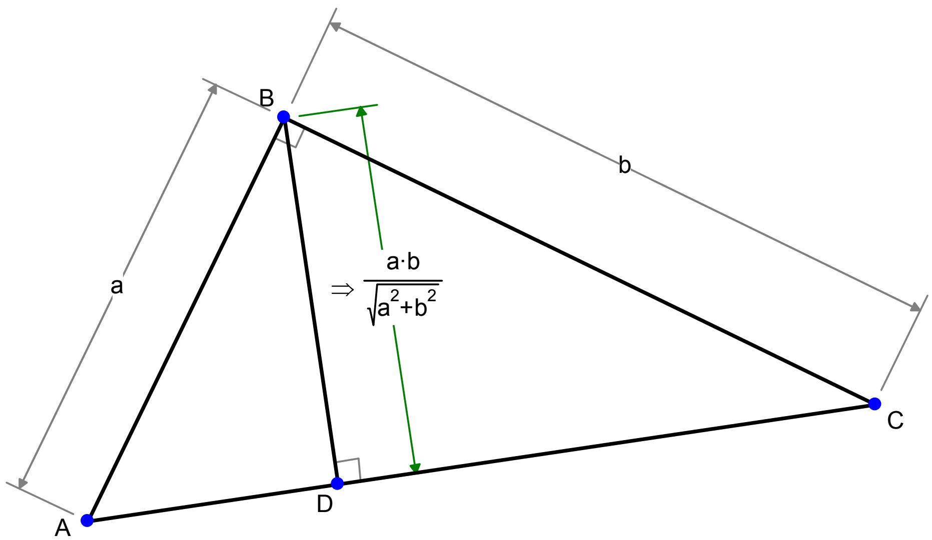
Circles and Tangents with Geometry Expressions

Introduction

Geometry Expressions automatically generates algebraic expressions from geometric figures. For example in the diagram below, the user has specified that the triangle is right and has short sides length a and b. The system has calculated an expression for the length of the altitude:



We present a collection of worked examples using Geometry Expressions. In most cases, a diagram is presented with little comment. It is hoped that these diagrams are sufficiently self explanatory that the reader will be able to create them himself.

The goal of these examples is to demonstrate the sort of problems which the software is capable of handling, and to suggest avenues of further exploration for the reader.

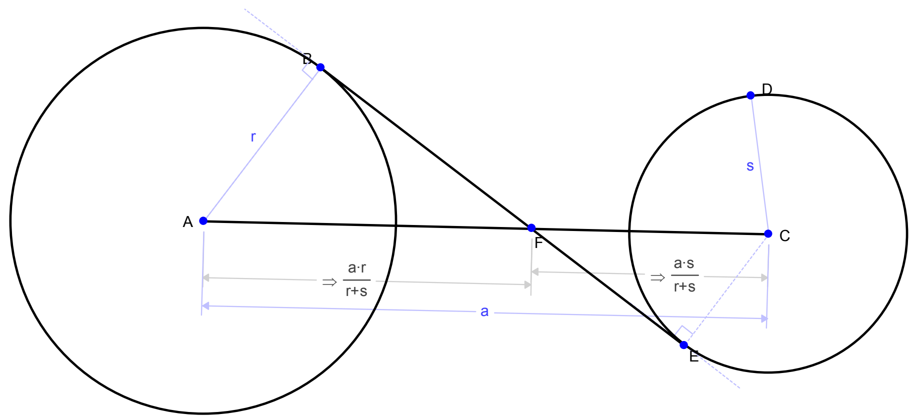
The examples are clustered by theme.

Circle common tangents

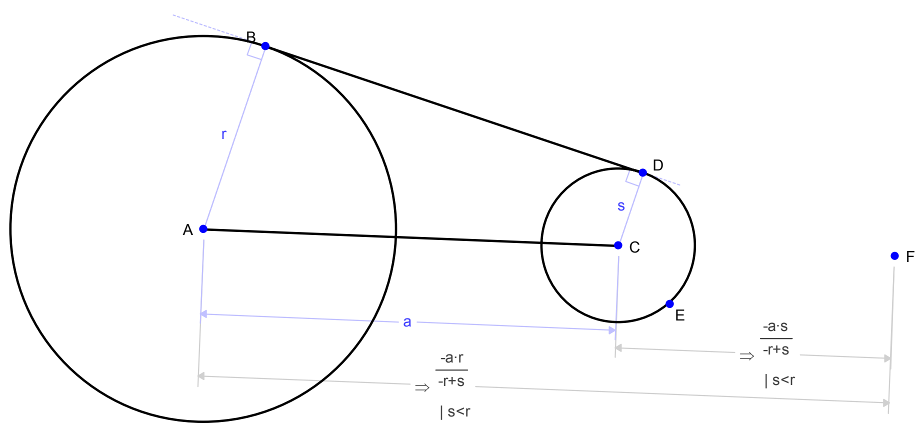
The following set of examples explores some properties of the common tangents of pairs of circles.

Example 1: Location of intersection of common tangents

Circles AB and CD have radii r and s respectively. If the centers of the circles are a apart, and E is the intersection of the interior common tangent with the line joining the two centers, what are the lengths AE and CE?

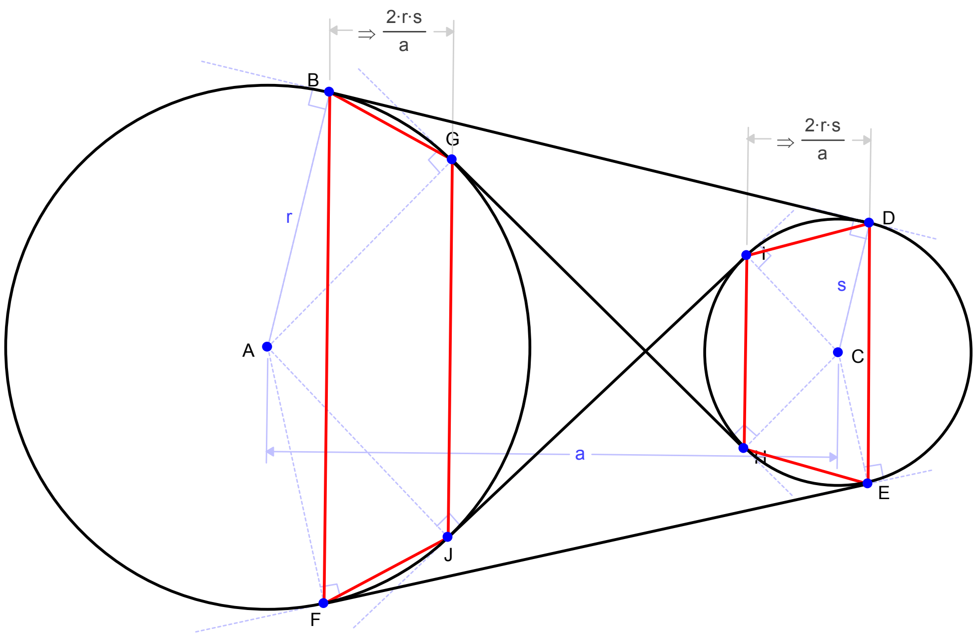


How about the exterior common tangent?



