 + b^2 - c^2}$

$\frac{\text{radius}(CF)}{\text{distance}(F,AB)} = \frac{2 \cdot a \cdot b}{a^2 + b^2 - c^2}$

74. Distance between orthocenter and circumcenter

F is the orthocenter of triangle ABC and G is the circumcenter.

$$AF^2 + BC^2 = 4GA^2$$

The diagram shows a triangle ABC inscribed in a circle. The orthocenter F is the intersection of altitudes AD, BE, and CF. The circumcenter G is the center of the circle. Side lengths are labeled a, b, and c. The diagram is part of a software interface with a toolbar at the top and a calculation area at the bottom.

4 · distance(G,A)²

$$\frac{4 \cdot a^2 \cdot b^2 \cdot c^2}{(a+b+c) \cdot (a+b-c) \cdot (a-b+c) \cdot (-a+b+c)}$$

distance(A,F)² + distance(B,C)²

$$\frac{4 \cdot a^2 \cdot b^2 \cdot c^2}{(a+b+c) \cdot (a+b-c) \cdot (a-b+c) \cdot (-a+b+c)}$$

75. Intersections of perpendicular bisectors with neighboring sides

The perpendicular bisectors of the sides AC, AB of the triangle ABC meet the sides AB, AC in D and E. Prove that the points B, C, D, E lie on a circle.

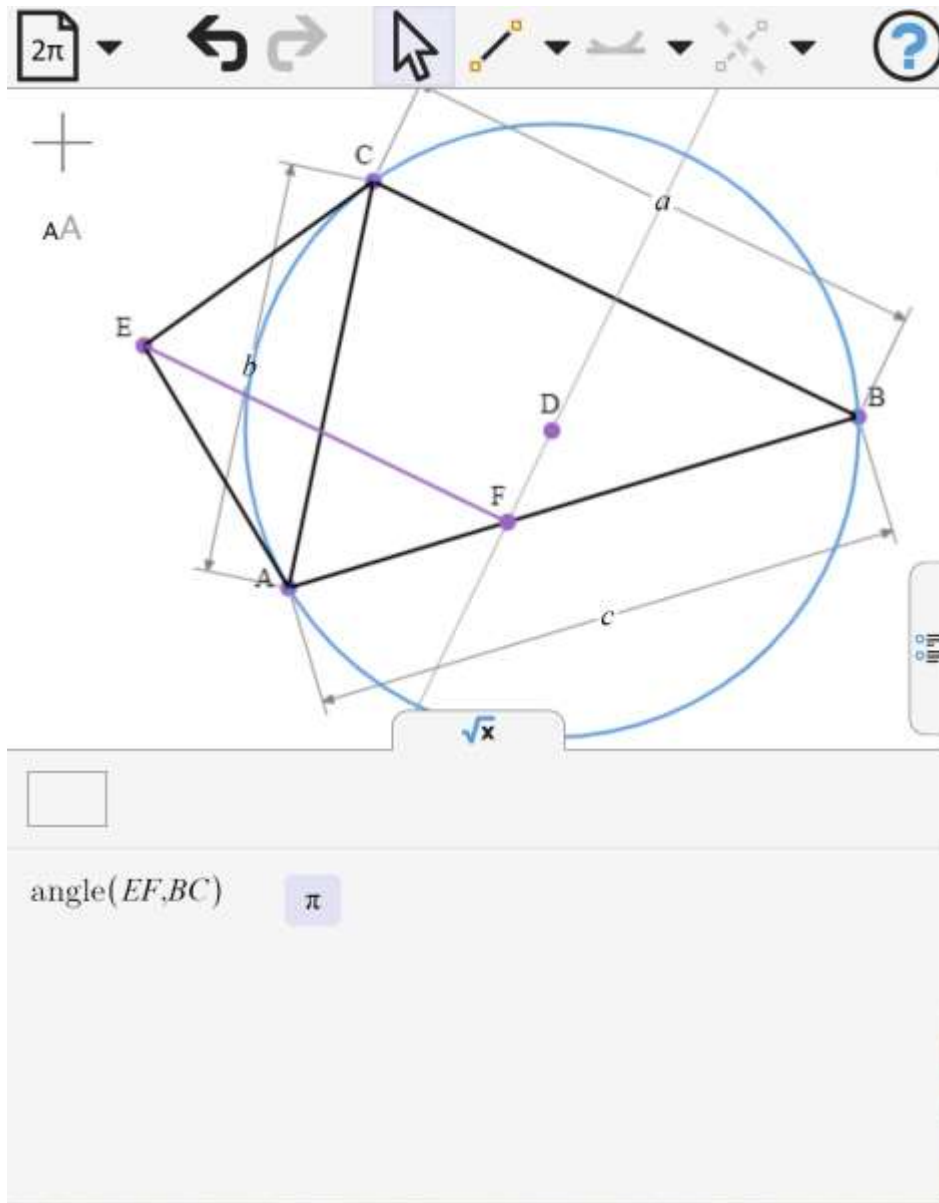
Furthermore the intersection of the perpendicular bisectors lies on the same circle.

The diagram illustrates a triangle ABC with vertices A , B , and C . The side lengths are labeled as a (opposite A), b (opposite B), and c (opposite C). The perpendicular bisectors of sides AC and AB are shown as dashed lines intersecting at point G . These bisectors intersect sides AB and AC at points D and E respectively. A blue circle is drawn passing through points B , C , D , and E . The center of this circle is marked as point F . The diagram is presented in a software environment with a toolbar at the top and a calculation panel at the bottom.

$\frac{\text{distance}(G,F)}{\text{radius}(C0)}$	1
$\frac{\text{distance}(E,F)}{\text{radius}(C0)}$	1

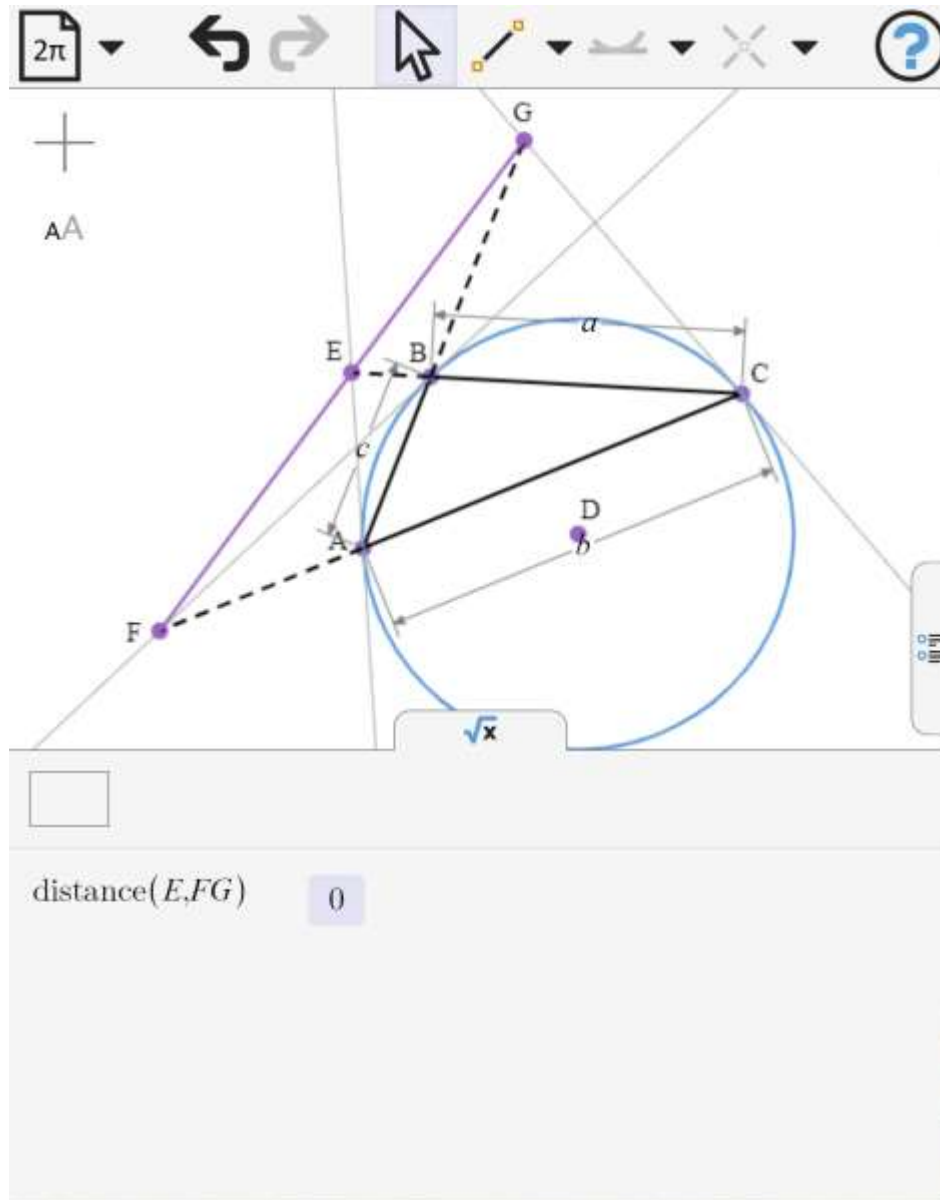
76. Line joining the intersection of two tangents of the circumcircle with the intersection between a perpendicular bisector and the adjacent side

The two tangents to the circumcircle of ABC at A and C meet at E . The perpendicular bisector of BC meets AB at F . Show that EF is parallel to BC .



77. Lemoine Axis

The lines tangent to the circumcircle of a triangle at the vertices meet opposite sides in three collinear points (the Lemoine axis of the triangle).



78. Point on the symmedian

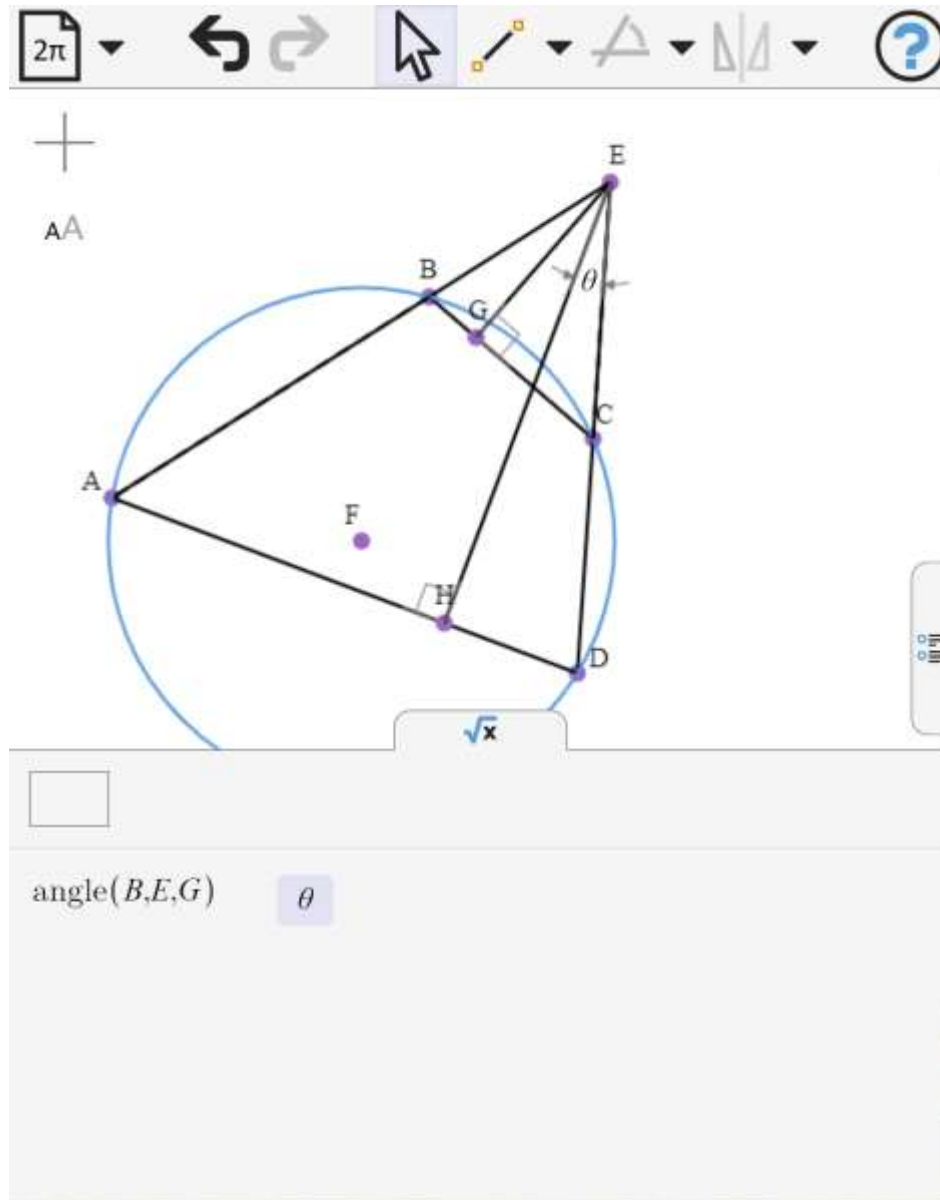
The distances from a point on the symmedian of a triangle to the two including sides are proportional to those sides. (The symmedian is the image of the median under reflection in the angle bisector.)

