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Introduction

Teachers know the difficulties in motivating many students to develop the habits of mind and critical thinking skills necessary to thoroughly understand the concepts of calculus. The purpose of this book is to use *Geometry Expressions* software in order to facilitate and enhance the calculus syllabus by allowing students to ground calculus concepts in a geometric way.

Major calculus concepts, such as derivative and integral of a function, have a clear geometric meaning. This encourages students to visualize the concepts and make connections between its geometric and algebraic representations. For example, a function can be represented geometrically by its graph; the derivative of the function is visually represented by the slope of the tangent line; the definite integral of the function is an area under the graph of the function. These geometric representations serve as a basis for a conceptual introduction of the concepts of derivative and integral. The formal definitions of the calculus concepts then lead to clarification of these geometric ideas.

For example Exploration 2.1, "Exploring Tangent Lines", introduces the conceptual idea of the derivative. After the derivative is formally defined, it is possible to interpret the slope of the tangent line in terms of the derivative. Similarly in Exploration 4.5, "The Trapezoid Method", an area of a curvilinear trapezoid is used to develop the concept of a definite integral. After an introduction of the Fundamental Theorem of Calculus, the area can be defined using a definite integral. Both of these examples are based on the concept of the differential of a function which has a clear geometric meaning, as shown in Figure 1.

CROSS REFERENCE TABLE

Problem Name	**Pp.**	**Pre-requisite knowledge**	**Key concept**	**Level**	**Class Time**	**AP Calculus AB and BC* Topic**
1.1) The Formal Definition of a Limit	22	Properties of absolute value Concavity of monotonic functions	ε-δ definition of limit	3	60 – 90 min.	*Limits of functions (including one-sided limits)
1.2) The Squeeze Theorem	34	Basic trigonometric ratios in a unit circle Inverse trigonometric functions Area of a circular sector Limit of a function at a point	$\lim_{x \to 0} \frac{\sin x}{x} = 1$	1	45 min.	*Limits of functions (including one-sided limits)
1.3) The Area of a Circle	43	The area of a regular polygon inscribed in a circle with given radius A bounded increasing sequence converges $\lim_{x \to 0} \frac{\sin(ax)}{x} = a$	Limits at infinity Horizontal asymptote	1	45 min.	Asymptotic and unbounded behavior
2.1) Exploring Tangent Lines	54	The slopes of tangent and normal lines to the curve are opposite reciprocals. Point-slope equation of a line Derivative formulas for basic functions	Equation of a tangent line Slope of the tangent line is equal to the derivative of the function	1	45 min.	Derivative at a point

4.3 The Second Fundamental Theorem of Calculus

Exploration 4.3: Consider the definite integral of a function $y = f(x)$ with a variable upper limit x. What is the rate of change of the integral and how does that relate to the integrand f?

SUMMARY

Mathematics Objective:

- Discover the 2nd Fundamental Theorem of Calculus: $\frac{d}{dx}\left[\int_a^x f(x)dx\right] = f(x)$.
- Establish the fact that differentiation and integration are inverse operations.

Vocabulary:

- Rate of change
- Antiderivative
- Integral
- Area as a definite integral

Pre-requisites:

- Derivative as a slope of the tangent line.
- Basic rules of differentiation
- Antiderivative
- Definite integral
- The Fundamental Theorem of Calculus

Problem Notes:

- Students explore the relationship between the slope of the tangent line to the graph of the antiderivative of $f(x)$ written as a definite integral with a variable upper limit and the function $f(x)$. They use geometric concepts of area and tangent line to establish the relationship between them. This relationship is known as the second fundamental theorem of calculus. While this theorem is often confusing to students, the geometrical representation can help them understand the meaning of this theorem: when you apply two operations that are the inverse of each other to a given function, you will get the original function.

- Students first explore the definite integral of a specific function with a variable upper limit, also known as the area function. They plot the graph of this function and analyze the rate of change of this function using the slope of the tangent line.
- Students then generalize their findings for a generic function.