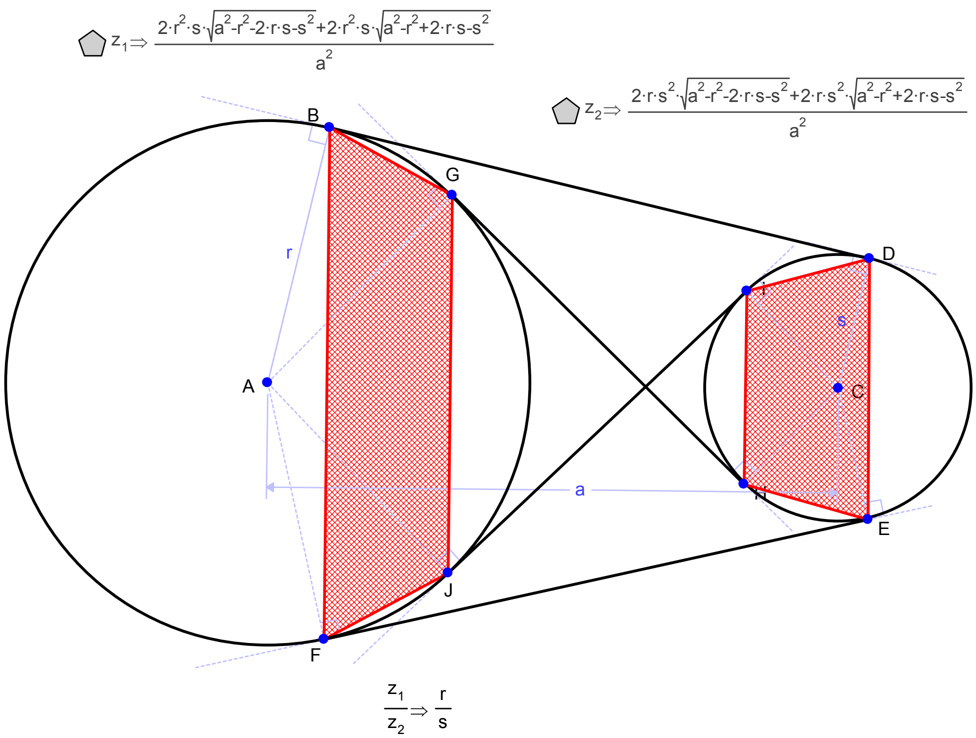
Example 2: Cyclic Trapezium defined by common tangents of 2 circles

Given circles radii r and s and distance a apart, what is the altitude of the trapezium formed by joining the intersections of the 4 common tangents with one of the circles?



Notice that this is symmetrical in r and s, and hence the trapezium in circle AB has the same altitude.

Look at the ratio of the areas of the trapezia in the previous diagram:

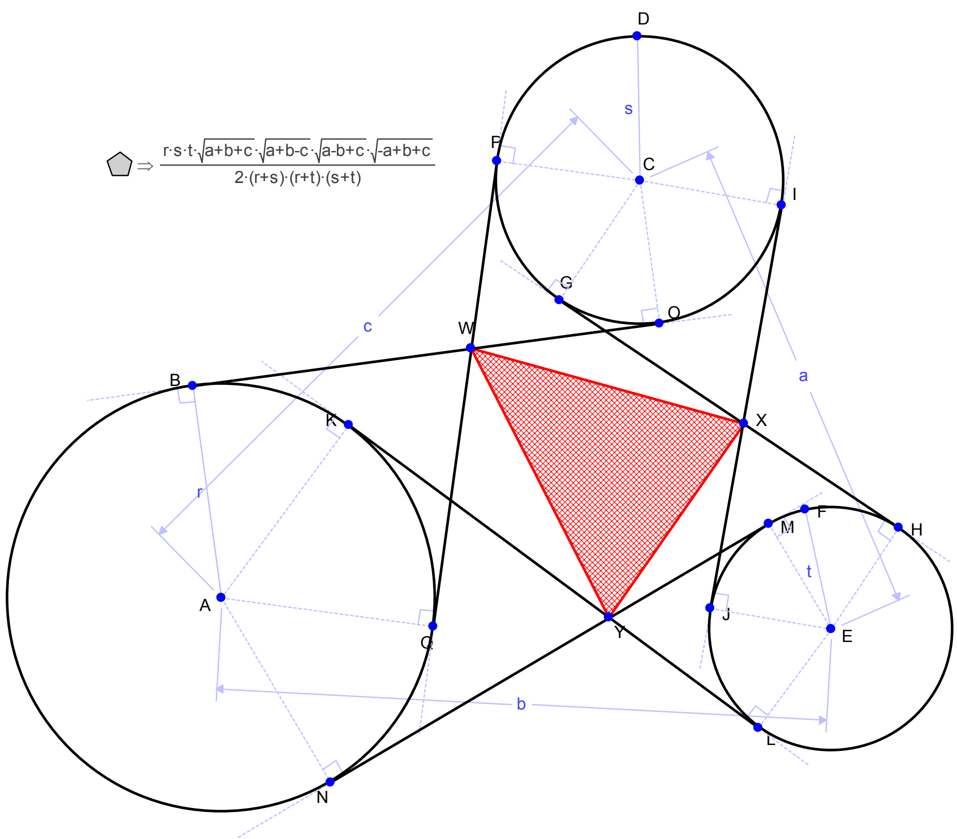


Notice that the ratio of trapezium areas is the same as the ratio of radii.

Example 3: Triangle formed by the intersection of the interior common tangents of two circles

Notice that if A is the area of the triangle formed by the centers of the circles, then area STU is:

$$\frac{2rstA}{(r+s)(s+t)(r+t)}$$

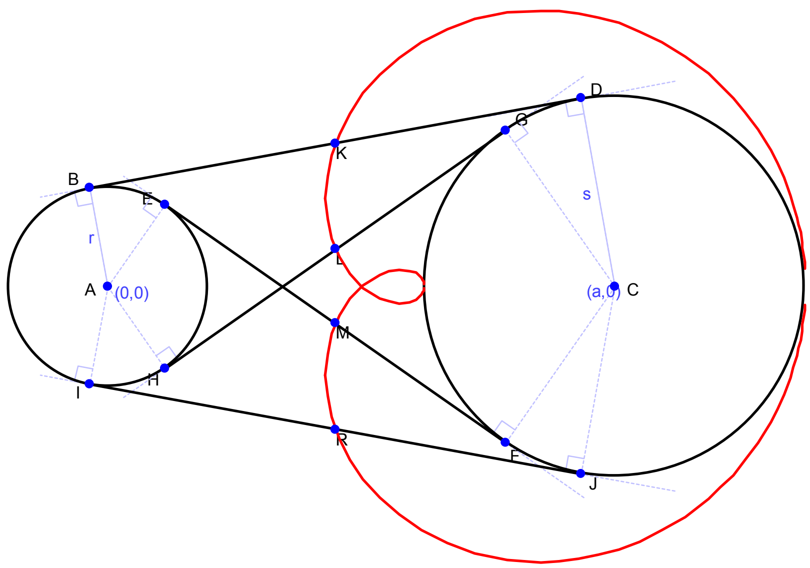


Notice that this ratio is independent of the size of a,b, and c.