The diagram shows a triangle ABC with vertices A , B , and C . Side BC is labeled a , side AC is labeled b , and side AB is labeled c . Medians AD and BE intersect at the centroid G . A symmedian line is drawn through G , intersecting side AC at point F . The distance from G to side AB is labeled t , and the distance from G to side BC is also labeled t . The diagram is part of a software interface with a toolbar at the top containing icons for a ruler, eraser, and other tools. Below the diagram is a formula input field with the expression:

$$\frac{\text{distance}(G,AB)}{\text{distance}(G,BC)} = \frac{c}{a}$$

79. Projections onto opposite sides of a cyclic quadrilateral

Let $ABCD$ be a cyclic quadrilateral and E the intersection of sides AB and BC . Let G and H be the projections of E onto BC and AD . Then angles BEG and HED are equal.



80. A parallelogram defined by the circumcircle of two vertices and the orthocenter of a triangle

The perpendicular at the orthocenter F to the altitude FC of the triangle ABC meets the circumcircle of FBC in H . Show that $ABHF$ is a parallelogram.

The diagram shows a triangle ABC with vertices A , B , and C . The orthocenter is F . The circumcircle of FBC is shown. The altitude from A to BC is AD , and the altitude from C to AB is CE . The perpendicular from F to FC meets the circumcircle at H . The center of the circumcircle is G . The side lengths are labeled a , b , and c .

Properties panel:

- angle(AB, FH) π
- distance(F, H) c

81. Line defined by the orthocenter and the circumcenter is bisected by the median

Let F be the orthocenter of triangle ABC and G its circumcenter. Let I be the midpoint of AF . Show that the segment IG is bisected by the median AH .

The diagram shows a triangle ABC inscribed in a circle. The orthocenter is F and the circumcenter is G . Median AH is drawn, where H is the midpoint of BC . I is the midpoint of AF . The diagram illustrates that AH bisects IG . The software interface includes a toolbar with icons for undo, redo, move, and other geometric tools. The command input area at the bottom contains the expression:

$$\frac{\text{distance}(I,J)}{\text{distance}(J,G)} \quad 1$$

82. Line joining the orthocenter with the midpoint of a side

Prove that FH (see above) passes through the diametric opposite of A on the circumcircle

distance(I,FH) 0

83. Internal and external angle bisectors

The internal and external bisectors of an angle of a triangle pass through the ends of the circumdiameter which is perpendicular to the side opposite the vertex considered

angle(EF, AC) $\frac{\pi}{2}$

distance(D, EF) 0

84. An altitude intersecting the circumcircle

The segment of the altitude extended between the orthocenter and the second point of intersection with the circumcircle is bisected by the corresponding side of the triangle

The diagram illustrates a triangle ABC inscribed in a circumcircle. The orthocenter H is the intersection of altitudes BE and CF . The circumcenter D is the center of the circle. The altitude BE is extended to meet the circumcircle at point G . The segment HG is bisected by the side AC at point F . Side lengths are labeled a , b , and c . The diagram is shown in a software interface with a toolbar and a command input field.

distance(G,H)
distance(H,F) 1

85. Circumcircle of two points and the orthocenter

The circumcircle of the triangle formed by two vertices and the orthocenter of a given triangle is equal to the circumcircle of the given triangle

radius(C1)

$$\frac{a \cdot b \cdot c}{\sqrt{a+b+c} \cdot \sqrt{a+b-c} \cdot \sqrt{a-b+c} \cdot \sqrt{-a+b+c}}$$

radius(C0)

$$\frac{a \cdot b \cdot c}{\sqrt{a+b+c} \cdot \sqrt{a+b-c} \cdot \sqrt{a-b+c} \cdot \sqrt{-a+b+c}}$$

86. Arc defined by intersection of altitudes with circumcircle

A vertex of a triangle is the midpoint of the arc determined on its circumcircle by the two altitudes, produced, issued from the two other vertices

The diagram shows a triangle ABC inscribed in a circumcircle. Altitudes AD and BE are drawn from vertices A and B respectively, meeting the opposite sides at D and E . The altitudes are extended to intersect the circumcircle at points F and E . The angle θ is indicated at vertex A . The side lengths b and c are labeled. The diagram is displayed in a software interface with a toolbar at the top and a properties panel at the bottom.

angle(A,D,F) $\pi - 2 \cdot \theta$

angle(A,D,E) $\pi - 2 \cdot \theta$

87. A line parallel to one circumcircle tangent through the intersection of the others

Through the point of intersection of the tangents DB, DC to the circumcircle (O) of the triangle ABC a parallel is drawn to the line touching (O) at A. If this parallel meets AB, AC in E, F show that D bisects EF.

The diagram shows a triangle ABC inscribed in a circumcircle with center O . Tangents DB and DC are drawn from an external point D to the circle at points B and C . A line passing through D is drawn parallel to the tangent to the circle at vertex A . This line intersects side AB at point E and side AC at point F . The goal is to prove that D bisects the segment EF .

The software interface includes a toolbar at the top with icons for a file, undo, redo, selection, and other tools. The diagram is labeled with vertices A, B, C, D, E, F, G and sides a, b, c . A text box at the bottom contains the expression $\frac{\text{distance}(E,F)}{\text{distance}(E,G)}$ with a value of 1 next to it.

88. Relationship between mix-linear circles and circumcircle

In a triangle ABC let p and q be the radii of two circles through A touching side BC at B and C respectively. Then $p \cdot q = R^2$ (where R is the circumradius).

The diagram illustrates a triangle ABC with vertices A, B, and C. Three circles are shown passing through vertex A. One circle is tangent to side BC at point B, and another is tangent to side BC at point C. The third circle is the circumcircle of the triangle, tangent to side BC at point D. The radius of the circumcircle is labeled R. The radius of the circle tangent to BC at B is labeled p, and the radius of the circle tangent to BC at C is labeled q. The side lengths of the triangle are labeled a, b, and c. Points E and F are also marked on the diagram.

Below the diagram, a mathematical expression is displayed:

$$\frac{\text{radius}(C1) \cdot \text{radius}(C2)}{\text{radius}(C0)^2} = 1$$

89. Intersections between parallels and tangents

The parallel to the side AC through the vertex B of the triangle ABC meets the tangent to the circumcircle (O) of ABC at C in B1, and the parallel through C to AB meets the tangent to (O) at B in C1. Prove that $BC^2 = BC_1 \cdot B_1C$

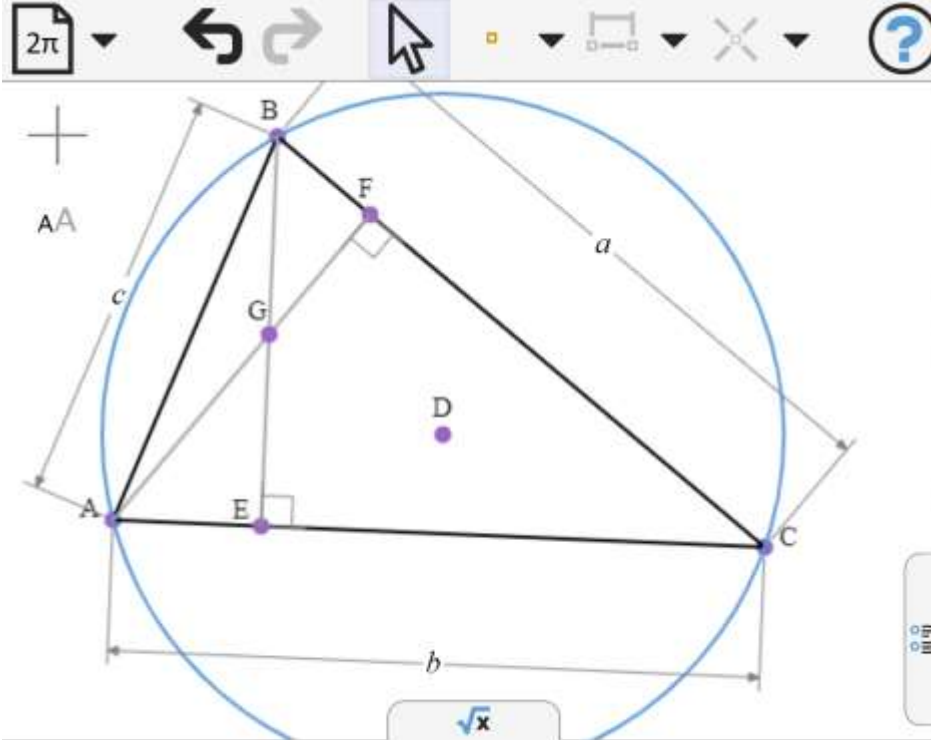
distance(B,F) · distance(C,E) a^2

The Euler Line

The circumcenter O , orthocenter H and centroid G of a given triangle are collinear and the line is called the Euler Line of the triangle.

90. Distance between orthocenter and circumcenter

Let D and G be the circumcenter and orthocenter of a triangle ABC . Let R be the radius of the circumcircle. Show that $DG^2 = 9R^2 - a^2 - b^2 - c^2$



$9 \cdot \text{radius}(C0)^2 - \text{distance}(G,D)^2$ $a^2 + b^2 + c^2$

91. Distance between orthocenter and centroid

With the same notations for the triangle ABC we have: $4AD^2 = \frac{4}{9}AB^2 + \frac{4}{9}AC^2 + \frac{4}{9}BC^2 + GH^2$

AA

\sqrt{x}

$4 \cdot \text{radius}(CO)^2 - \text{distance}(G,H)^2$

$\frac{4 \cdot (a^2 + b^2 + c^2)}{9}$

92. Circumradius related to side lengths and distance between orthocenter and circumcenter

With the usual notations for the triangle ABC we have:

$$9AD^2 = AB^2 + AC^2 + BC^2 + GD^2$$

The diagram shows a triangle ABC inscribed in a circle. The orthocenter is G and the circumcenter is D . The altitudes from B , C , and A are BE , CF , and AH respectively. The circumradius is labeled a . The diagram is part of a software interface with various tool icons at the top and a formula input field at the bottom.

Input field: $9 \cdot \text{distance}(A,D)^2 - \text{distance}(G,D)^2$

Output field: $a^2 + b^2 + c^2$

93. Circumradius in terms of distances between vertices and orthocenter

With the notation of the previous example,

$$12AD^2 = AB^2 + AC^2 + BC^2 + AG^2 + BG^2 + CG^2$$

The diagram shows a triangle ABC inscribed in a circle. The vertices are labeled A , B , and C . The orthocenter is G , and the circumcenter is D . The altitudes from B and C are BE and CF respectively, meeting at G . The side lengths are $a = BC$, $b = AC$, and $c = AB$. The circumradius is $R = AD$. The diagram is part of a software interface with various tool icons.

Below the diagram, there is a text input field containing the following expression:

$$12 \cdot \text{distance}(A,D)^2 - \text{distance}(A,G)^2 - \text{distance}(B,G)^2 - \text{distance}(C,G)^2$$

The expression is followed by a highlighted box containing the answer:

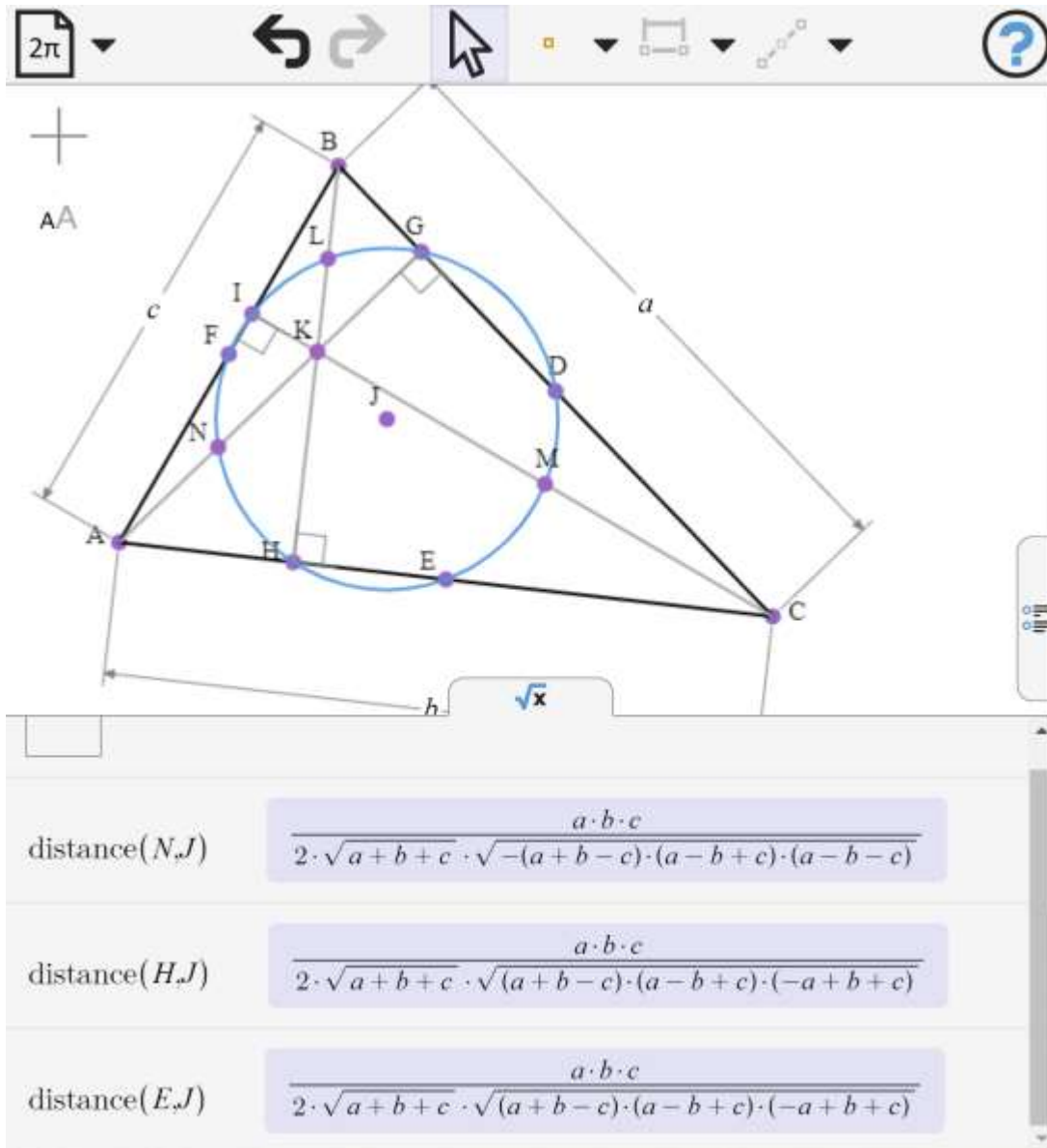
$$a^2 + b^2 + c^2$$

The Nine Point Circle

The midpoints of the segments joining the orthocenter of a triangle to its vertices are called the Euler Points of the triangle. The three Euler Points determine the Euler Triangle.

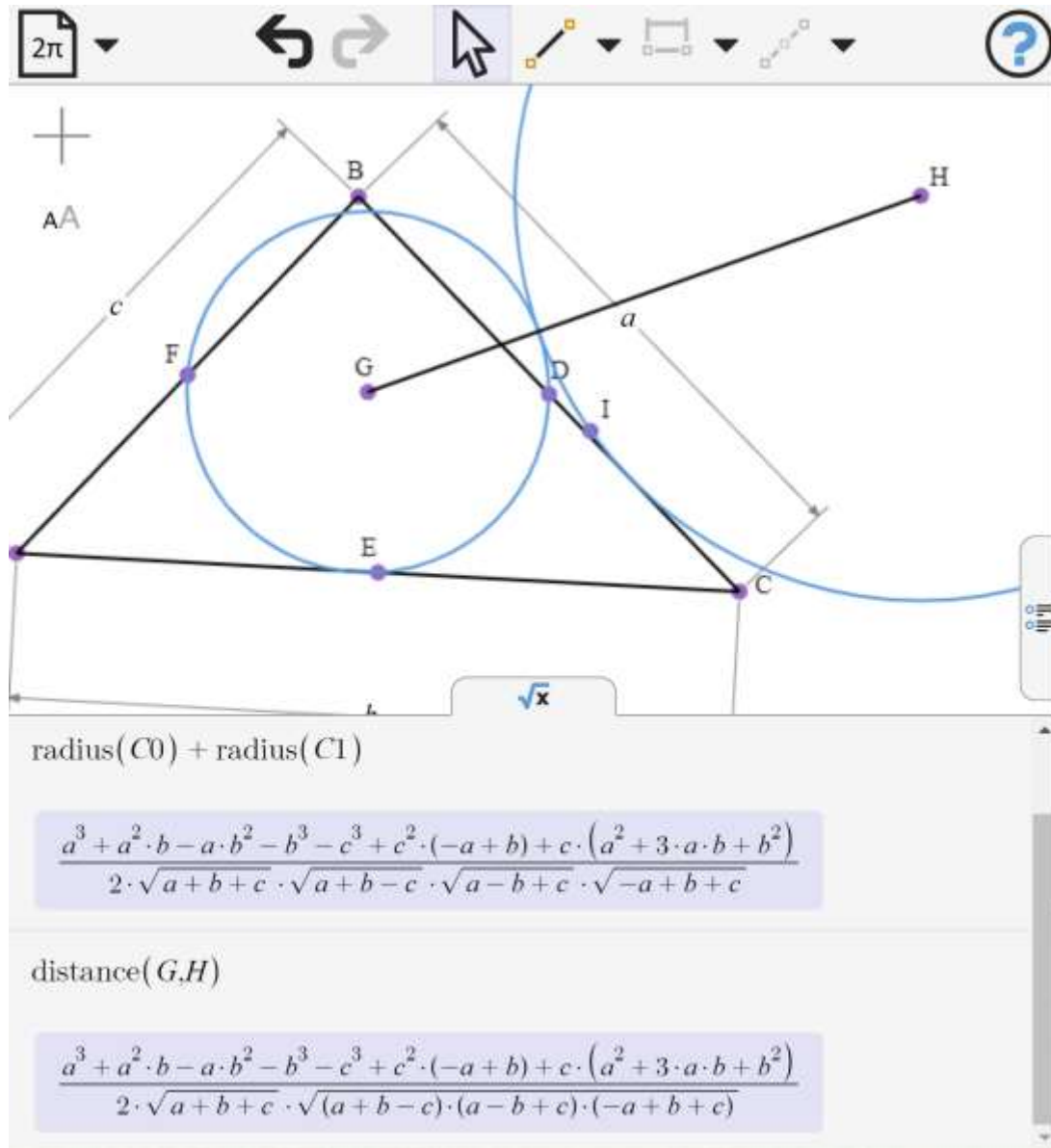
94. The Nine Point Circle Theorem

In a triangle, the midpoints of the sides, the feet of the altitudes and the Euler Points lie on the same circle. (The Euler Points are the midpoints of the segments joining the vertices to the orthocenter).



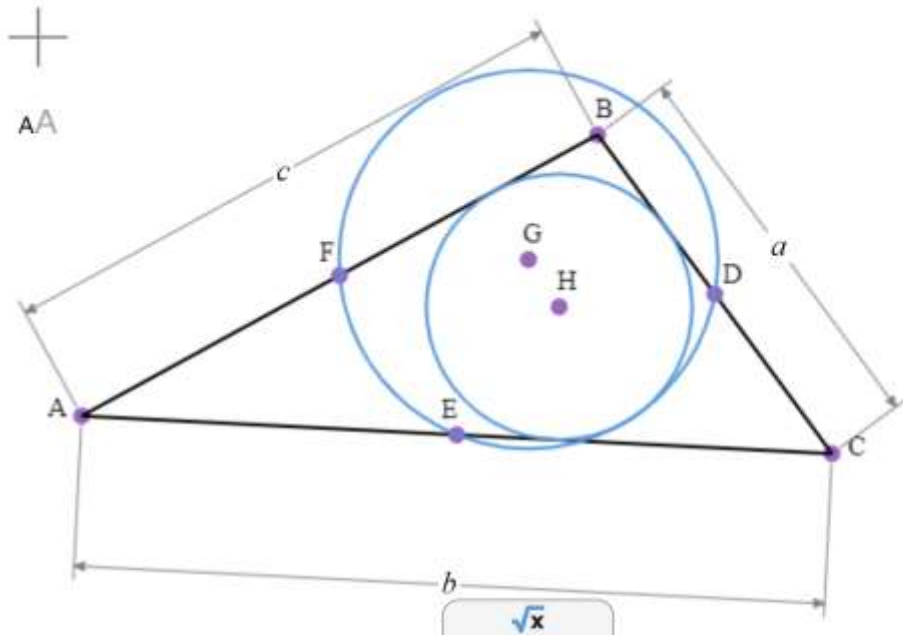
95. Feuerbach's Theorem

The nine-point circle of a triangle touches each of the four tritangent circles of the triangle.



For an excircle, we show that the distance between the circle centers is the sum of their radii.

For the incircle, we show that the distance between the circle centers is the difference in the radii.



radius(C0) - radius(C1)

$$\frac{a^3 - a^2 \cdot b - a \cdot b^2 + b^3 + c^3 + c^2 \cdot (-a - b) + c \cdot (-a^2 + 3 \cdot a \cdot b - b^2)}{2 \cdot \sqrt{a + b + c} \cdot \sqrt{a + b - c} \cdot \sqrt{a - b + c} \cdot \sqrt{-a + b + c}}$$

distance(G,H)

$$\frac{a^3 - a^2 \cdot b - a \cdot b^2 + b^3 + c^3 + c^2 \cdot (-a - b) + c \cdot (-a^2 + 3 \cdot a \cdot b - b^2)}{2 \cdot \sqrt{a + b + c} \cdot \sqrt{(a + b - c) \cdot (a - b + c) \cdot (-a + b + c)}}$$

96. Nine Point Circle and Circumcircle

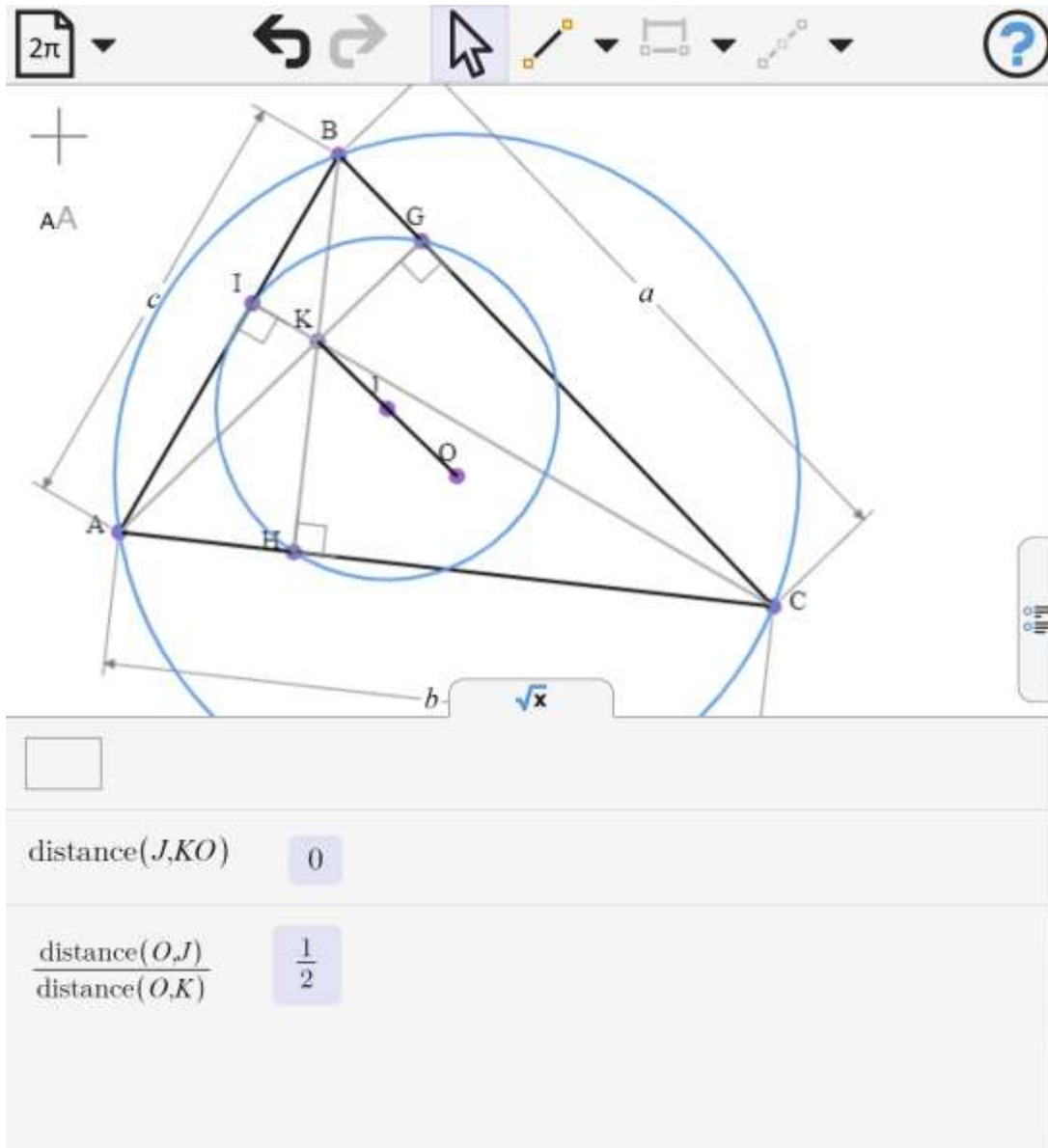
The radius of the nine-point circle is equal to half the circumradius of the triangle

radius(C1) $\frac{a \cdot b \cdot c}{\sqrt{a+b+c} \cdot \sqrt{a+b-c} \cdot \sqrt{a-b+c} \cdot \sqrt{-a+b+c}}$

radius(C0) $\frac{a \cdot b \cdot c}{2 \cdot \sqrt{a+b+c} \cdot \sqrt{a+b-c} \cdot \sqrt{a-b+c} \cdot \sqrt{-a+b+c}}$

97. Nine Point circle center

The nine-point circle center lies on the Euler line midway between the circumcenter and the orthocenter.