Technology skills:

- Draw: function, polygon, arc, line segment
- Constrain: point proportional along the curve
- Construct: tangent line
- Calculate: area, slope

STEPS-BY-STEP INSTRUCTIONS

THE DEFINITE INTEGRAL AS A FUNCTION

1. Draw a function $y = x^2$.
   a. Use **Toggle grid and axes** to show the axes without grid.
   b. Choose **Draw** → Function. Select Type → Cartesian. In the Y = prompt type x^2.
2. Plot the region bounded by the graphs of $y = x^2$, $y = 0$, $x = a$, and a variable upper boundary x.
   a. Plot points A and B on the x-axis and points C and D on the curve.
   b. Select point A and x-axis, and choose **Constrain** → Point proportional. Type a in the edit box. Select point C and the curve, and choose **Constrain** → Point proportional. Type a in the edit box.
   c. Select point B and x-axis, and choose **Constrain** → Point proportional. Type x in the edit box. Select point D and y-axis, and choose **Constrain** → Point proportional. Type x in the edit box.
   d. Choose **Draw** → Segment and plot segments AB, AC and BD. Choose **Draw** → Arc, and plot an arc CD.

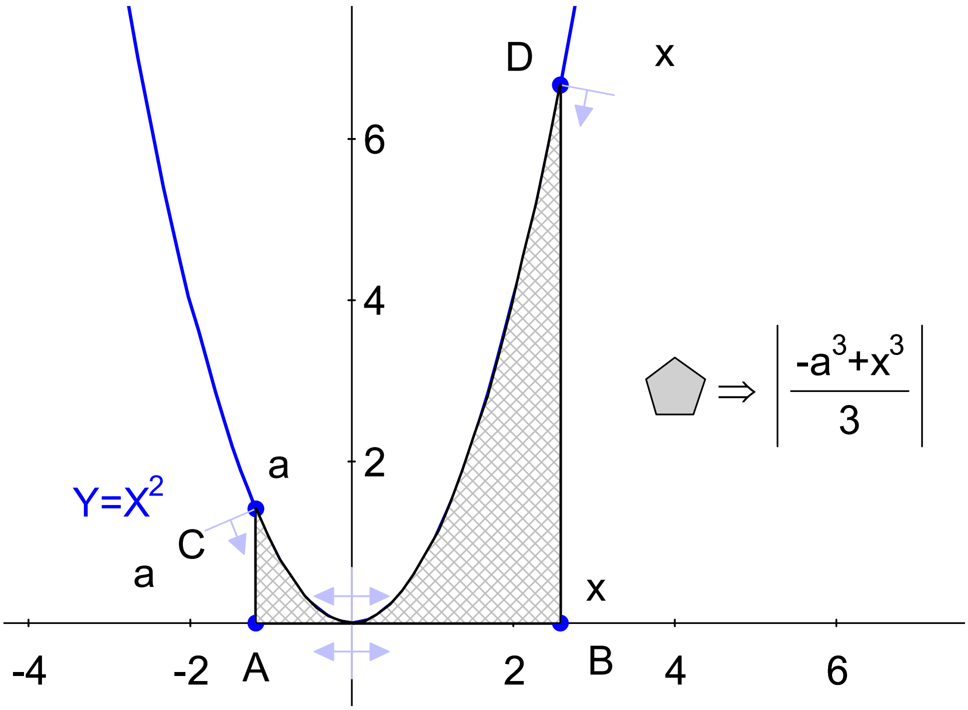
e. Select points A, B, C, and D and choose **Draw** → Polygon.

Q1. What is the area of this plane region?

A1: $A(x) = \int_a^x t^2 dt = \frac{x^3}{3} - \frac{a^3}{3}$. *The area is a function of x, a = const.*

2. Verify your answer with the help of software.

- Select the polygon interior and choose **Calculate** → Area.



THE RATE OF CHANGE OF THE AREA FUNCTION

Q2. The area of the region is a function of *x*. How does the area change as *x* changes? What is the derivative of this function?

A2: As x increases, the area increases. $\frac{d}{dx}\left(\frac{x^3 - a^3}{3}\right) = x^2$.

3. To check the answer to question 2, plot the graph of the area of the region bounded by the graphs of $y = x^2$, $y = 0$, $x = a$, and a variable upper boundary x. Find the slope of this graph at a point x.