Example 4: Locus of centers of common tangents to two circles

W take the locus as the radius r of the left circle varies. The midpoints of all four common tangents lie on the same fourth order curve

$$\Rightarrow 4\cdot X^4+8\cdot X^2\cdot Y^2+4\cdot Y^4-12\cdot X^3\cdot a-12\cdot X\cdot Y^2\cdot a+a^4-a^2\cdot s^2+Y^2\cdot\left(4\cdot a^2-4\cdot s^2\right)+X^2\cdot\left(13\cdot a^2-4\cdot s^2\right)+X\cdot\left(-6\cdot a^3+4\cdot a\cdot s^2\right)=0$$



We can use Maple to solve for the intersections with the x axis:

```
> subs(Y=0,4*X^4+8*Y^2*X^2+4*Y^4-12*a*X^3-
12*a*Y^2*X+a^4-s^2*a^2+(4*a^2-4*s^2)*Y^2+(13*a^2-
4*s^2)*X^2+(-6*a^3+4*s^2*a)*X );
```

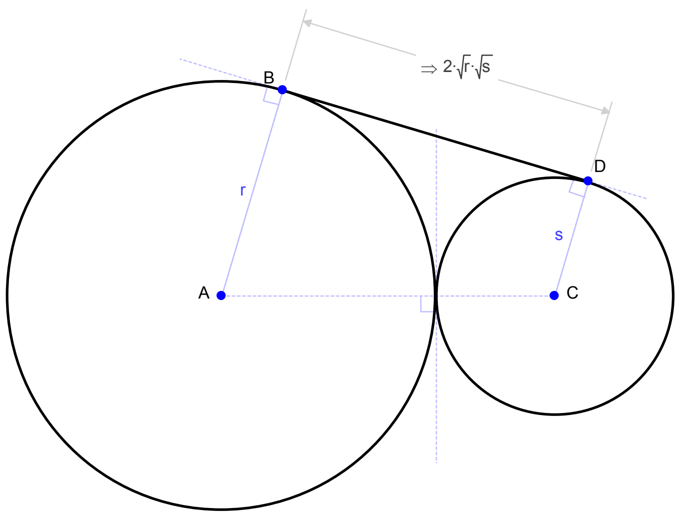
$$4\,X^4 - 12\,a\,X^3 + a^4 - s^2\,a^2 + (13\,a^2 - 4\,s^2)\,X^2 + (-6\,a^3 + 4\,s^2\,a)\,X$$

```
> solve(%,X);
```

$$a - s,\ a + s,\ \frac{1}{2}\,a,\ \frac{1}{2}\,a$$

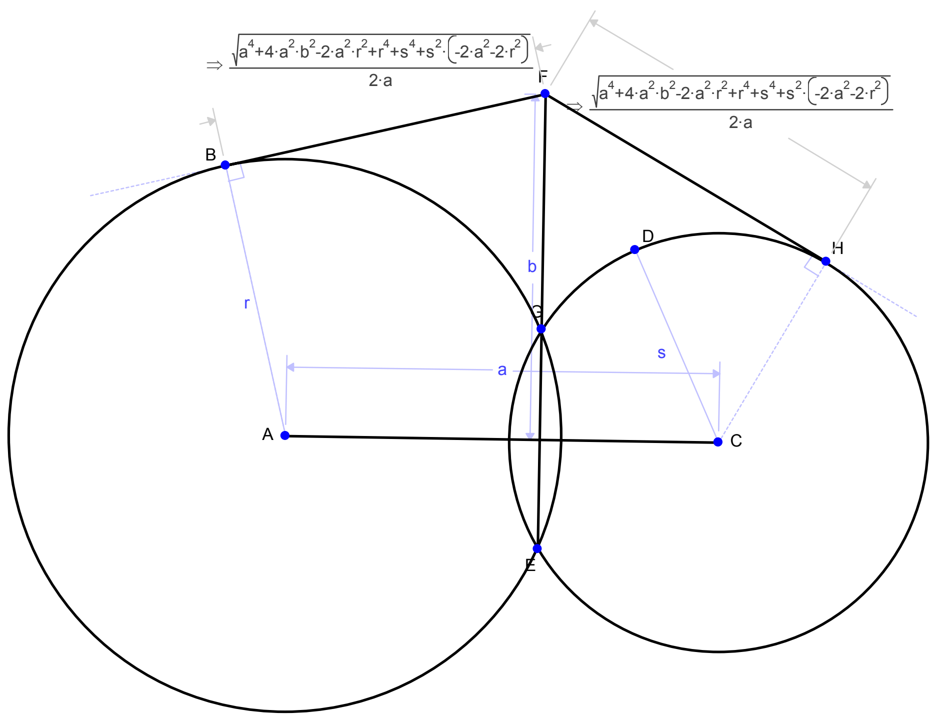
Example 5: Length of the common tangent to two tangential circles

A succinct formula:



Example 6: Tangents to the Radical Axis of a Pair of Circles

The radical axis of a pair of circles is the line joining the points of intersection. The lengths of tangents from a given point on this axis to the two circles are the same.