We show that J lies on HO and that $OK = 2 \cdot OJ$.

98. Nine point circle center collinearity

Show that the foot of the altitude of a triangle on a side, the midpoint of the segment of the circumdiameter between this side and the opposite vertex and the nine point center are collinear

The diagram illustrates a triangle ABC with vertices A , B , and C . The circumcircle is shown with center K and radius a . The orthocenter is H , and the nine-point center is G . The nine-point circle is shown with center G and radius r . The diagram shows the collinearity of G , H , and K . The distance between G and H is shown to be 0 .

distance(G,H,K)

99. Nine point circle center halfway between Euler point and opposite side midpoint

The center of the nine-point circle is the midpoint of a Euler point and the midpoint of the opposite side.

The diagram shows a triangle with vertices A , B , and C . The orthocenter is H , the circumcenter is G , and the centroid is K . The Euler line is the line passing through H , G , and K . The nine-point circle is shown as a blue circle passing through the midpoints of the sides (F , D , H), the feet of the altitudes (I , J , L), and the midpoints of the segments from each vertex to the orthocenter (K , L , M). The center of the nine-point circle is E , which is the midpoint of the segment GH . The diagram is displayed in a software interface with a toolbar at the top and a command input area at the bottom.

distance(G,EL) 0

Both points lie on the 9-point circle, so all we need to show is that the center lies on the chord between the points.

100. Distance between the reflection of a vertex in the opposite side of a triangle and its orthocenter

Let K be the orthocenter of triangle ABC , and G the center of the nine point circle. If L is the reflection of the vertex A in the opposite side BC , show that KL is equal to 4 times the distance of the nine-point center from BC

distance(K,L)
distance(G,BC) 4

101. Length of the tangent from a vertex to the nine point circle

Show that the square of the tangent from a vertex of a triangle to the nine point circle is equal to the altitude issued from that vertex multiplied by the distance of the opposite side from the circumcenter.

The diagram shows a triangle ABC with vertices $A(0,0)$, $B(b,c)$, and $C(a,0)$. The circumcenter is G and the incenter is I . The orthocenter is H . The nine-point circle passes through the midpoints of the sides and the feet of the altitudes. A tangent line is drawn from vertex A to the nine-point circle at point H .

The software interface includes a toolbar with icons for 2π , undo, redo, mouse, square, line, and a question mark. Below the diagram, there are input fields for mathematical expressions:

distance(A,BC) · distance(I,BC) $\frac{a \cdot b}{2}$

distance(A,H)² $\frac{a \cdot b}{2}$

102. Reflection of the circumcenter in the side of a triangle

Show that the reflection of the circumcenter in the side of a triangle coincides with the reflection of the vertex opposite the side in the nine-point center of the triangle.

The diagram shows a triangle ABC with vertices A , B , and C . The circumcenter is G and the nine-point center is I . The circumcircle is shown in grey, and the nine-point circle is shown in blue. The reflection of G across side AC is J . The reflection of B across side AC is P . The diagram illustrates that P lies on the line segment BI and that $BP = PI$.

Below the diagram, there is a software interface with a toolbar and a list of distance calculations:

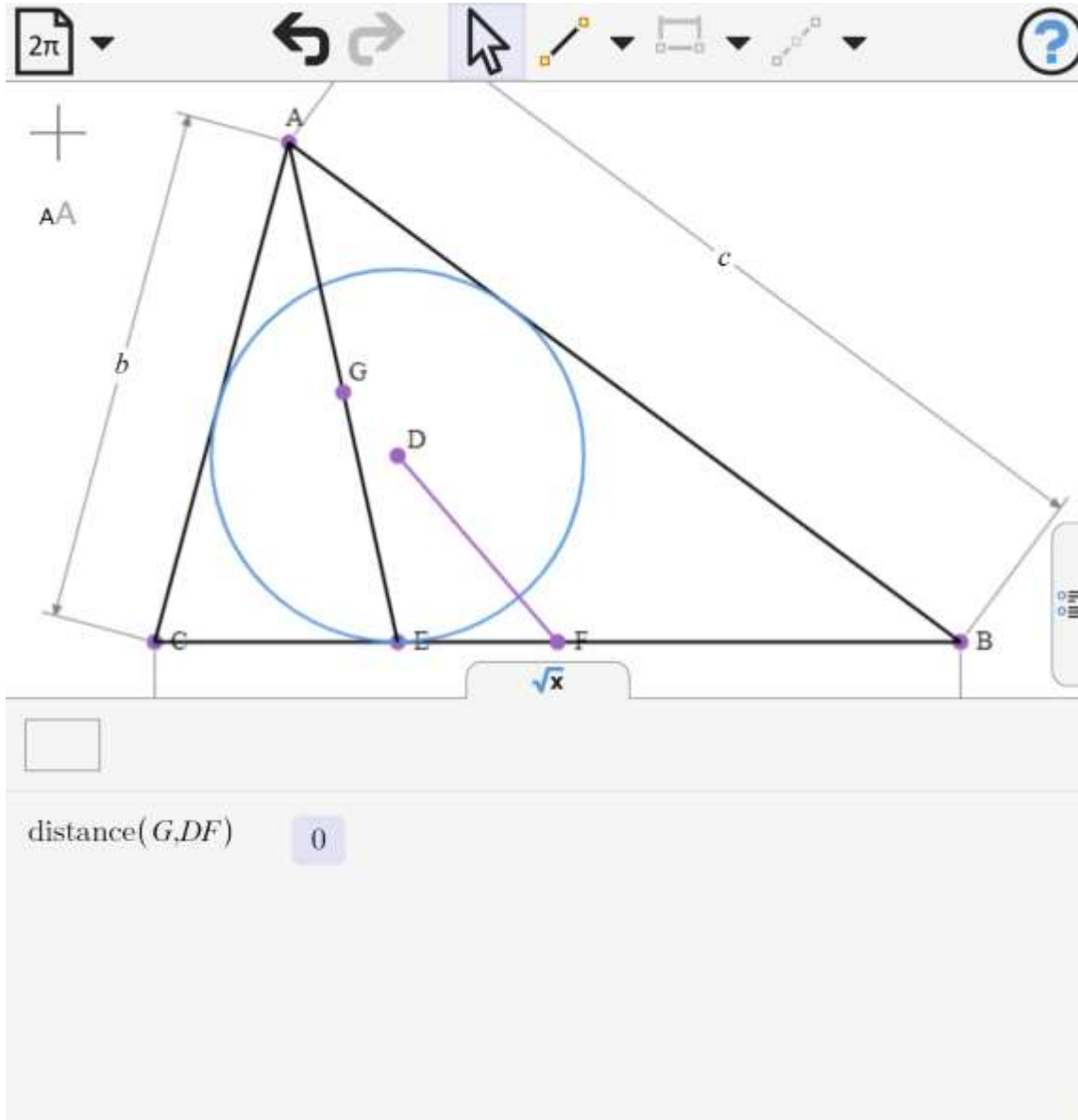
- distance(G,B) - distance(G,J) 0
- distance(G,BJ) 0

P is the image of O under reflection in AC . We show that N lies on BP and that $BN=PN$.

Incircles and Excircles

103. The midpoint of line joining a vertex with the point of contact of the incircle on the opposite side

Let incircle (with center I) of triangle ABC touch the side BC at X and M be the midpoint of this side. Then line MI bisects AX .



G is the midpoint of AE , we check that it lies on the line between F and D

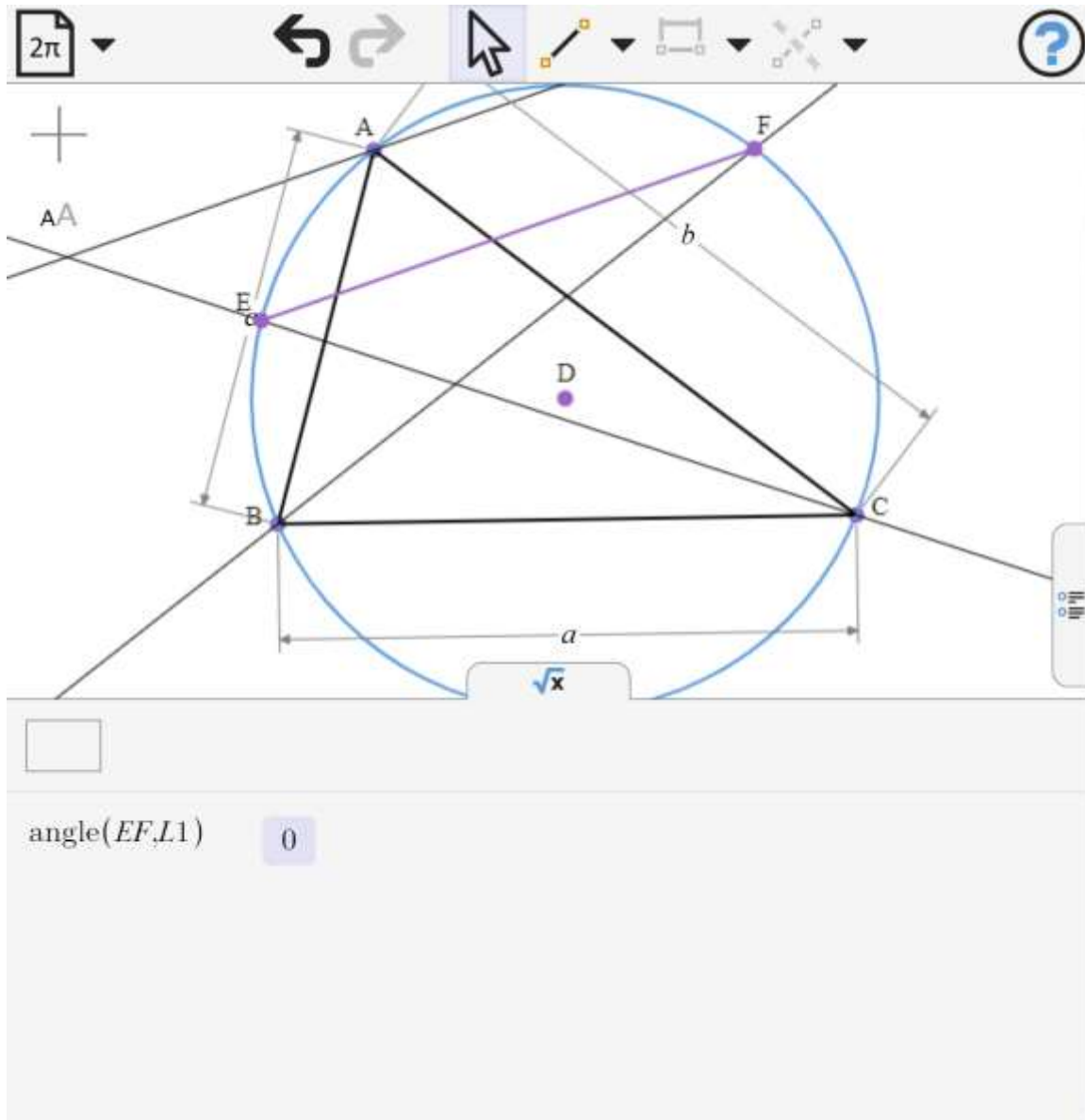
104. Product of the distances from a vertex to the incenter and the center of the opposite excircle

The product of the distances of two tritangent centers of a triangle from the vertex of the triangle collinear with them is equal to the product of the two sides of the triangle passing through the vertex considered.

distance(A,D) · distance(A,F) b · c

105. A relationship between angle bisectors and the circumcircle

Show that the external bisector of an angle of a triangle is parallel to the line joining the points where the circumcircle is met by the external (internal) bisectors of the other two angles of the triangle



106. Product of incircle and excircle radii

The product of the four tritangent radii of a circle is equal to the square of its area

The diagram illustrates a triangle ABC with side lengths a , b , and c . The incircle is tangent to the sides at points A , B , and C . The excircles are tangent to the sides at points H , D , and F . The centers of the circles are labeled H , D , and F .

