The Farmer and the Mathematician II:

more mathematical investigations using *Geometry Expressions*

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Thanks to the US Geological Survey, Department of the Interior, and *Microsoft Research Maps* (while it existed) for providing free public access to their vast data store of aerial photographs used throughout this book.

Cover Attributes
Thanks to the US Department of Agriculture and Gene Alexander of the Natural Resources Conservation Service for making his picture public domain.

ISBN 1-882564-26-X

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Forward

The Farmer is back and he's been collecting a lot of questions for his new mathematician. (The mathematician from our last book is "on vacation - practicing his zero slope!") Our farmer is still using his Center Pivot Irrigation System – those long stretches of wheeled scaffolding, sometimes up to a half-mile in length, rotating around a fixed point attached to his water source. And he's become quite good at scanning *Google Earth* to see what his fellow farmers are up to.

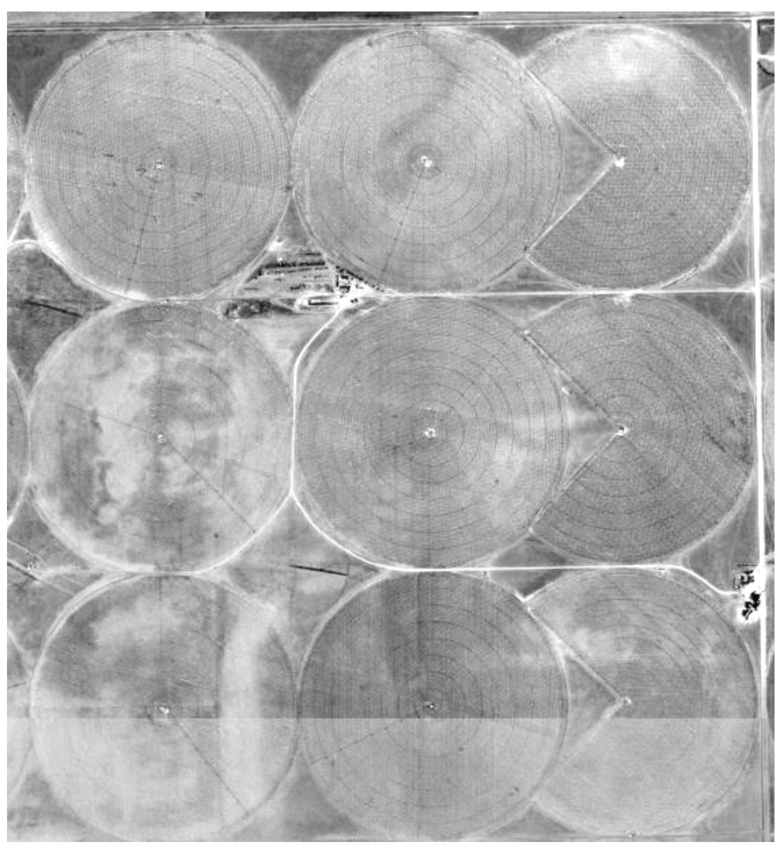
Again, as in the first book, *The Farmer and the Mathematician,* we demonstrate to the student that Mathematics is not just for calculating the price of produce in the grocery store. Here we have a fresh batch of examples of mathematics in action using our symbolic geometry software, *Geometry Expressions,* with images from *Google Earth*.

The process of mathematical modeling is both critical to using mathematics in the world around us, and to developing an understanding and appreciation for the true utility of mathematics. A mathematician or engineer would first be given a problem to solve. They would also be given **constraints**, or a set of rules and conditions that must be followed. From that, the scientist would attempt to create a mathematical representation of the problem. Often, real problems are much too complex and it is necessary to constrain, or simplify them further. This is very similar to the approach that a student would take to gain insight into a problem by solving a simpler, related one. *Geometry Expressions* has been written to mirror this process. It is a **constraint-based** geometric and algebraic modeling program that allows the student to investigate problems in both a numeric and symbolic representation.

So join the Farmer and his mathematical instructor, Sophie, as they fly over Texas, investigating interesting geometrical ideas and stumbling over important mathematical concepts. The content is appropriate and adaptable for students in a range of courses from Algebra and Geometry up through and including Calculus.