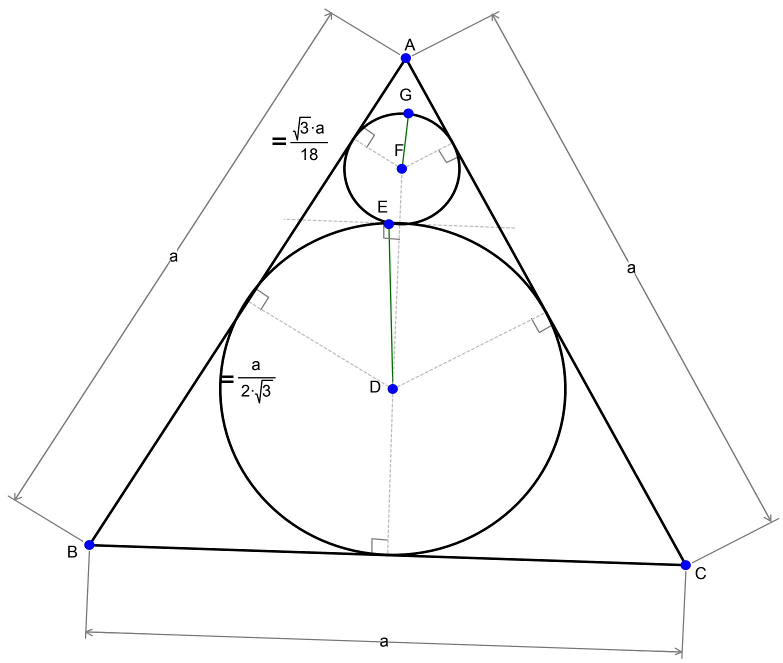


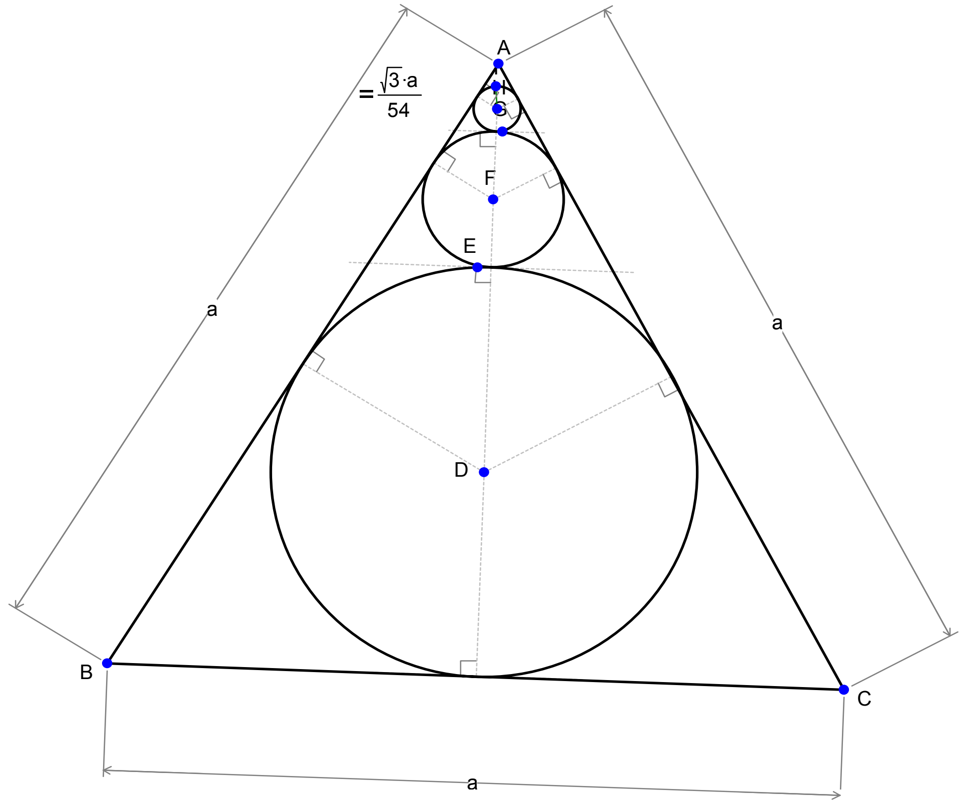
Arbelos

A set of examples studying circles squeezed between other circles.

Example 7: Various Circles in an Equilateral Triangle

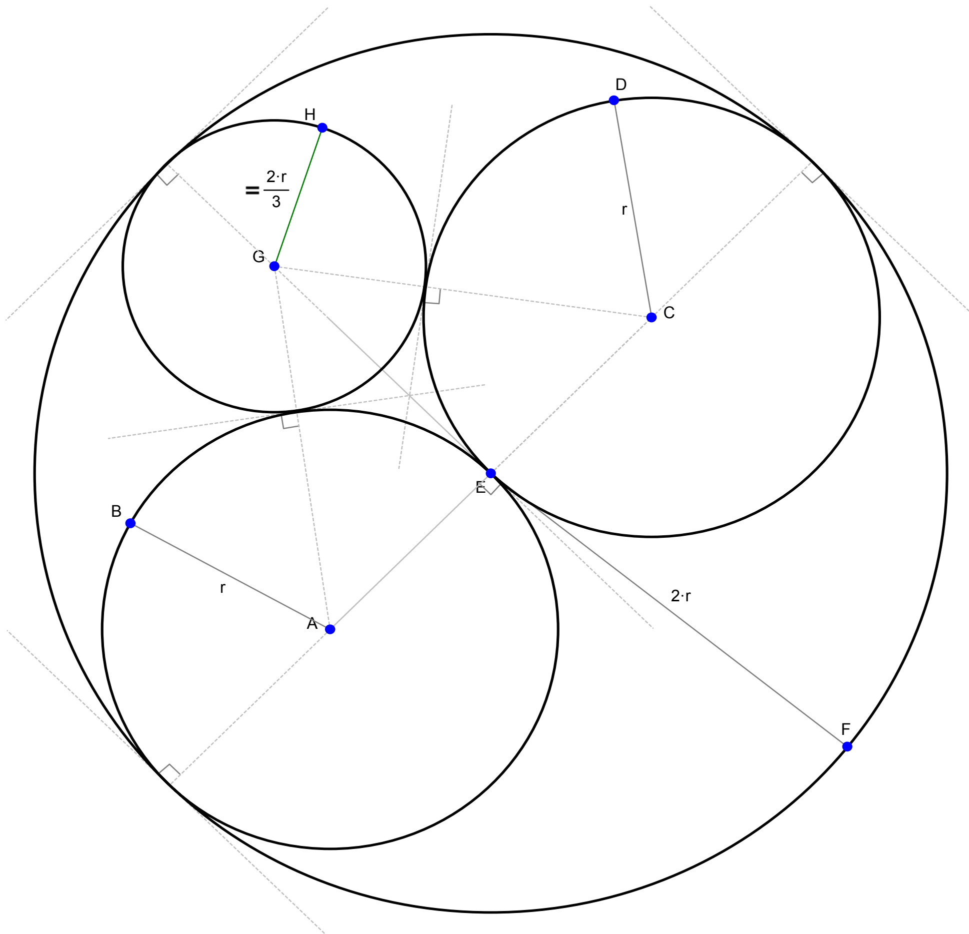
We look at the radii of various circles in an equilateral triangle:



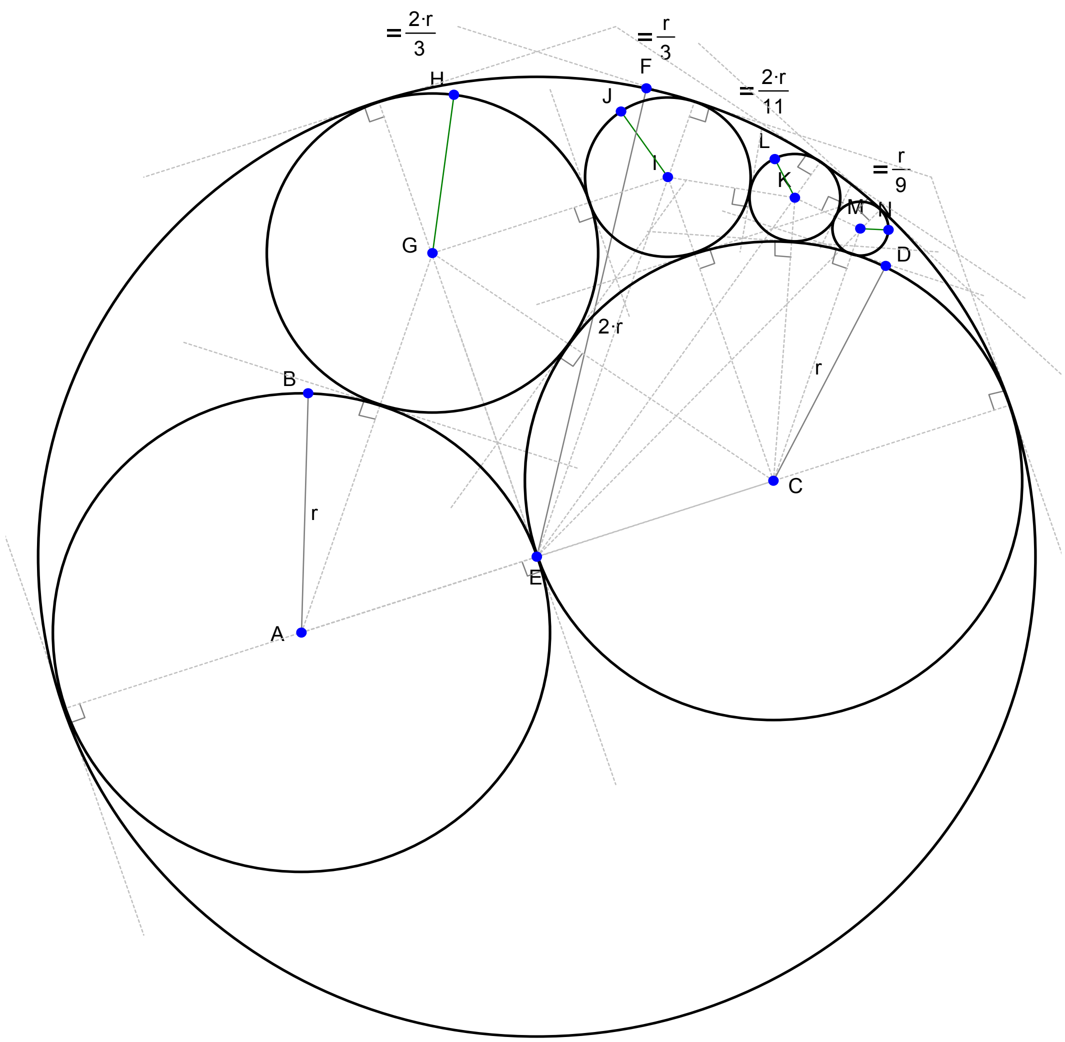


What would the next length in the sequence be?

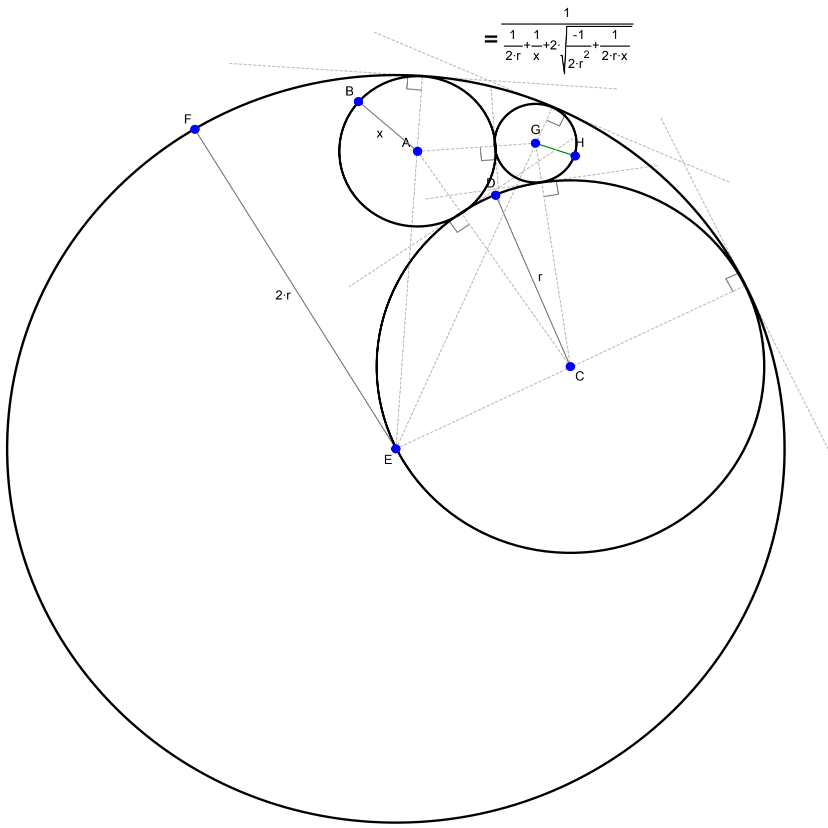
Example 8: Two circles inside a circle twice the radius, then a third



And if we keep on going:



The general case looks like this:



We can copy this expression into Maple to generate the above sequence:

```
> 1/(1/2*1/r+1/x+2*sqrt(-1/2*1/(r^2)+1/2/x/r));
```

$$\frac{1}{\frac{1}{2}\frac{1}{r}+\frac{1}{x}+\sqrt{-2\,\frac{1}{r^2}+\frac{2}{x\,r}}}$$

```
> subs(r=1,%);
```

$$\frac{1}{\frac{1}{2}+\frac{1}{x}+\sqrt{-2+\frac{2}{x}}}$$

```
> f:=x->1/(1/2+1/x+sqrt(-2+2/x));
```

$$f := x \to \frac{1}{\frac{1}{2}+\frac{1}{x}+\sqrt{-2+\frac{2}{x}}}$$

```
> f(1);
```

$$\frac{2}{3}$$

```
> f(2/3);
```

$$\frac{1}{3}$$

```
> f(1/3);
```

$$\frac{2}{11}$$

```
> f(2/11);
```

$$\frac{1}{9}$$

```
> f(1/9);
```

$$\frac{2}{27}$$

```
> f(2/27);
```

$$\frac{1}{19}$$

```
>
```

A little analysis of the series can lead us to postulate the formula 2/(n^2+2) for the n'th circle:

Let's feed the n-1th term into Maple:

```
> f(2/((n-1)^2+2));
```

$$\frac{1}{\frac{3}{2}+\frac{1}{2}(n-1)^2+\sqrt{(n-1)^2}}$$

In order to get the expression to simplify, we make the assumption that n>1:

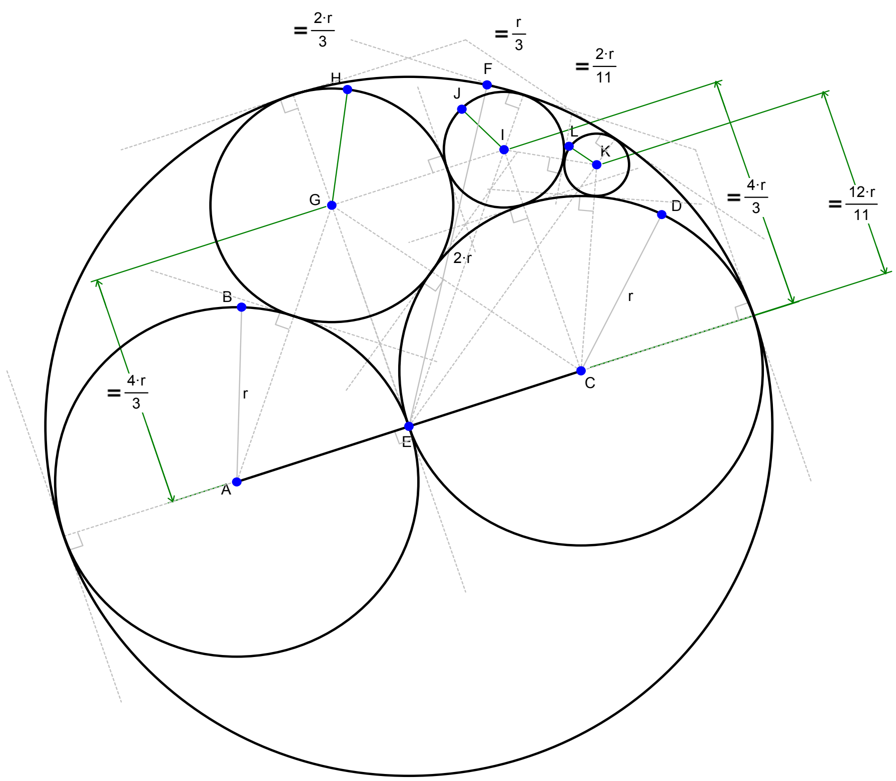
```
> assume(n>1);
> simplify(f(2/((n-1)^2+2)));
```

$$2\,\frac{1}{2+n\sim^2}$$

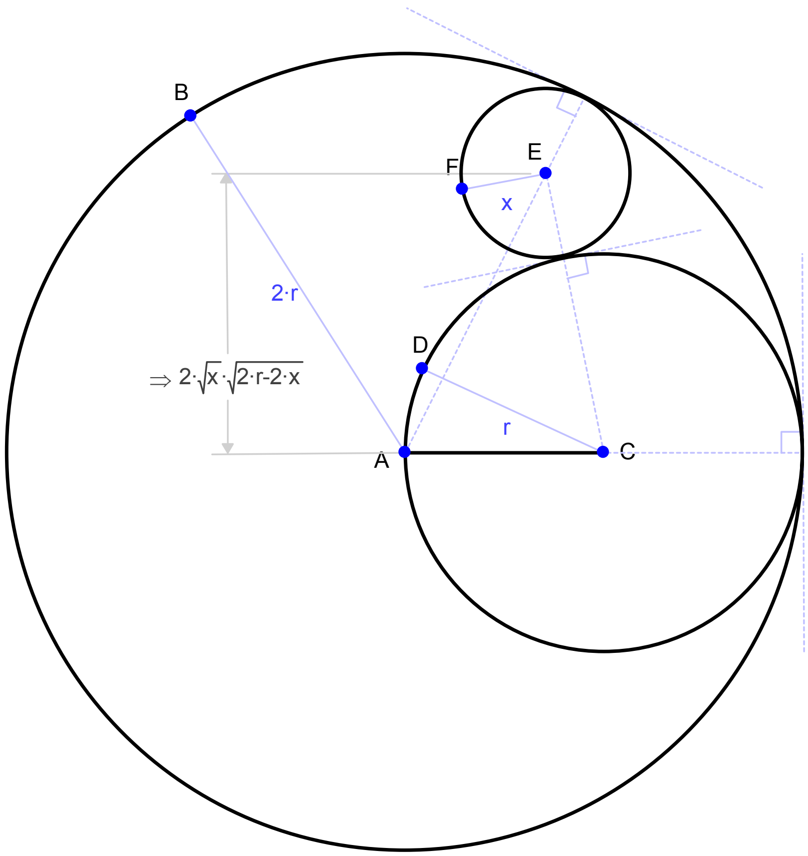
We see that this is the next term in the series. By induction, we have shown that the n'th circle has radius $\frac{2}{2+n^2}$

Example 9: A theorem old in Pappus' time

A theorem which was old in Pappus' days (3rd century AD) relates the radii to height of the circles in figures like the above:



Applying the general model, we get a formula:



Again, we can copy this into Maple for analysis:

$$2\sqrt{x}\,\sqrt{2\,r-2\,x}$$

```
> subs(x=2/(n^2+2),r=1,%);
```

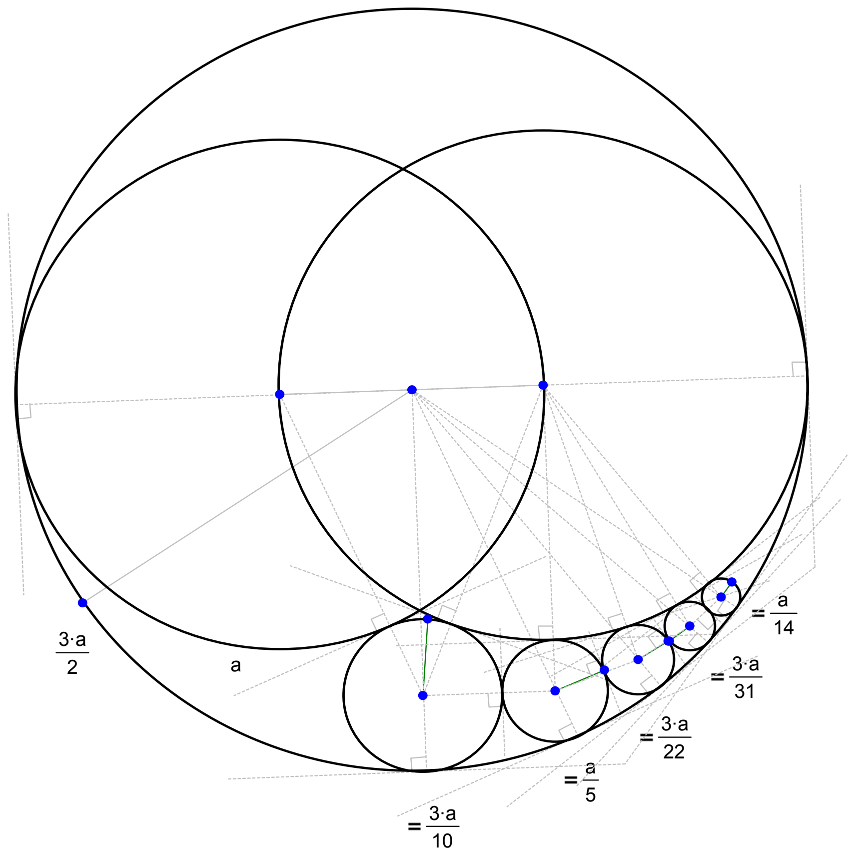
$$2\sqrt{2}\sqrt{\frac{1}{2+n\sim^2}}\sqrt{2-\frac{4}{2+n\sim^2}}$$

```
> simplify(%);
```

