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Foreword

I can remember having arguments like this as a young kid and either I was a bizarre child (which is highly likely), or this is a typical situation played out in many childhood arguments. The sense of triumph as one combatant first proposes the insurmountable obstacle of the infinite, introduces one of the most important ideas of mathematics, and perhaps unknowingly the larger questions of philosophy, science, literature, and even religion. The apparent victory above is first achieved by skipping the inevitability of this argument going on for a *very* long time. But this quickly gives way to the absurdity that there is indeed a number “larger” than infinity and the can of worms opened from that competitive impulse brings up issues that have been debated for thousands of years.

Calculus is the study of the infinite and since much of secondary mathematics is designed to prepare one for the study of calculus, wrestling with the ideas of the infinite, even if informally, is extremely important for a student’s mathematical development. That is the purpose of this book.

Among the first to discuss these ideas was the Greek philosopher / mathematician Zeno, and not long after him, Archimedes came the closest to “discovering” Calculus without the tools of modern mathematics. Though none of

Zeno's actual writings survive, Aristotle recorded accounts of Zeno's thoughts on the infinite, time and space in what have come to be known as Zeno's Paradoxes. One of those specifically involves the idea of a race in which a slower runner is given a head start and investigates the possibilities of the faster runner "catching up". This has come to be known as "Achilles and the Tortoise". These characters are our hosts as we use modern software to investigate Archimedes methods, ideas of the infinite, and Zeno's Paradoxes in an introduction to Calculus, without using Calculus.

Lesson Three

Objectives

The student will be able to use the coordinate graphing capabilities of Geometry Expressions to create a two dimensional representation of Zeno's paradox.

The student will be able to recognize and use connections among mathematical ideas.

Lesson Notes

In this lesson the student creates a graph to illustrate Zeno's paradox.

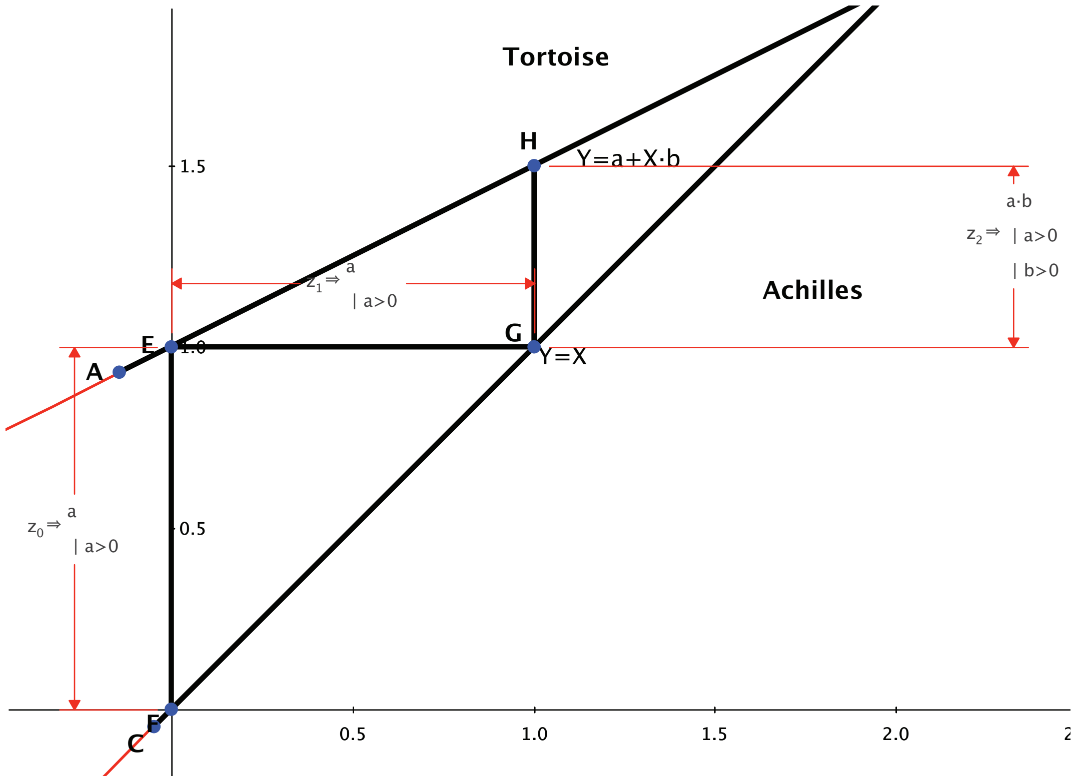
4. The lines appear to intersect, which should indicate that Achilles does indeed catch the tortoise.