The software interface shows the following formulas:

area(A,B,C) $\frac{\sqrt{a+b+c} \cdot \sqrt{a+b-c} \cdot \sqrt{a-b+c} \cdot \sqrt{-a+b+c}}{4}$

radius($C0$) · radius($C1$) · radius($C2$) · radius($C3$)

$\frac{(a+b+c) \cdot (a+b-c) \cdot (a-b+c) \cdot (-a+b+c)}{16}$

107. Points of contact of incircle and excircle

The points of contact of a side of a triangle with the incircle and the excircle relative to this side are two isotomic points

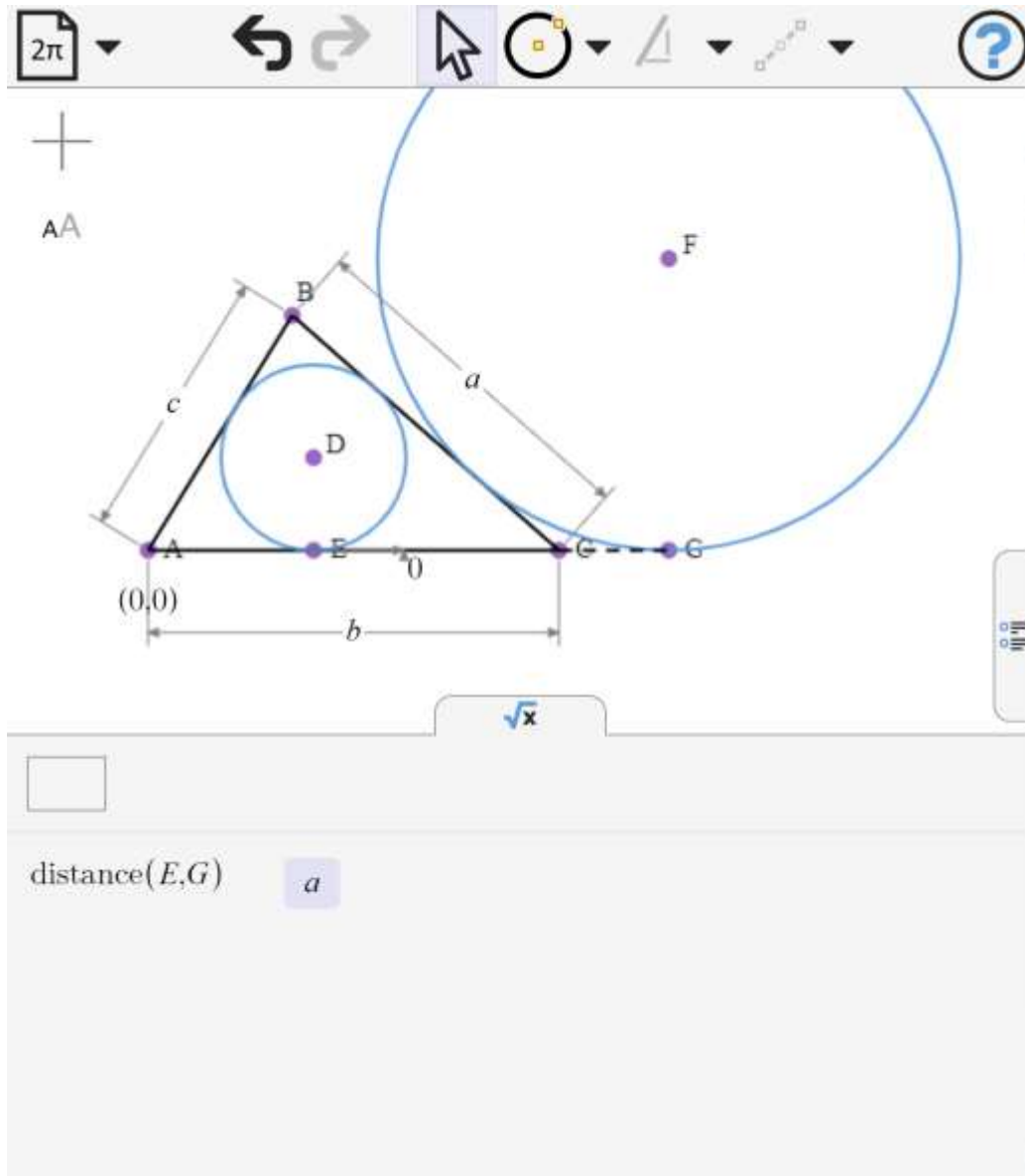
The diagram shows a triangle with vertices A, B, and C. Side BC has length a , side AC has length b , and side AB has length c . The incircle, centered at D , is tangent to side BC at point E . The excircle, centered at F , is tangent to side BC at point G . The points E and G are isotomic points on side BC.

The software interface includes a toolbar with icons for 2π , undo, redo, mouse, circle, and other tools. Below the diagram is a calculation panel with a text input field and two rows of data:

distance(G,C)	$\frac{a-b+c}{2}$
distance(B,E)	$\frac{a-b+c}{2}$

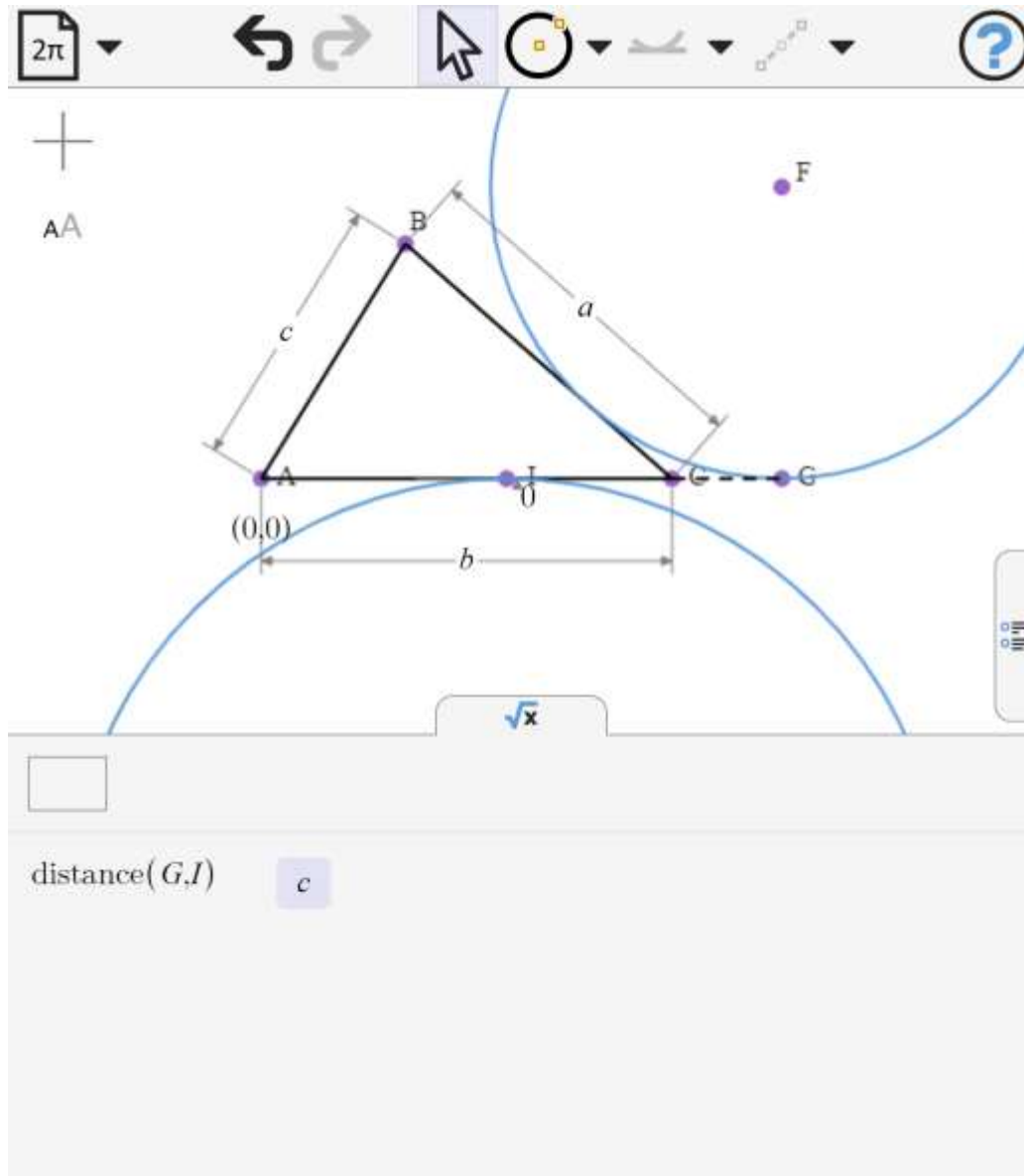
108. Distance between point of contact of incircle and excircle

(I) is the incircle, (J) the excircle defined by side BC. The distance between the points of contact of (I) and (J) with AC (extended) is the same as the length of BC.



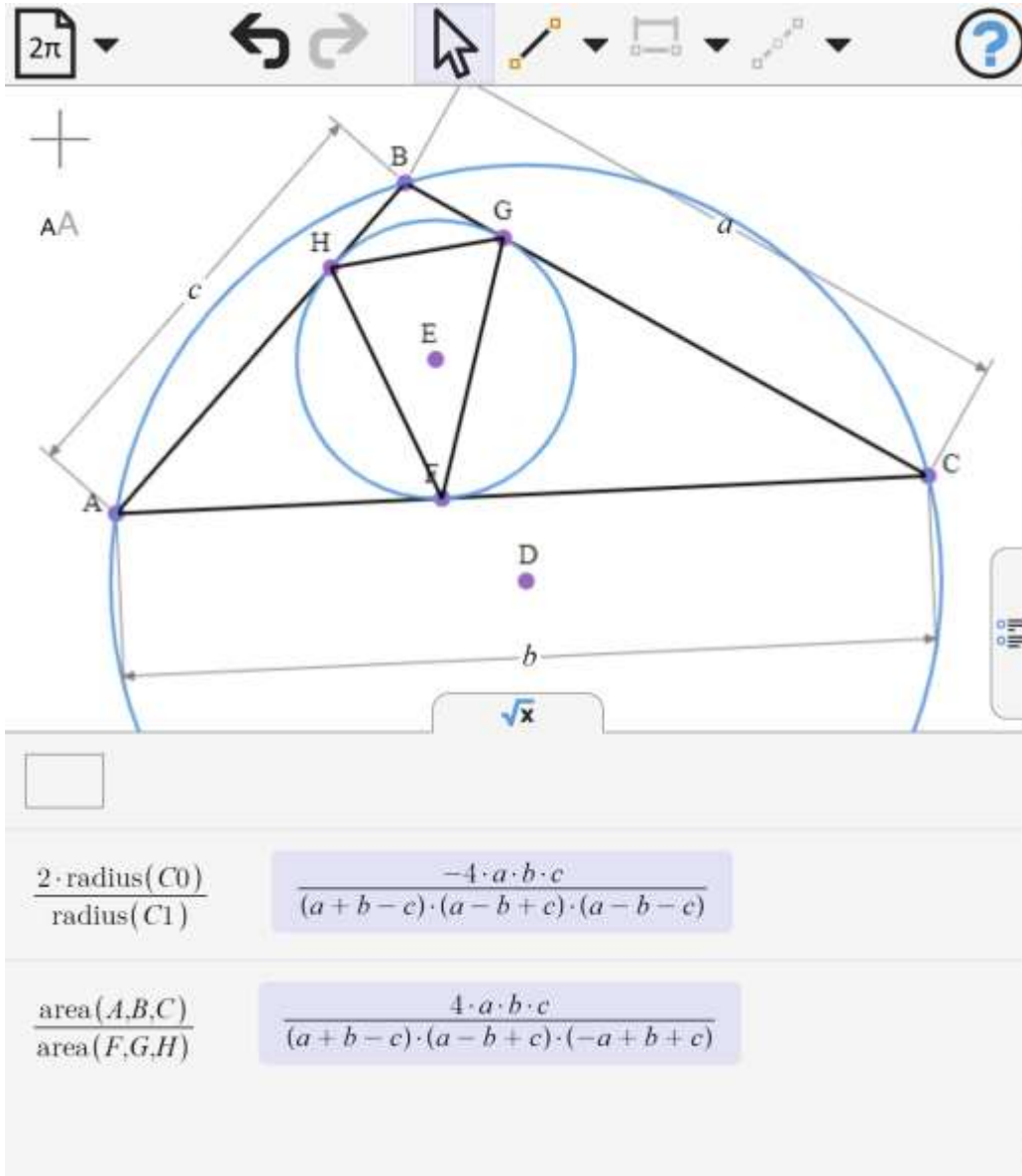
109. Distance between points of contact of two excircles

(I) is the excircle defined by AB, (J) the excircle defined by side BC. The distance between the points of contact of (I) and (J) with BC (extended) is the same as the length of AC.



110. Ratio of areas of a triangle and the triangle formed by the points of contact of its incircle

The ratio of the area of a triangle to the area of the triangle determined by the points of contact of the sides with the incircle is equal to the ratio of the circumdiameter of the given triangle with its inradius.