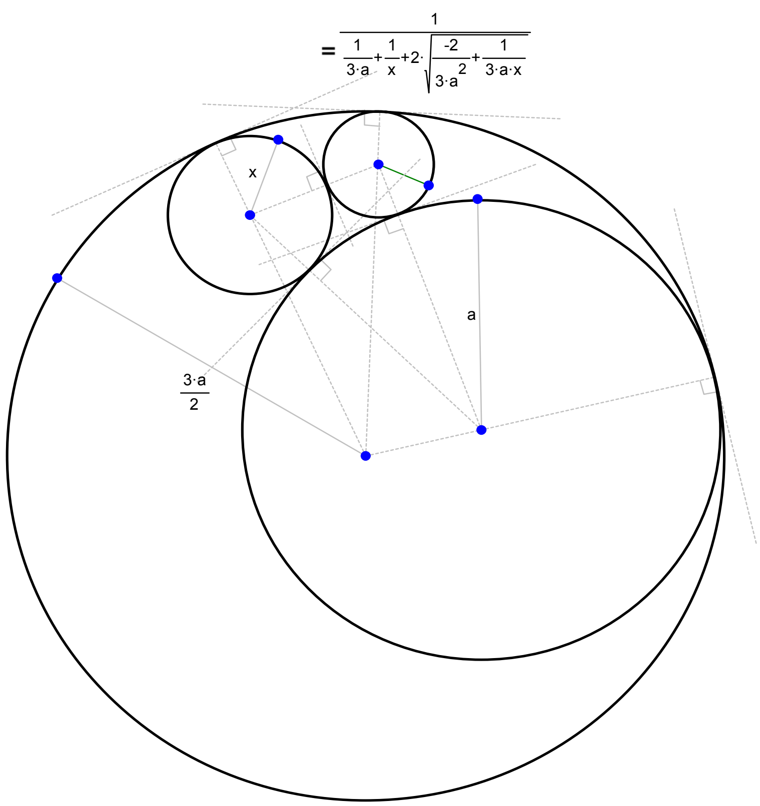
$$4\frac{n\sim}{2+n\sim^2}$$

We see that the height above the centerline for these circles is the radius multiplied by 2n.

Example 10: Another Family of Tangential Circles



We can follow through a similar analysis to the previous section:




```
> f:=x->1/(1/3+1/x+2/3*sqrt(-6+3/x));
```

$$f := x \rightarrow \frac{1}{\frac{1}{3}+\frac{1}{x}+\frac{2}{3}\sqrt{-6+\frac{3}{x}}}$$

```
> f(1);
```

$$\frac{1}{\frac{4}{3}+\frac{2}{3}I\sqrt{3}}$$

```
> f(3/10);
```

$$\frac{1}{5}$$

```
> f(%);
```

$$\frac{3}{22}$$

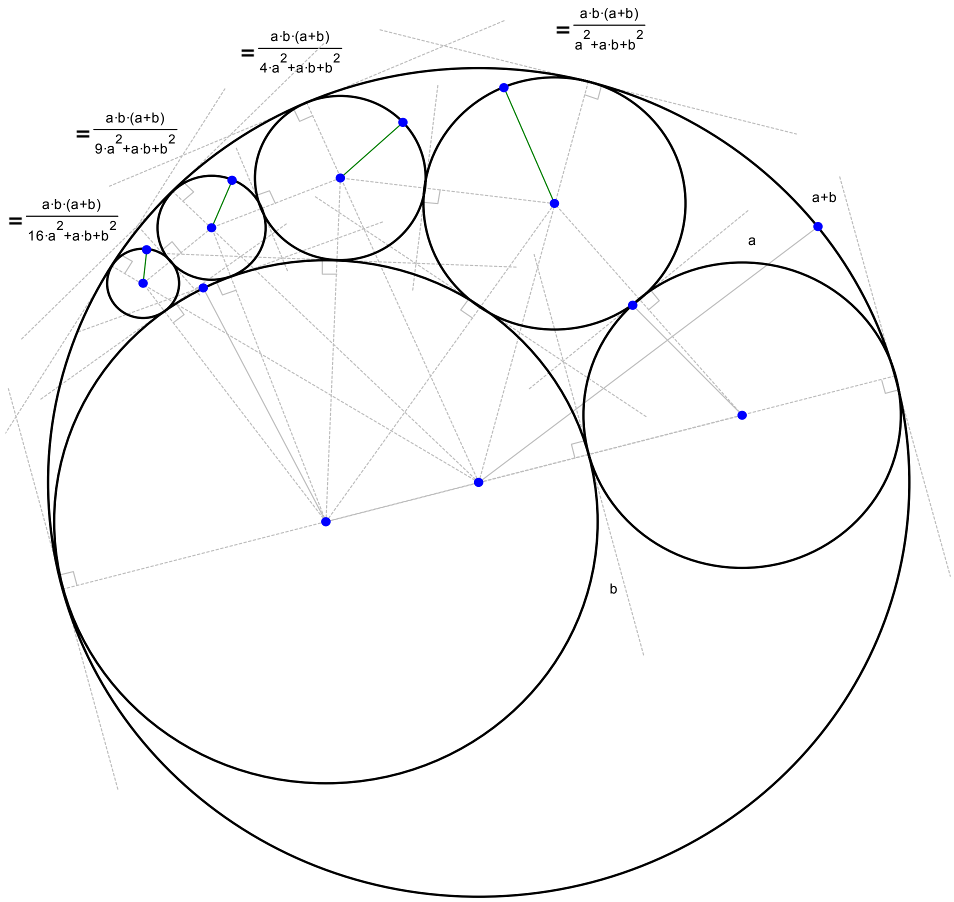
```
>

> assume(n>1);
> simplify(f(3/((n-1)^2+6)));
```