7. **a units of distance**

8. **a units of time**. Achilles equation is y = x, so his distance and time are equivalent.

10. **ab units**. The tortoise's position is given by the equation **a + bx**. The time interval that has transpired is a units, so replacing a for x gives: **a + ba**. The First **a** is the initial position of the tortoise, so the new distance traveled is just **ba**.

11.



12. **Distance covered by Achilles: ab units**

Time for Achilles to cover that distance: ab units

Distance covered by Tortoise in that time: ab^2 units

This last answer is a good algebraic exercise. This time the x-coordinate is a + ab from the information above. So, by substituting and simplifying:

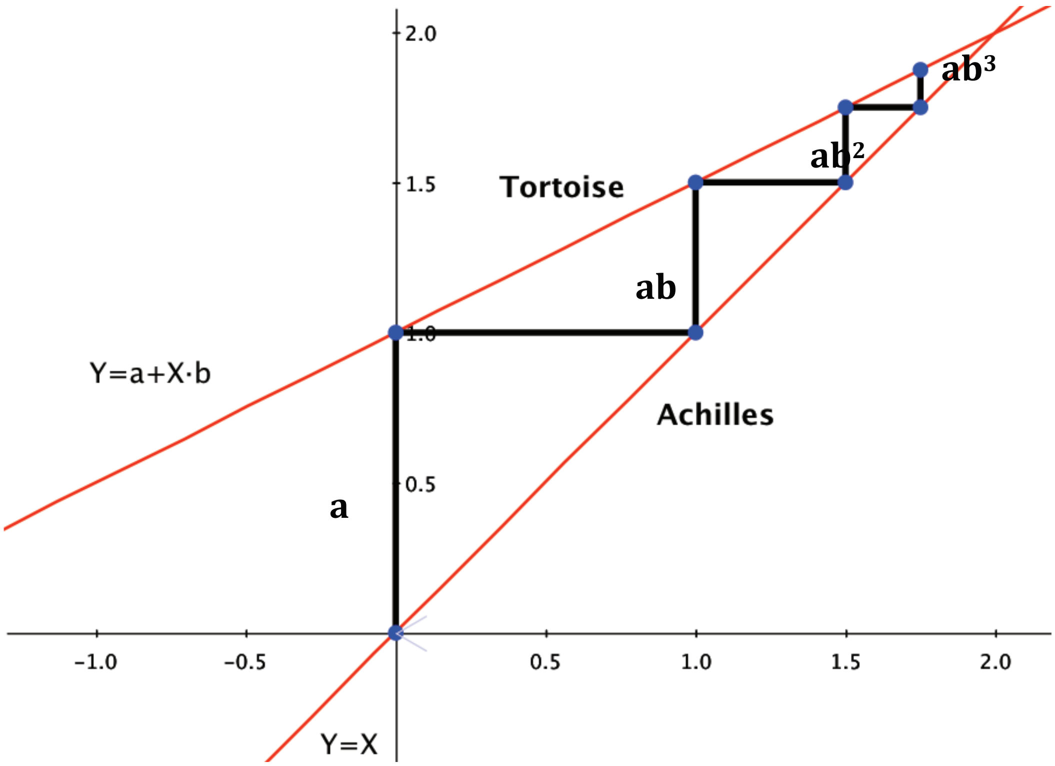
$$a + bx$$

$$a + b(a + ab)$$

$$a + ab + ab^2$$

The first two terms are the previous distance, so the final term is the new distance covered.

13.