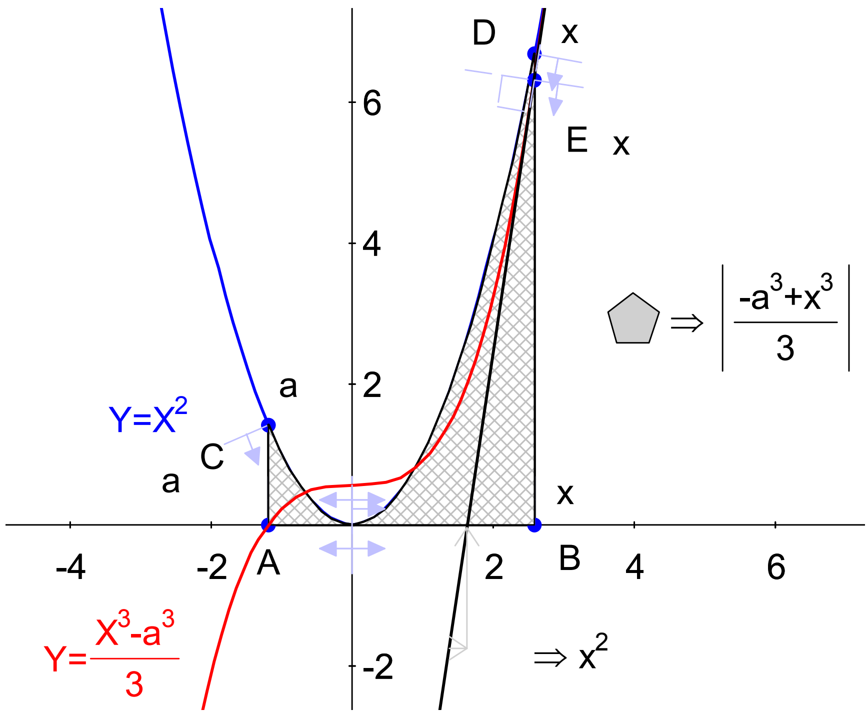
   a. Click on the expression for the area and choose **Edit** → Copy As → String.

   b. Choose **Draw** → Function. Choose Cartesian for Type and paste the area function in the Y= prompt using Ctrl-V. Delete "abs" in the expression of the function. (Or, before copying the string, set the **Output Properties** →Use Assumptions →Yes from the right-click context menu. This gets rid of the absolute value symbols.)

   c. Select the graph of the area function and choose **Construct** → Tangent to curve.

   d. Select the point of tangency (point E) and the curve and choose **Constrain** → Point proportional. In the open edit box type x.

   e. Select the tangent line and choose **Calculate** → Slope.



Q3. How does the slope of the tangent line to the area function compare to the original function that bounds the region?

A3: They are equal.

Q4. Will your answer hold true for other functions? Explain your answer.

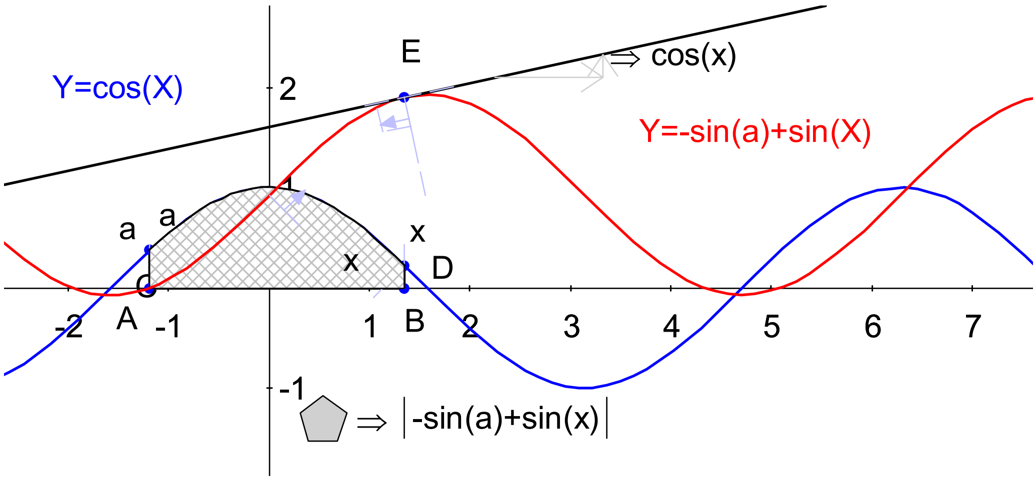
A4: Answers will vary.

4. Verify your decision by choosing another function and repeating the work above.

   a. Double-click on the expression Y = x^2 and type a different function.

   b. Click on the expression for the area and choose **Edit** → Copy As → String.

   c. Choose **Draw** → Function. Choose Cartesian for Type and paste the area function in the Y= prompt using Ctrl-V. Delete "abs" in the expression of the function.



Q5. Write a general expression for the area of the plane region between the graph of the function $y = f(x)$ and x-axis on the interval $[a, x]$.

A5: $A(x) = \int_a^x f(t)dt$

Note: to avoid the confusion of using x in two different ways, the variable of integration is switched to t.

Q6. Write a general expression for the derivative of the area function.