111. Distances between vertices and points of contact with the incircle

If X, Y and Z are the points of contact between the incircle and the triangle opposite A, B, C respectively. Show that $AZ \cdot BX \cdot CY = r$ times the area of the triangle

radius(C1) · area(A,B,C) $\frac{(a+b-c) \cdot (a-b+c) \cdot (-a+b+c)}{8}$

distance(A,F) $\frac{-a+b+c}{2}$

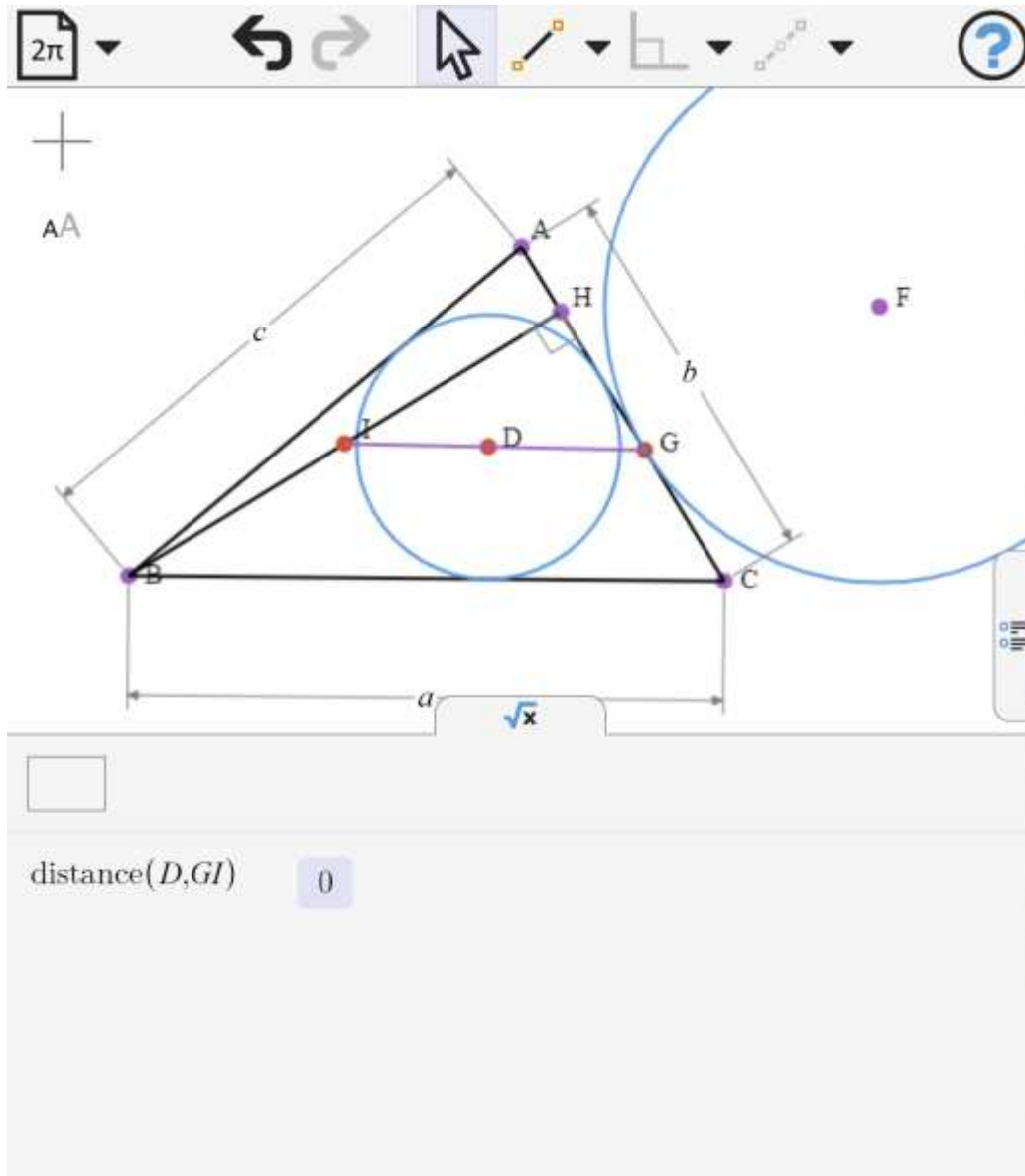
distance(C,G) $\frac{a+b-c}{2}$

distance(B,H) $\frac{-(-a+b-c)}{2}$

Visually we can see that the identity holds, but when we ask Geometry Expressions to compute the product of the lengths, because it goes back to the unsimplified expression, it does not give the good answer!

112. Incentre, excircle point of contact and altitude midpoint are collinear

Show that the midpoint of an altitude of a triangle, the point of contact of the corresponding side with the excircle relative to that side and the incenter of the triangle are collinear



113. Product of the distances from the vertices to the incenter

Show that the product of the distances of the incenter of a triangle from the three vertices of the triangle is equal to $4Rr^2$, where R is the radius of the circumcircle, and r is the radius of the incircle.

The diagram shows a triangle with vertices A, B, and C. A circumcircle is drawn around the triangle, with center D. An incircle is drawn inside the triangle, touching the sides at points F and G. The incenter is labeled E. The side lengths are labeled a, b, and c. The circumradius is R, and the inradius is r. The diagram is shown in a software interface with various tool icons at the top and a formula input field at the bottom.

distance(A,E) · distance(B,E) · distance(C,E)
 $4 \cdot \text{radius}(C0) \cdot \text{radius}(C1)^2$ 1 x

114. External angle bisectors

The external bisectors of the angles of a triangle meet the opposite sides in 3 collinear points

The diagram illustrates a triangle ABC with vertices A , B , and C . The side lengths are labeled as a (opposite A), b (opposite B), and c (opposite C). The external angle bisectors are shown as lines extending from each vertex. The bisector from A meets the extension of BC at point D . The bisector from B meets the extension of AC at point E . The bisector from C meets the extension of AB at point F . A purple line passes through points D , E , and F , demonstrating that they are collinear. The diagram is shown in a software interface with a toolbar at the top and a command input area at the bottom. The command input area shows the command `distance(E,DF)` with a value of `0`.

Intercept Triangles

Let L, M, N be three points on the sides BC, CA, AB of triangle ABC . Then triangle LMN and the triangle determined by lines AL, BM and CN are called the intercept triangles of triangle ABC for points L, M, N .

115. Feinman's triangle

Let D, E, F be points on the sides BC, CA, AB of a triangle ABC such that $BD/BC = CE/CA = AF/AB = 1/3$.

Show the area of the triangle determined by the lines AD, BE, CF is one seventh the area of triangle ABC .

The diagram shows a triangle ABC with vertices A (bottom left), B (top), and C (bottom right). The side lengths are labeled a (opposite A), b (opposite B), and c (opposite C). The height from A to BC is labeled h . Points D, E, F are located on sides BC, CA, AB respectively, such that $BD/BC = CE/CA = AF/AB = 1/3$. Lines AD, BE, CF are drawn and intersect at points G, H, I to form the inner triangle GHI . The diagram is part of a software interface with a toolbar at the top and a calculation area at the bottom.

area(G,H,I)
area(A,B,C) $\frac{1}{7}$

116. Steiner's triangle with same ratio on each side

Let D, E, F be points on the sides BC, CA, AB of a triangle ABC such that $BD/BC = CE/CA = AF/AB = r$.

Show the ratio of area of the triangle determined by the lines AD, BE, CF to the area of triangle ABC is

$$\frac{(2r-1)^2}{r^2-r+1}$$

The diagram illustrates a triangle ABC with vertices A , B , and C . Points D , E , and F are located on the sides BC , CA , and AB respectively. Lines AD , BE , and CF are drawn, intersecting at an inner triangle GHI . The side lengths are labeled as a , b , and c . A height h is shown from vertex A to the base BC . The diagram is presented in a software interface with various icons at the top and bottom.

Below the diagram, the ratio of the area of the inner triangle GHI to the area of the outer triangle ABC is given by the following equation:

$$\frac{\text{area}(G,H,I)}{\text{area}(A,B,C)} = \frac{(-1+2\cdot r)^2}{1-r+r^2}$$

117. Steiner's triangle

Using the above notation if $\frac{BD}{DC} = p, \frac{CE}{EA} = q, \frac{AF}{FB} = r$, then the ratio of areas is $\frac{(pqr-1)^2}{(qp+p+1)(rp+r+1)(rq+q+1)}$

The proportional distances used by GXWeb represent the ratios $\frac{BD}{BC}, \frac{CE}{CA}, \frac{AF}{AB}$. Hence, we need to set

$$\frac{BD}{BC} = \frac{p}{1+p}, \frac{CE}{CA} = \frac{q}{1+q}, \frac{AF}{AB} = \frac{r}{1+r}.$$

area(G,H,I)
area(A,B,C)

$$\frac{(1-p \cdot q \cdot r)^2}{(-1-q-g \cdot r) \cdot (-1+r \cdot (-1-p)) \cdot (1+p+p \cdot q)}$$

118. Two equiareal triangles

Let E,F,G be points on sides BC,CA,AB of a triangle ABC. Show that if G,H,I are points on sides AB, AC and BC such that EG is parallel to AC, DH is parallel to BC and FI is parallel to AB, then triangles EFG and GHI have the same areas.

The diagram shows a triangle ABC with vertices A, B, and C. Points E, F, and G are on sides BC, CA, and AB respectively. Lines EG, FH, and DI are drawn, forming an inner triangle GHI. The diagram includes various labels: 'c' for side AB, 'b' for side AC, and 'theta' for angle A. Points D, H, and I are also marked on the sides. The diagram is part of a software interface with a toolbar at the top and a calculation panel at the bottom.

area(G,H,I) - area(D,E,F) 0

area(G,H,I) $\frac{-b \cdot c \cdot (-1 + p + q - p \cdot q + r \cdot (1 - p - q)) \cdot \sin(\theta)}{2}$

area(D,E,F) $\frac{b \cdot c \cdot (1 - p - q + p \cdot q + r \cdot (-1 + p + q)) \cdot \sin(\theta)}{2}$

119. A triangle defined by parallels

Three parallel lines drawn through the vertices of a triangle ABC meet the respectively opposite sides in the points E, F, G . Show that area EFG is twice area ABC .

The diagram illustrates the construction of triangle EFG from triangle ABC . Three parallel lines are drawn through vertices A , B , and C . The line through A is parallel to BC and intersects BE and CF at E and F respectively. The line through B is parallel to AC and intersects AC at G and AE at E . The line through C is parallel to AB and intersects AB at G and BF at F . The resulting triangle EFG is shaded in purple. The diagram also shows the side lengths b and c , and the angle θ at vertex A . A dashed line represents the altitude from A to BC .

area(D,E,F) $b \cdot c \cdot \sin(\theta)$

area(A,B,C) $\frac{b \cdot c \cdot \sin(\theta)}{2}$

Equilateral triangles

120. The Napoleon Triangle

If equilateral triangles are erected externally (or internally) on the sides of any triangle, their centers form an equilateral triangle.

The diagram illustrates the construction of the Napoleon triangle. It shows a central triangle ABC with vertices A , B , and C . Three equilateral triangles are constructed externally on its sides: ABE on side AB , BCF on side BC , and CAM on side CA . The centers of these three equilateral triangles are labeled O , M , and N respectively. The triangle formed by these centers, OMN , is the Napoleon triangle. The diagram includes various construction lines, congruence markings, and labels for vertices and centers.

Below the diagram, the distance between the centers M and N is given by the following formula:

$$\sqrt{\frac{a^2}{6} + \frac{b^2}{6} + \frac{c^2}{6} + \frac{\sqrt{3} \cdot \sqrt{a+b+c} \cdot \sqrt{a+b-c} \cdot \sqrt{a-b+c} \cdot \sqrt{-a+b+c}}{6}}$$

distance(M,N)

The distance between the centers M and O is given by the following formula:

$$\sqrt{\frac{a^2}{6} + \frac{b^2}{6} + \frac{c^2}{6} + \frac{\sqrt{3} \cdot \sqrt{a+b+c} \cdot \sqrt{a+b-c} \cdot \sqrt{a-b+c} \cdot \sqrt{-a+b+c}}{6}}$$

distance(M,O)

The distance between the centers N and O is given by the following formula:

$$\sqrt{\frac{a^2}{6} + \frac{b^2}{6} + \frac{c^2}{6} + \frac{\sqrt{3} \cdot \sqrt{a+b+c} \cdot \sqrt{a+b-c} \cdot \sqrt{a-b+c} \cdot \sqrt{-a+b+c}}{6}}$$

121. Lines joining vertices to centers of equilateral triangles

Continuing the above example, show that AM , BN and CO are concurrent

The diagram illustrates the construction of three equilateral triangles on the sides of a triangle ABC . Equilateral triangles ABE , BCF , and ACD are constructed on the exterior of ABC . The centers of these equilateral triangles are labeled O , M , and N respectively. Lines AM , BN , and CO are drawn and intersect at a common point P . The diagram includes various construction lines, congruence markings (single, double, and triple tick marks), and labels for points A , B , C , E , O , M , P , and N .

distance(P, BN) 0

122. Fermat point

Let equilaterals BCD, ABF and ACE be erected externally on the sides of triangle ABC. Show that $AD=CF=BE$

The diagram shows a central triangle ABC with vertices A, B, and C. Three equilateral triangles are constructed externally on its sides: BCD on side BC, ABF on side AB, and ACE on side AC. Lines are drawn from each vertex of triangle ABC to the opposite vertex of the external equilateral triangle: AD from A to D, BE from B to E, and CF from C to F. The diagram includes side length labels: 'a' for BC, 'b' for AC, and 'c' for AB. Congruence markings (single, double, and triple tick marks) are used to indicate equal lengths between corresponding sides of triangles. A software interface is overlaid on the diagram, featuring a toolbar at the top with icons for undo, redo, pan, and zoom, and a calculator at the bottom. The calculator shows the following expressions:

distance(C,E)

$$\frac{\sqrt{2} \cdot \sqrt{a^2 + b^2 + c^2} + \sqrt{3} \cdot \sqrt{a+b+c} \cdot \sqrt{a+b-c} \cdot \sqrt{a-b+c} \cdot \sqrt{-a+b+c}}{2}$$

distance(A,D)

$$\frac{\sqrt{2} \cdot \sqrt{a^2 + b^2 + c^2} + \sqrt{3} \cdot \sqrt{a+b+c} \cdot \sqrt{a+b-c} \cdot \sqrt{a-b+c} \cdot \sqrt{-a+b+c}}{2}$$

Pedal Triangles

From a point P three perpendicular lines are drawn to the sides of a triangle. The triangle whose vertices are the feet of these perpendiculars is called the pedal triangle of point P with respect to the given triangle

123. Pedal triangle of the orthocenter

Let K be the area of the pedal triangle of the orthocenter of ABC. What is the ratio of K to ABC?