Lesson 1: Positioning Pacmen

Farmer: Well, I've been waiting for a long time for this and it has finally come! It's very common to encounter rectangular fields and last time we spent most of our time talking about squares. Some farmers have taken to using full circles and parts of circles that I like to call "Pacmen". Look at this field:



Mathematician: Ooh! I love that game, though I prefer Ms. Pacman.

Farmer: Hey, who are you?

Mathematician: My name is Sophie; I'm taking over for the other guy while he is on vacation - practicing his zero slope! The technical term for partial circles is ***sectors*** and they present some really interesting questions worth investigating.

Farmer: Ok, well, let's see what you've got!

Interesting Investigation

1. North Texas still seems to be a gold mine for good examples. Using Google Earth, try navigating to these places first:

   a. 36º 4' 1" N, 102º 50' 59" W

   b. 35º 59' 38" N, 102º 21' 24" W

   c. 34° 10' 50" N, 101° 35' 3" W

To Navigate in Google Earth, Type the coordinates into the **"Fly to"** box. If you don't know a shortcut for the degree symbol, don't worry, it is implied. Separate the latitude and longitude with a comma. For example, 24º31'1" N, 102º 50'59" W, would be entered as: 24 31'1" N, 102 50'59" W.

2. Spend some time searching Google Earth for some other places where farmers have used "Pacmen" shapes to fill rectangular fields. Feel free to explore other areas of the country. Enter the coordinates of one additional example below:

Latitude	Longitude

3. If there is time, exchange coordinates with a different group and look at their discovery.

Farmer: Hey, I noticed that some of the Pacmen have their mouths open wider than others. What's going on with that?

Mathematician: Excellent question! What you are dealing with is called **Aspect Ratio**.

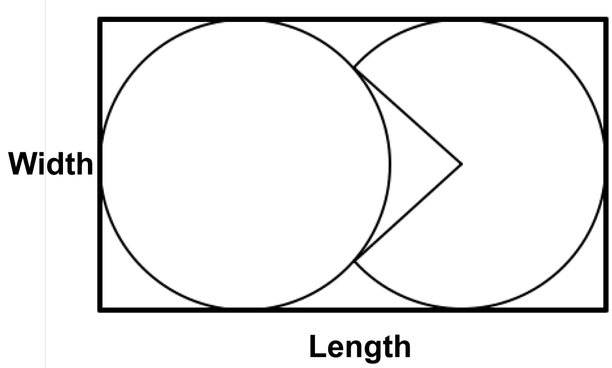
Farmer: I have heard of that before!

Mathematician: Yes, you probably have heard it referring to movies, televisions, and computer monitors. The aspect ratio is the ratio of the length of a rectangle to its width.

Farmer: So what aspect of that is so interesting?

Mathematician: Oh, I see my colleague's sense of humor is rubbing off on you! The aspect ratio will affect the width of the sector opening, and therefore will also affect how much of their total area is usable. Remember this is called the **density**!

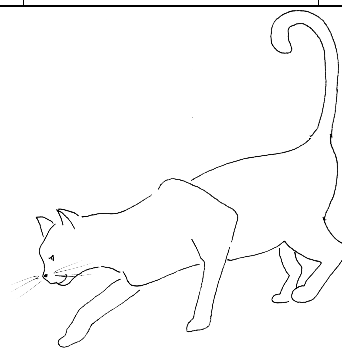
$$\textbf{Aspect Ratio} = \frac{\textbf{Length}}{\textbf{Width}}$$



4. We can use the ruler tool in Google Earth to approximate the aspect ratios. Choose **TOOLS-RULER** from the menu, or use the icon. If you click at the beginning and end of a distance you would like to measure, it should draw a line segment and supply the actual distance in the dialogue box. If you "missed" a

bit, you can click on the endpoints of the segment to adjust their position. To make another measurement, choose **CLEAR** and repeat the process. Use this tool to measure and calculate the aspect ratios for **1 a, b, and c** (one full circle and the adjoining sector; the **Length** is always the longer side). Calculate the ratio for the location you found as well. Complete the chart below.

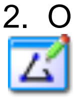
	Length	**Width**	**Aspect Ratio**
1a			
1b			
1c			
Your location			



Engaging Exploration

Let's get to the reason we are here, which is to use *Geometry Expressions*. You should have found that the last location in the above chart had an aspect ratio of close to 1.5. Let's look at some possible values of the aspect ratio.