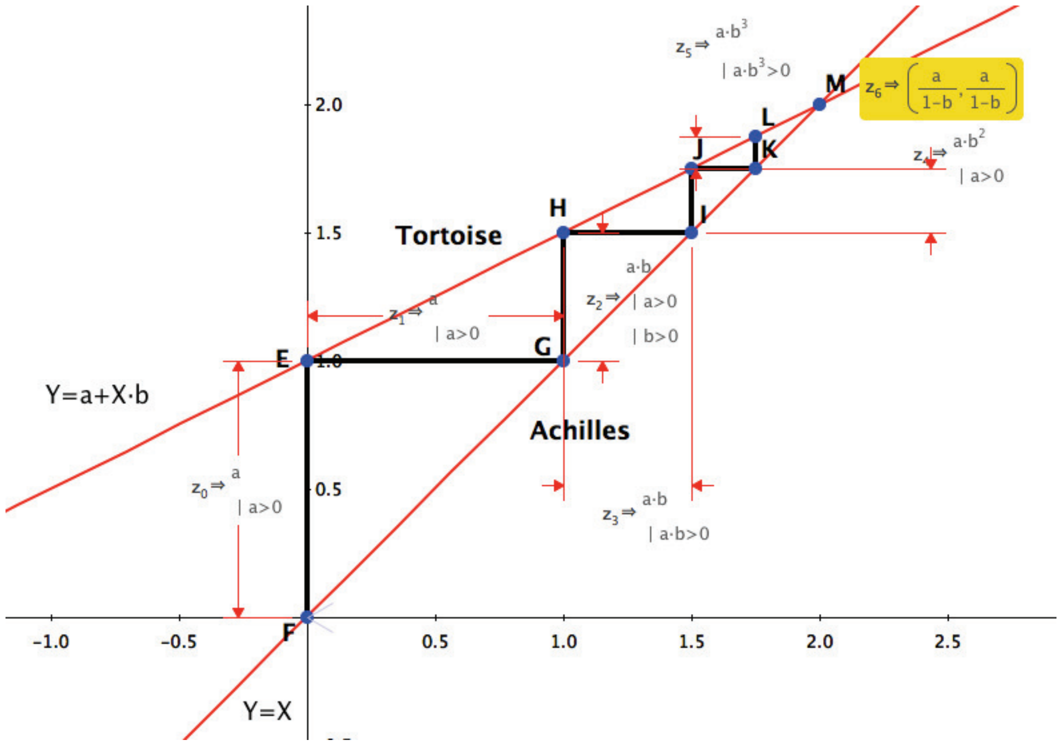
14. **Total Distance (4 time periods):** $a + ab + ab^2 + ab^3$

15. **Next term:** ab^4

16. **Starting term:** a **Common Ratio:** b

17. **Sum:** $\dfrac{a}{1-b}$

18.



Quick Questions

2. The intersection point moves further and further away (in both time and distance)

3. You are increasing the speed of the Tortoise relative to Achilles speed. So, it takes longer for Achilles to close the gap between them.

4. The lines are parallel.

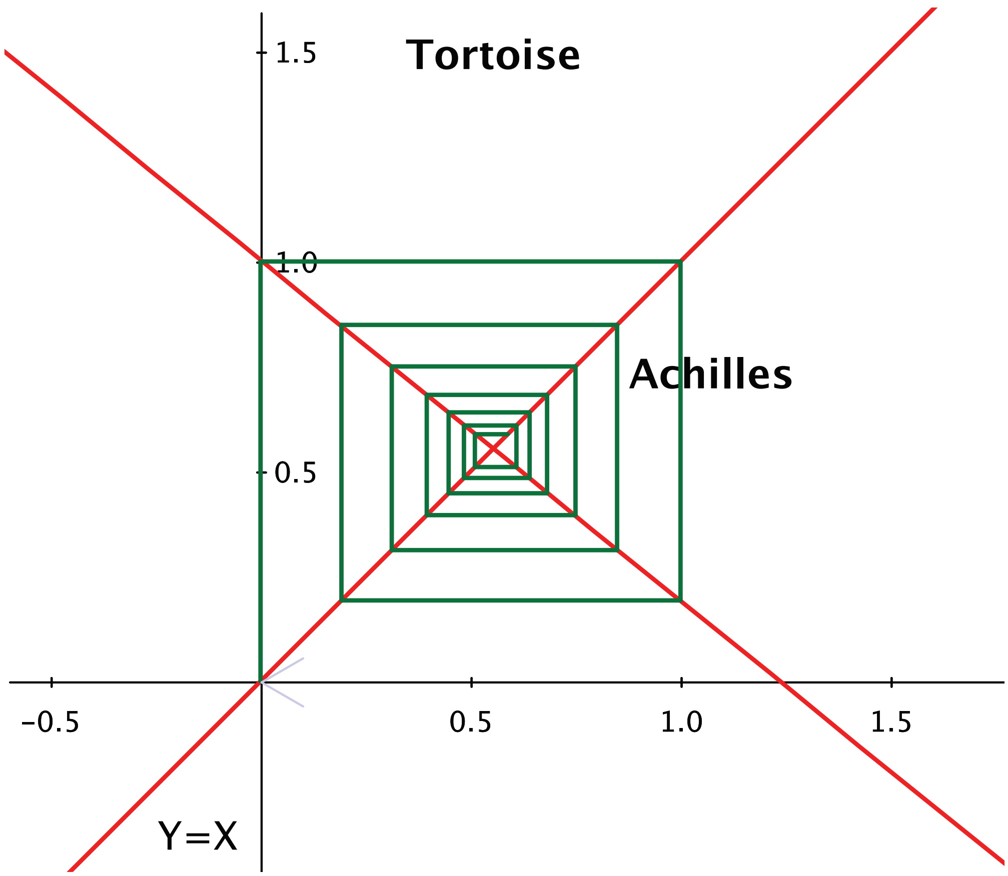
5. The Tortoise is moving at the same speed as Achilles and there is no paradox, Achilles will never catch up.