The screenshot shows the WolframAlpha interface with the following content:

Input:

$$\frac{-a^6 + a^4 b^2 + a^2 b^4 - b^6 - c^6 + c^4(a^2 + b^2) + c^2(a^4 - 2a^2 b^2 + b^4)}{4a^2 b^2 c^2}$$

Alternate forms:

$$\frac{-a^6 + a^4 b^2 + a^4 c^2 + a^2 b^4 - 2a^2 b^2 c^2 + a^2 c^4 - b^6 + b^4 c^2 + b^2 c^4 - c^6}{4a^2 b^2 c^2}$$

Expanded form:

$$\frac{a^4}{4b^2 c^2} - \frac{b^4}{4a^2 c^2} - \frac{c^4}{4a^2 b^2} + \frac{a^2}{4b^3} + \frac{b^2}{4a^3} + \frac{a^2}{4c^2} + \frac{c^2}{4a^2} + \frac{b^2}{4c^2} - \frac{1}{2}$$

Roots:

$$c = -\sqrt{a^2 - b^2}, -a b \sqrt{a^2 - b^2} + 0$$

$$c = \sqrt{a^2 - b^2}, a b \sqrt{a^2 - b^2} + 0$$

Wolfram Alpha lets us see this in a number of different forms, including a factored form

124. Pedal triangle of the centroid

Let K be the area of the pedal triangle of the centroid of ABC and R the circumradius of ABC .

Show that $\frac{K}{ABC} = \frac{AB^2+BC^2+AC^2}{36R^2}$

The diagram shows a triangle ABC inscribed in a circle with circumradius R . The vertices are labeled A , B , and C . The side lengths are a (opposite A), b (opposite B), and c (opposite C). The centroid G is the intersection of the medians. The orthocenter H is the intersection of the altitudes. The pedal triangle of G is IJK , where I , J , and K are the feet of the perpendiculars from G to the sides BC , AC , and AB respectively. Right-angle symbols are shown at I , J , and K . A square root symbol \sqrt{x} is visible near the bottom of the diagram.

Below the diagram, the following equation is displayed in a software interface:

$$\frac{\text{area}(IJK)}{\text{area}(ABC)} = \frac{a^2 + b^2 + c^2}{36 \cdot \text{radius}(CO)^2}$$

The value 0 is shown in a small box next to the equation.

125. Pedal triangle of the circumcenter

Let K be the area of the pedal triangle of the circumcenter of ABC . Show that the area of ABC is $4K$.

area(G,I,J) $\frac{\sqrt{a+b+c} \cdot \sqrt{a+b-c} \cdot \sqrt{a-b+c} \cdot \sqrt{-a+b+c}}{16}$

area(A,B,C) $\frac{\sqrt{a+b+c} \cdot \sqrt{a+b-c} \cdot \sqrt{a-b+c} \cdot \sqrt{-a+b+c}}{4}$

126. Pedal triangle of the incenter

Let K be the area of the pedal triangle of the incenter of ABC . Show that $\frac{K}{ABC} = \frac{r}{2R}$

The diagram illustrates a triangle ABC inscribed in a circle with circumradius R . The incenter I is the center of the inscribed circle. The pedal triangle GJI is formed by dropping perpendiculars from I to the sides BC , CA , and AB . The orthocenter H is also shown. The diagram includes side lengths a , b , and c , and various geometric markers like right angles and perpendicular bisectors.

Below the diagram, the area of the pedal triangle GJI and the area of the original triangle ABC are given by the following formulas:

area(GJI) $\frac{\sqrt{a+b+c} \cdot \sqrt{a+b-c} \cdot \sqrt{a-b+c} \cdot \sqrt{-a+b+c}}{16}$

area(ABC) $\frac{\sqrt{a+b+c} \cdot \sqrt{a+b-c} \cdot \sqrt{a-b+c} \cdot \sqrt{-a+b+c}}{4}$

127. Pedal triangle of a point on a concentric circle to the circumcircle

Let C be a circle concentric to the circumcircle with radius k times the radius of the circumcircle. Let H be a point on C . Let IJK be the pedal triangle of H . Show $\frac{\text{area}(IJK)}{\text{area}(ABC)} = \frac{1-k^2}{4}$

The diagram illustrates the geometric setup for the problem. It shows a triangle ABC with vertices $A(0,0)$, $C(a,0)$, and $B(b,c)$. A circumcircle is drawn around the triangle. A smaller concentric circle C with radius k is also shown. A point H is located on circle C . The pedal triangle IJK is formed by dropping perpendiculars from H to the sides of ABC . The diagram also shows the orthocenter H of the triangle ABC and the centroid G . A point D is marked on the vertical line through H , and the distance from the origin to D is labeled k . The angle θ is shown between the vertical line through H and the line segment HI .

The software interface includes a toolbar with icons for undo, redo, move, and other tools. The bottom panel shows the area ratio formula:

$$\frac{\text{area}(IJK)}{\text{area}(ABC)} = \frac{1-k^2}{4}$$

128. A triangle with one angle twice the other

Let angle ABC be twice ACB, and let D be the midpoint of side BA and E the foot of the altitude from A. Show that $AB=2 \cdot DE$

The diagram shows a triangle ABC with vertices A at the top, C at the bottom left, and B at the bottom right. The base CB is labeled with length a . Point D is the midpoint of side AB , and point E is the foot of the altitude from A to the base CB . The angle at C is labeled θ , and the angle at B is labeled $2 \cdot \theta$. A right-angle symbol is shown at E .

Below the diagram, the software interface shows the following distance formulas:

distance(D,E) $\frac{a}{2 \cdot (3 - 4 \cdot \sin(\theta)^2)}$

distance(A,B) $\frac{a}{3 - 4 \cdot \sin(\theta)^2}$

129. Points equal distance, but opposite directions along triangle sides from the base

Let D and E be two points on two sides BA and BC of triangle ABC such that AD=CE. Let F be the intersection of AC and DE. Show that $FD \cdot AB = EF \cdot BC$

distance(F,D) · distance(A,B)

$$\frac{\sqrt{a} \cdot \sqrt{c} \cdot \sqrt{a \cdot b^2 \cdot c + d^2 \cdot (a^2 - b^2 + 2 \cdot a \cdot c + c^2)} + d \cdot (a^3 - a \cdot b^2 + a^2 \cdot c + b^2 \cdot c - a \cdot c^2 - c^3)}{a + c}$$

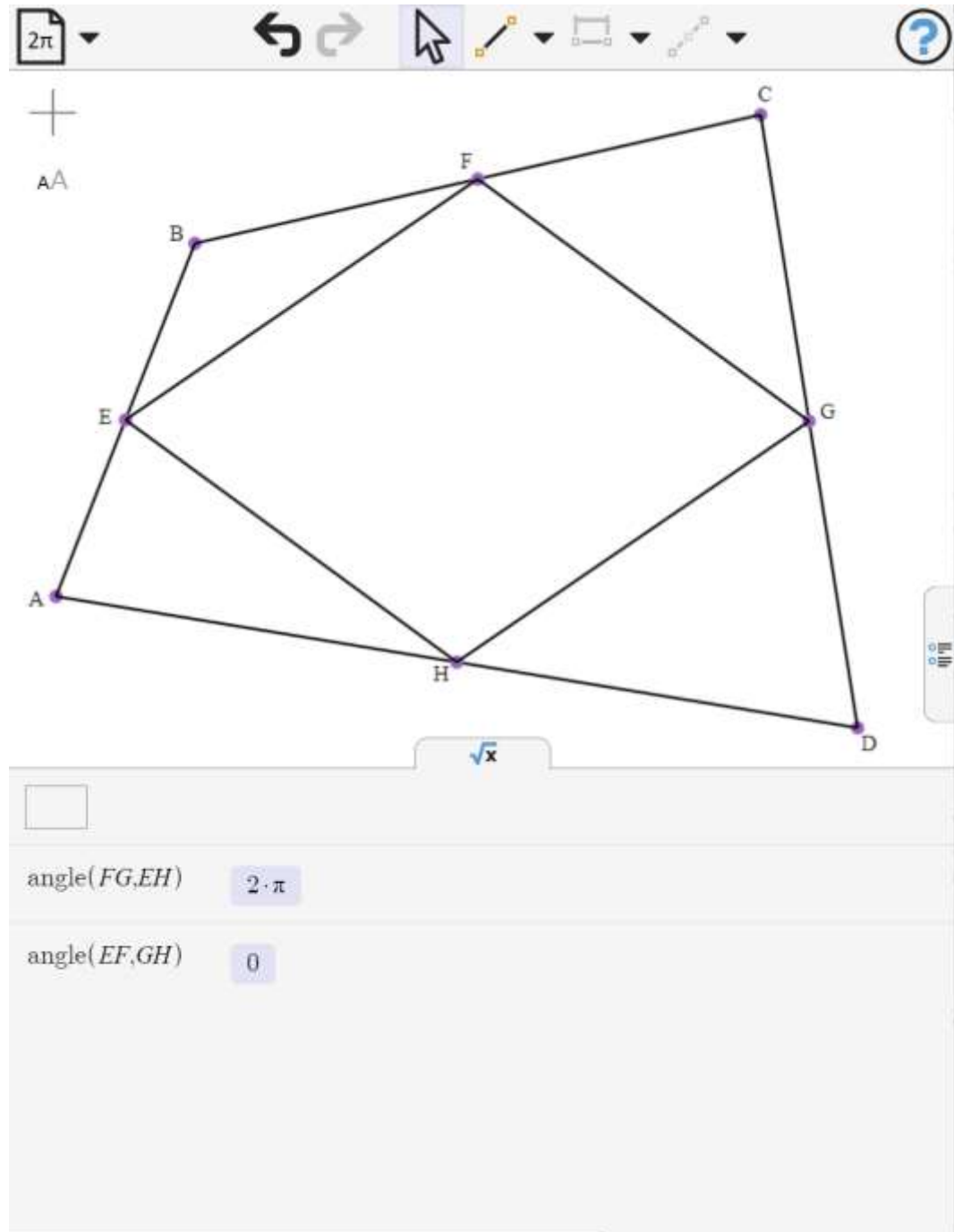
distance(E,F) · distance(B,C)

$$\frac{\sqrt{a} \cdot \sqrt{c} \cdot \sqrt{a \cdot b^2 \cdot c + d^2 \cdot (a^2 - b^2 + 2 \cdot a \cdot c + c^2)} + d \cdot (a^3 - a \cdot b^2 + a^2 \cdot c + b^2 \cdot c - a \cdot c^2 - c^3)}{a + c}$$

Quadrilaterals

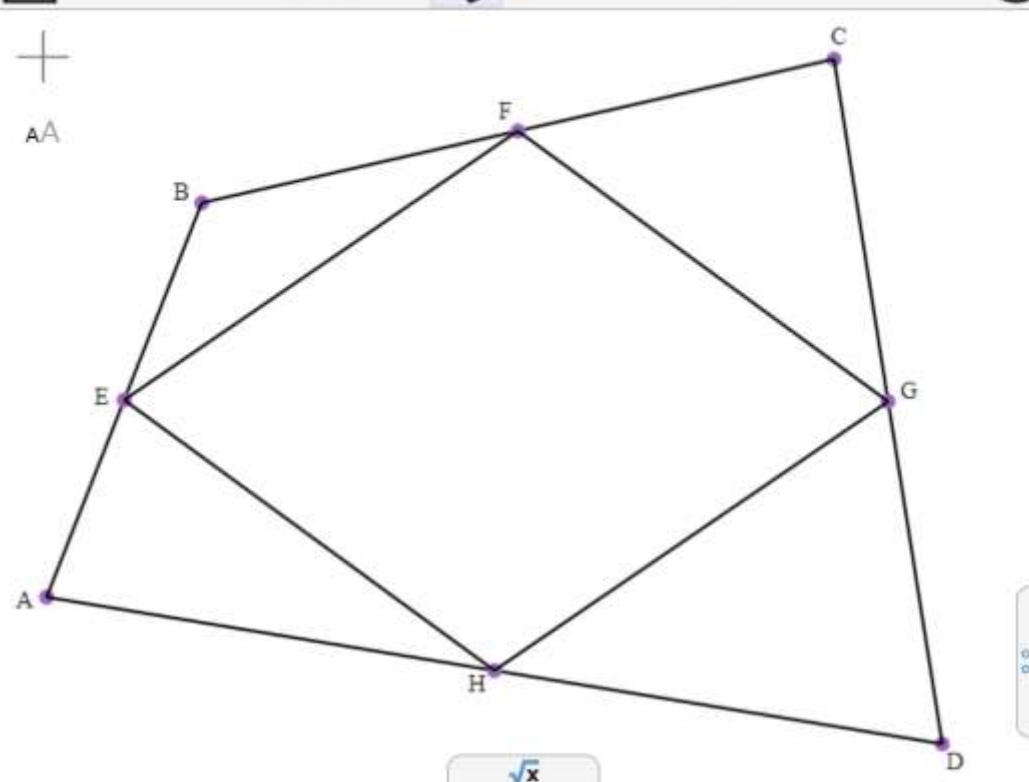
130. Midpoints of a quadrilateral

The figure formed when the midpoints of the sides of a quadrilateral are joined in order is a parallelogram



131. Area of the midpoint parallelogram

The area of the parallelogram whose vertices are the midpoints of the sides of a quadrilateral is equal to half the area of the quadrilateral.