1. Assuming we use one full circle and one sector, what is true of the sector if the aspect ratio is ***exactly*** 1.5?

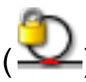
2. Open a new **GX** document and save it as ***lesson1.gx***. Use the polygon tool () to create a quadrilateral. Out of the many possible ways to construct a rectangle, the most efficient is to select one pair of opposite sides and make them parallel by constraining the distance between them. Select the pair of sides and choose . Enter **w** (for width). Select the remaining pair of sides and constrain their length to be **r*w**. Use **CNTRL-H** to hide the interior of the quadrilateral.

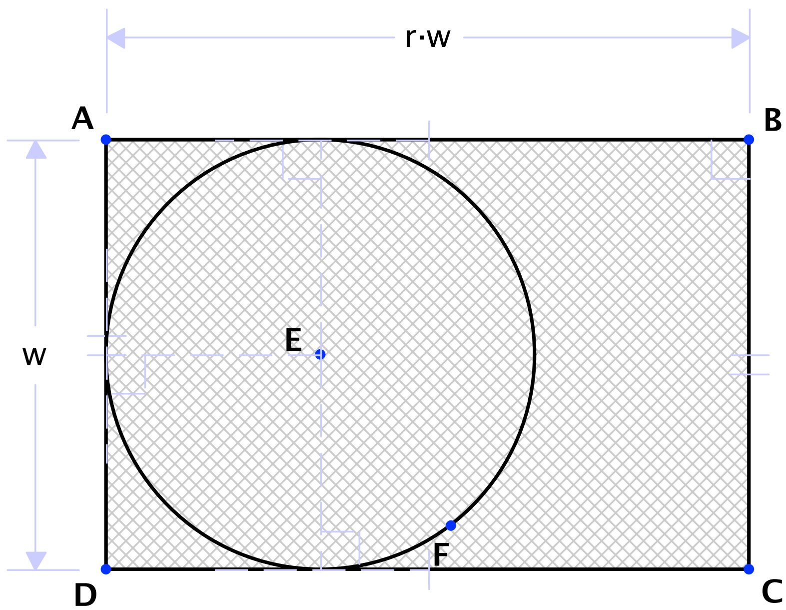
3. Drag the various parts (points and sides)of the quadrilateral .

    a. What is the name of this quadrilateral as it is currently constructed?

    b. What constraint do we need to add to make it a rectangle? Add this to your drawing.

4. What will **r** represent in this context?

5. Construct a circle and constrain it to be tangent () to the width and the parallel lengths.



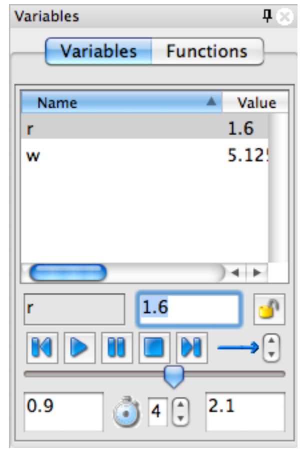
6. Construct a second circle and constrain it to be tangent to the opposite width, along with the two longer sides.

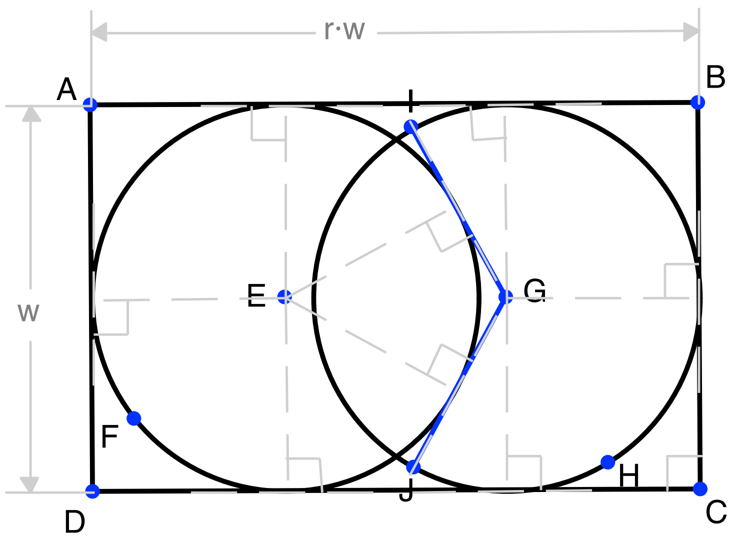
7. In the **Variables** tool panel, select the variable **r**. Change the range of values to have a minimum of **0.9** and a maximum of **2.1**. Drag the slider through the range of values and observe the results.