6. The Tortoise is actually moving faster than Achilles and the distance between them will grow without bound. In terms of a geometric series, the common ratio is greater than one, so the series diverges and has an infinitely large sum.

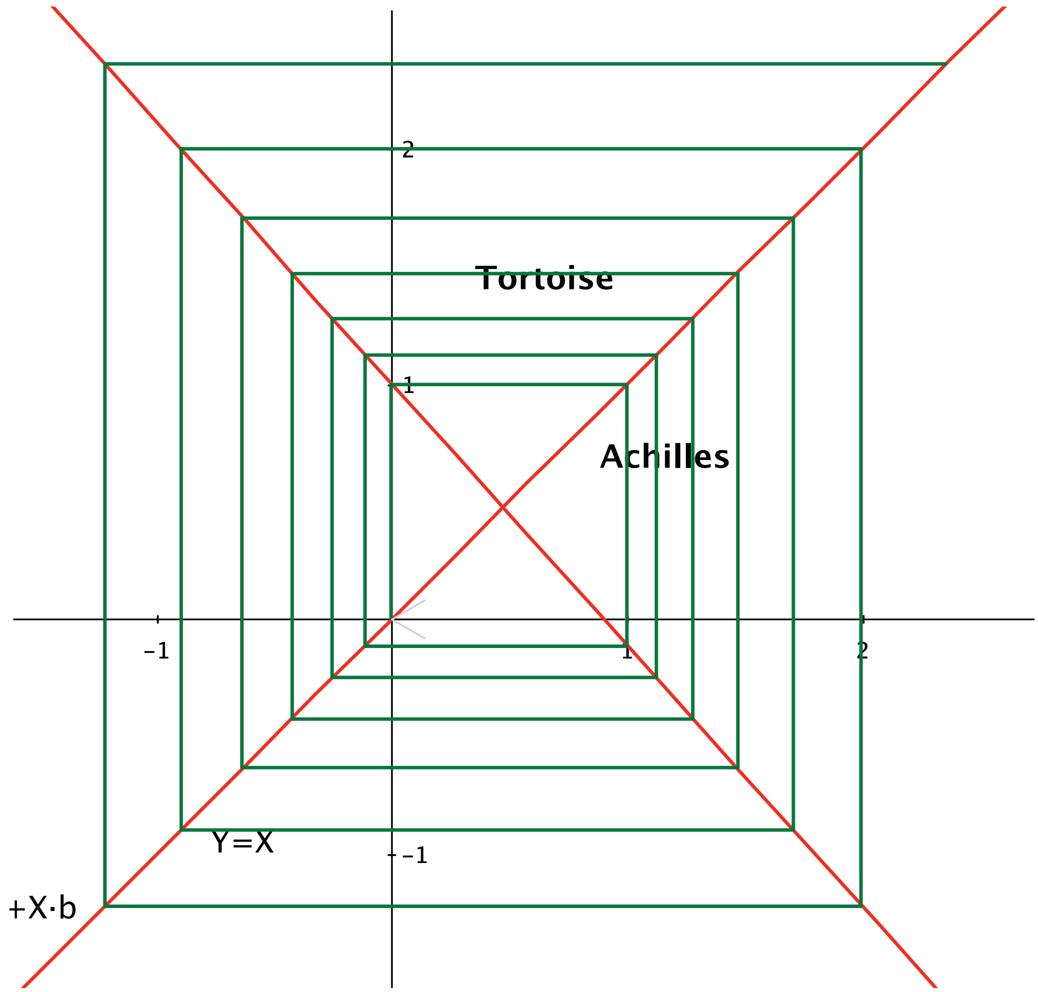
7. In terms of a geometric series, the common ratio is greater than one, so the series diverges and has an infinitely large sum. Because the Tortoise is travelling faster, he will increase the distance between them.

Interesting Investigation

Result when b is approximately -0.8



Result when b is approximately -1.1



Lesson Nine

Objectives

The student will use Geometry Expressions to generalize the area under a curve for other power functions.

The student will be able to recognize and use connections between mathematical ideas.