The diagram illustrates a quadrilateral $ABCD$ with vertices A , B , C , and D . The midpoints of the sides are labeled E (on AB), F (on BC), G (on CD), and H (on DA). The quadrilateral formed by these midpoints, $EFGH$, is a parallelogram. The diagram is shown in a software interface with a toolbar at the top and a calculation area at the bottom.

area(E,F,G,H)
area(A,B,C,D) $\frac{1}{2}$

132. Sum of squares of the diagonals related to side lengths

The sum of squares of the sides of a quadrilateral is equal to the sum of squares of the diagonals increased by four times the square of the segment joining the midpoints of the diagonals.

The diagram shows a quadrilateral with vertices A, B, C, and D. Diagonals AC and BD intersect at point E. A segment EF is drawn, where F is the midpoint of diagonal BD. The side lengths are labeled as follows: AB = d, BC = a, CD = b, and DA = c. The segment EF is labeled as e.

Below the diagram, a software interface shows the following equation:

$$\text{distance}(A,C)^2 + \text{distance}(B,D)^2 + 4 \cdot \text{distance}(E,F)^2 = a^2 + b^2 + c^2 + d^2$$

133. Diagonals related to lines joining midpoints

The sum of squares of the diagonals of a quadrilateral is equal to twice the sum of squares of the lines joining the midpoints of the opposite sides of the quadrilateral.

$$\frac{\text{distance}(E,F)^2 + \text{distance}(G,H)^2}{\text{distance}(B,D)^2 + \text{distance}(A,C)^2} = \frac{1}{2}$$

134. Area of an intersection parallelogram

Let points E,F,G,H be $\frac{1}{3}$ of the way along segments DA, AB, BC, CD of parallelogram ABCD. Let I be the intersection of AH and BE, let J be the intersection of CF and BE, let K be the intersection of DG and CF, and let L be the intersection of AH and DG. Then the ratio of areas of ABCD to IJKL is 1:13.

The diagram shows a parallelogram ABCD with vertices A (bottom-left), B (bottom-right), C (top-right), and D (top-left). The bottom side AB has length a and the left side AD has length b . The angle at vertex A is θ . Points E, F, G, and H are located on sides DA, AB, BC, and CD respectively, such that $AE = \frac{1}{3}AD$, $BF = \frac{1}{3}AB$, $CG = \frac{1}{3}BC$, and $DH = \frac{1}{3}CD$. Lines AH, BE, CF, and DG are drawn. Their intersections form an inner parallelogram IJKL, where I is the intersection of AH and BE, J is the intersection of BE and CF, K is the intersection of CF and DG, and L is the intersection of DG and AH.

Below the diagram, a software interface shows a formula input area with the following expression:

$$\frac{\text{area}(IJKL)}{\text{area}(ABCD)} = \frac{1}{13}$$

135. Von Abuel's Theorem

Squares are drawn externally on the sides of a quadrilateral, Show that the segments joining the centers of the opposite squares are equal and perpendicular.

The diagram shows a quadrilateral $ABCD$ with vertices A , B , C , and D . Squares are constructed externally on each side. The centers of these squares are labeled M (on AB), N (on BC), P (on CD), and O (on DA). Lines connect M to O and P to N . The software interface includes a toolbar at the top with various geometric tools and a command window at the bottom.

The command window displays the following results:

$\frac{\text{distance}(M,O)}{\text{distance}(N,P)}$	1
$\text{angle}(MO, NP)$	$\frac{\pi}{2}$

136. The square on the hypotenuse

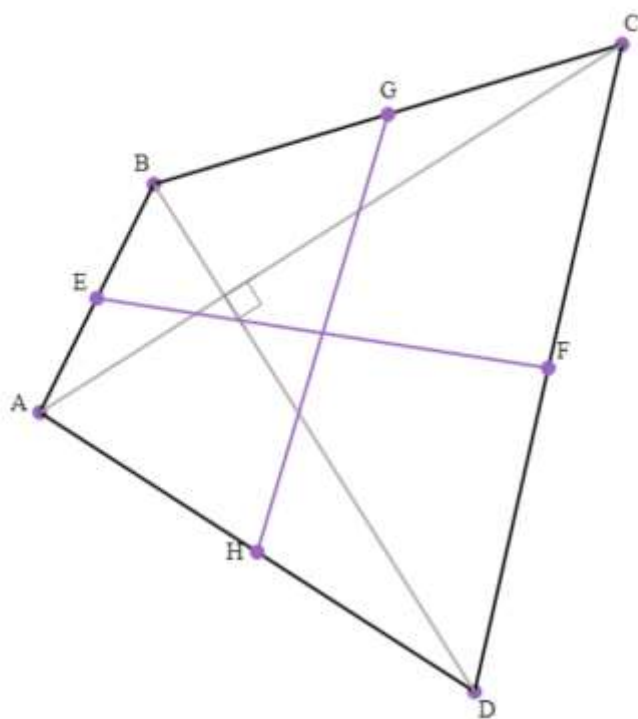
On the hypotenuse AB of a right angled triangle ABC, a square is constructed. Let F be the intersection of its diagonals. Angle ACF = angle FCB.

The diagram illustrates a right-angled triangle ABC with the right angle at C . The hypotenuse is AB . A square $ABDE$ is constructed on the hypotenuse AB . The diagonals AD and BE of the square intersect at point F . The angle $\angle ACF$ is highlighted in purple. The side AC is labeled b and the side BC is labeled a . Tick marks on the square's sides indicate they are equal. The software interface includes a toolbar with icons for undo, redo, move, and other geometric tools. The command input area at the bottom shows the command `angle(A,C,F)` with the value $\frac{\pi}{4}$.

angle(A,C,F) $\frac{\pi}{4}$

137. An orthodiagonal quadrilateral

In an orthodiagonal quadrilateral (quadrilateral with perpendicular diagonals) the lines joining the centers of opposite sides are equal.

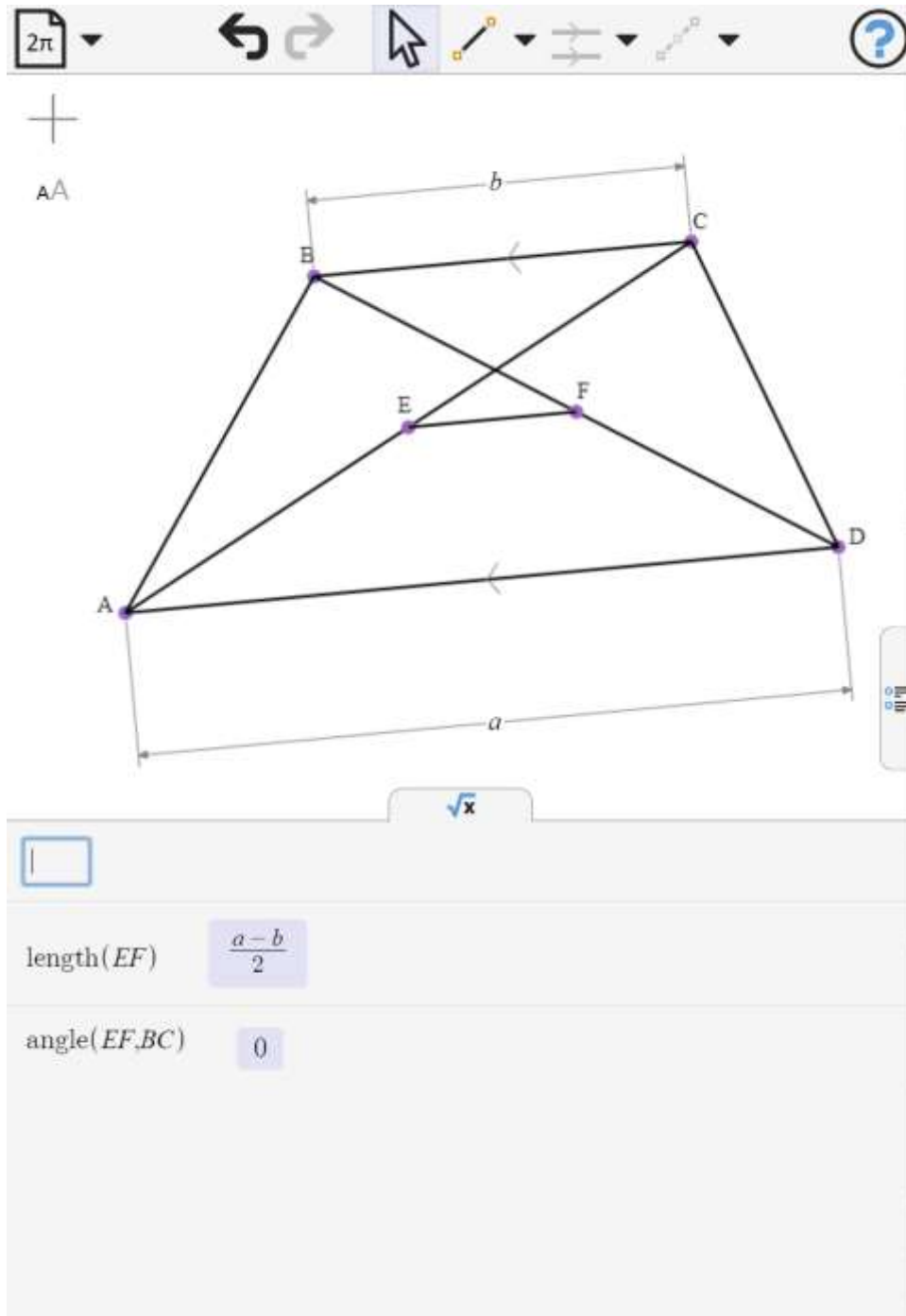


The diagram shows a quadrilateral with vertices A, B, C, and D. The diagonals AC and BD are perpendicular, intersecting at a right angle. Points E, F, G, and H are the midpoints of sides AB, BC, CD, and DA respectively. Lines connecting these midpoints (EF, FG, GH, HE) form a purple rhombus. A right angle symbol is shown at the intersection of the diagonals.

distance(E,F)
distance(G,H) 1

138. Midpoints of trapezoid diagonals

Let E, F be the midpoints of the diagonals of a trapezoid. Then EF is parallel to the two parallel sides of the trapezoid and its length is half the difference between the lengths of the parallel sides.



139. Three tangents and a diameter

Through point F on the circle with diameter BC , a tangent to the circle is drawn meeting the tangents at B and C in points D and E . Show that $DF \cdot EF = AB^2$

distance(D,F) · distance(E,F) r^2

140. Angle bisector and circumcircle

The angle bisector at C bisects the arc AB of the circumcircle of triangle ABC.

The diagram illustrates a circle with an inscribed triangle ABC . The angle bisector of $\angle C$ intersects the circumcircle at point D . A point E is located on the minor arc AB . Lines AD , DE , and BE are drawn. The angle bisector of $\angle C$ is shown bisecting arc AB at D . The angle bisector of $\angle C$ is shown bisecting arc AB at D . The angle bisector of $\angle C$ is shown bisecting arc AB at D .

angle(A,D,E) θ

angle(B,D,E) θ

141. A circle through the midpoint of the hypotenuse

Let D be the midpoint of the hypotenuse of the right angled triangle ABC . A circle passing through A and D meets AB in F . G is the point on the circle such that FG is parallel to BC . Show that $BC = 2FG$.

distance(B,C) $\sqrt{b^2 + c^2}$

distance(G,F) $\frac{\sqrt{b^2 + c^2}}{2}$