$$3\,\frac{1}{6+n\sim^{2}}$$

```
> simplify(%);
```

Example 11: Yet Another Family of Circles

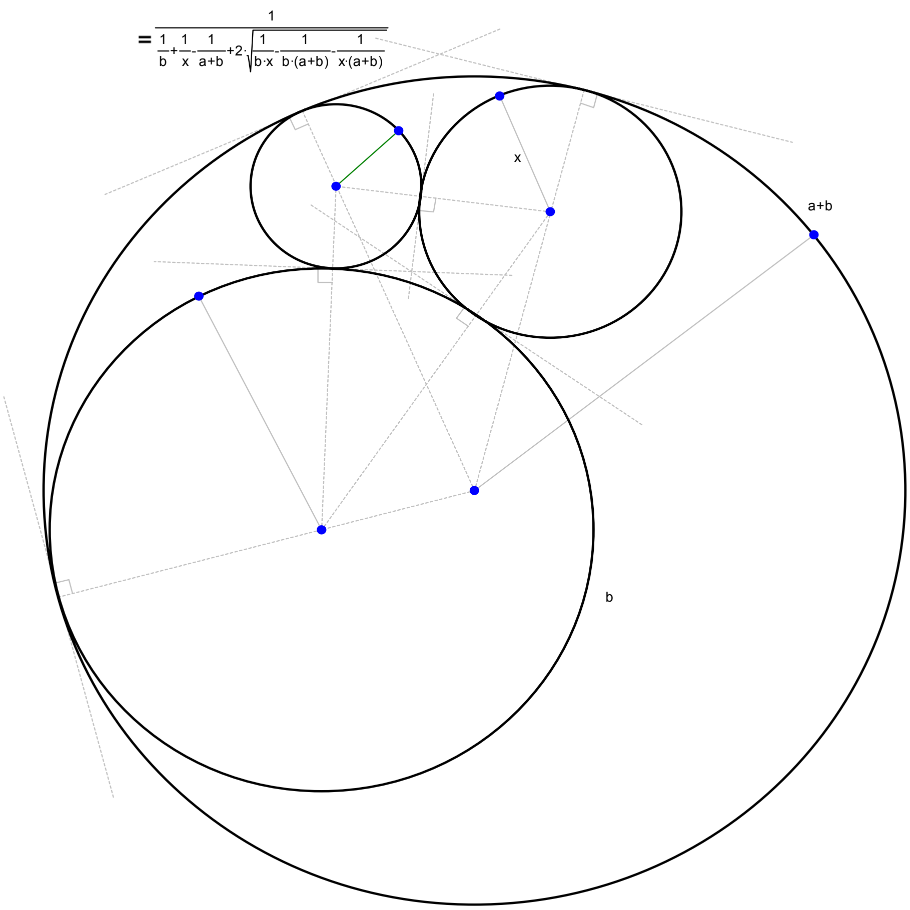
We generalize the situation from a couple of examples ago. We look at the family generated by two circles of radius a and b inside a circle of radius a+b:



The pattern is pretty obvious this time: the radius of the nth circle is:

$$\frac{ab(a+b)}{n^2a^2+ab+b^2}$$

To prove this, we derive the formula for the general circle radius x, and analyze in Maple:



Now we try feeding in one of the circle radii into this formula in maple (first making the assumption that the radii are positive (along with n>1 for later use):

```
> assume(a>0,b>0,n>1);
> f:=x->1/(1/b+1/x-1/(a+b)+2*sqrt(1/(x*b)-1/((a+b)*b)-
1/((a+b)*x)));
```

$$f := x \to \frac{1}{\frac{1}{b}+\frac{1}{x}-\frac{1}{a+b}+2\sqrt{\frac{1}{x\,b}-\frac{1}{(a+b)\,b}-\frac{1}{(a+b)\,x}}}$$

```
> f((a+b)*b*a/(9*a^2+b*a+b^2));
```

$$1\Bigg/\left(\frac{1}{b\sim}+\frac{9\,a\sim^2+b\sim\,a\sim+b\sim^2}{(a\sim+b\sim)\,b\sim\,a\sim}-\frac{1}{a\sim+b\sim}+2\sqrt{\frac{9\,a\sim^2+b\sim\,a\sim+b\sim^2}{(a\sim+b\sim)\,b\sim^2\,a\sim}-\frac{1}{(a\sim+b\sim)\,b\sim}-\frac{9\,a\sim^2+b\sim\,a\sim+b\sim^2}{(a\sim+b\sim)^2\,b\sim\,a\sim}}\right)$$

```
> simplify(%);
```

$$\frac{(a\sim+b\sim)\,b\sim\,a\sim}{16\,a\sim^2+b\sim\,a\sim+b\sim^2}$$

Let's try the general case, feeding in the formula for the n-1st radius:

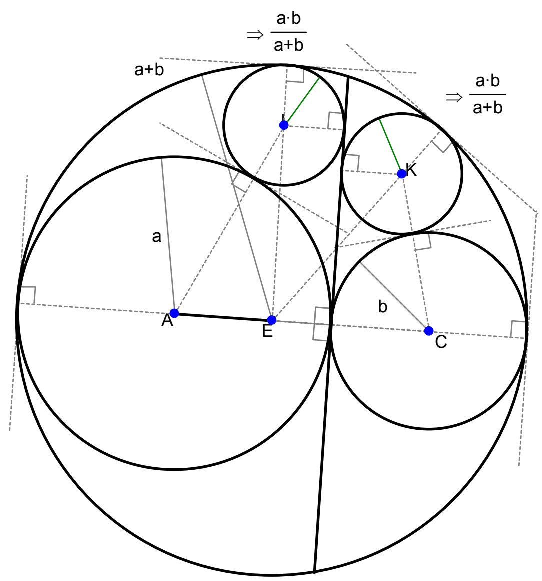
```
> simplify(f((a+b)*b*a/((n-1)^2*a^2+b*a+b^2)));
```

$$\frac{b\sim\,a\sim\,(a\sim+b\sim)}{b\sim\,a\sim+a\sim^2\,n\sim^2+b\sim^2}$$

By induction, we have proved the general result.

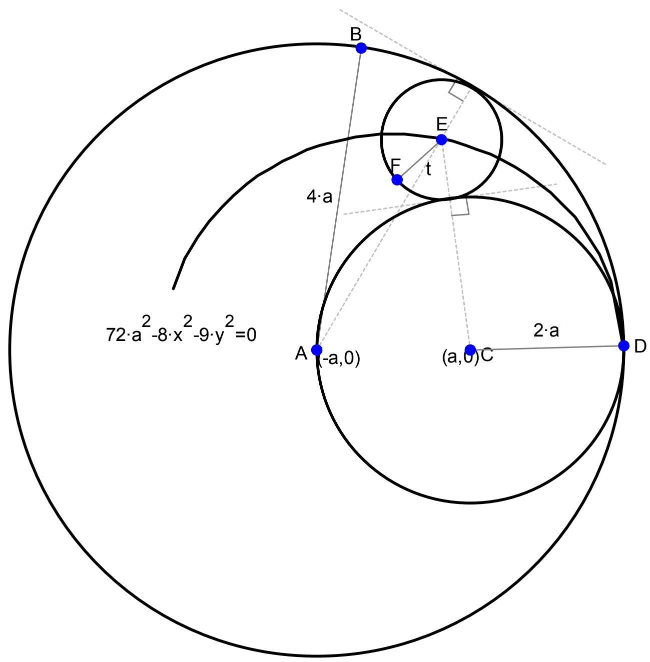
Example 12: Archimedes Twins

The given circles are mutually tangential with radius a, b and a+b. Archimedes twins are the circles tangential to the common tangent of the inner circles. We see from the symmetry of the radius expression that they are congruent.



Example 13: Squeezing a circle between two circles

Take a circle radius 2a centered at (a,0) and a circle radius 4a centered at (-a,0). Now look at the locus of the center of the circle tangent to both.



It's an ellipse. From the drawing we can see that the semi major axis in the x direction is 3a. What is the semi major axis in the y direction?