The student will be able to use reasoning and methods of proof.

The student will be able to make and investigate mathematical conjectures.

Lesson Notes

In this lesson, we take the work from the previous lesson on square root functions and generalize it to other power functions.

Quick Question

$$\tfrac{1}{2}bh = \tfrac{1}{2}t \cdot t = \frac{t^2}{2}$$

Informative Instructions

3. $\dfrac{t^3}{2}$ 4. $\dfrac{t^4}{2}$

5. It appears as if the area of the triangle is $\dfrac{t^{b+1}}{2}$ for any power function $y=ax^b$

For example, for $y=ax^6$, the area of the triangle should be $\dfrac{t^7}{2}$.

7.

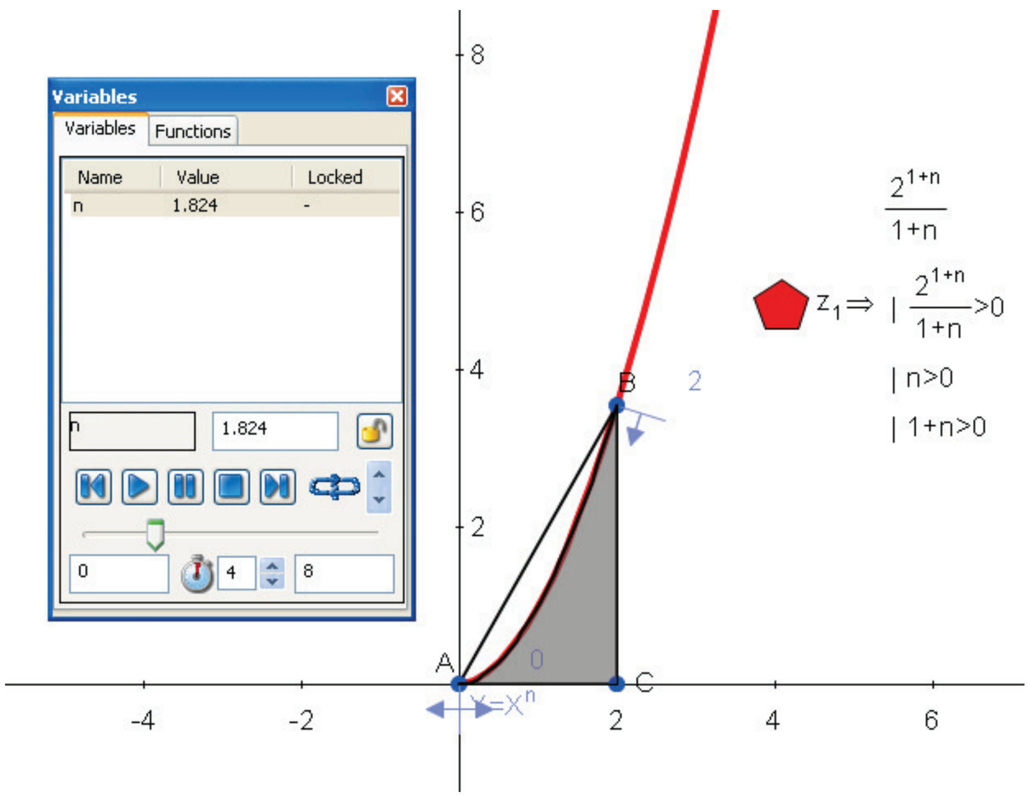
Function	Area of triangle	Area under curve
$y=x$	$\dfrac{t^2}{2}$	$\dfrac{t^2}{2}$
$y=x^2$	$\dfrac{t^3}{2}$	$\dfrac{t^3}{3}$
$y=x^3$	$\dfrac{t^4}{2}$	$\dfrac{t^4}{4}$
$y=x^4$	$\dfrac{t^5}{2}$	$\dfrac{t^5}{5}$
$y=x^5$	$\dfrac{t^6}{2}$	$\dfrac{t^6}{6}$
$y=x^6$	$\dfrac{t^7}{2}$	$\dfrac{t^7}{7}$
$y=x^n$	$\dfrac{t^{n+1}}{2}$	$\dfrac{t^{n+1}}{n+1}$

Quick Questions

1. a. $f(x) = x^{\frac{1}{2}}$ b. $\dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} = \dfrac{2x^{\frac{3}{2}}}{3}$

2. The area will increase as the power increases.

3.



4. Because 1 stays fixed on the function, the area will actually decrease as the curve gets steeper.

5. The area decreases more dramatically in this example as the curve flattens out more quickly for a larger portion of the interval from 0 to 1.