8. Explain the implications for the farmer if the aspect ratio is set to **1**. Use complete sentences.

9. Explain the implications for the farmer if the aspect ratio is set to **2**. Use complete sentences.

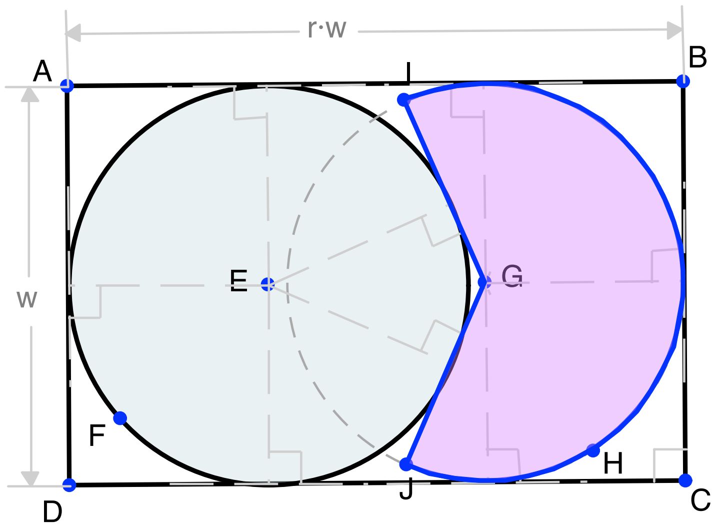
10. It seems as if in many of the fields, the Pacmen have been constructed so the segments are tangent to the whole circle. Lets construct our drawing this way for now. In the **Variables** tool panel, highlight the current value of **r** and change it to something around **1.6**.





11. Select  and draw two radii of the second circle. Constrain each radius to be tangent to the first circle. **Note:** when drawing the radius, be careful not to drag the endpoint too close to the full circle or you may accidentally constrain it wrongly.

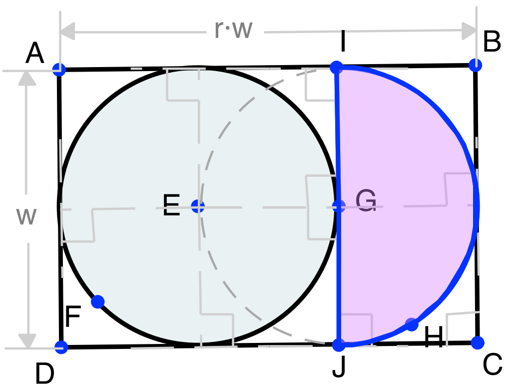
12. Choose **DRAW-ARC** () Click on the endpoint **I** of radius **GI** and then drag it around the circle in a clockwise direction to point **J** to create the arc.

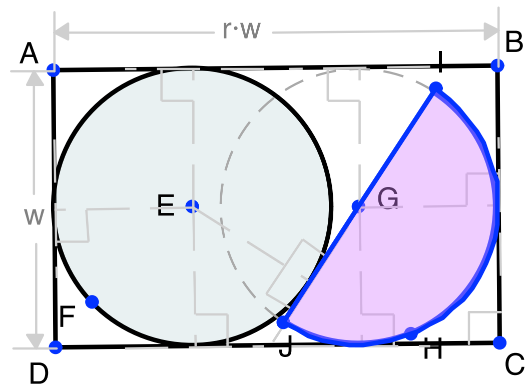


13. Select the arc and the two radii and choose  to create the area of the sector. To make your drawing appear a little nicer, select the underlying circle and change its line style to a dashed line. See your teacher if you need help doing this. If you don't see your teacher anyway to check that you have successfully completed the drawing!

Teacher Check:__________

14. Before you experiment with your drawing, we need to make one adjustment to avoid some awkwardness. It is possible for the tangent radii that we drew to





go in the opposite direction. The figure on the left has **r** at exactly 1.5. Slide it up and you'll probably get something like the figure on the right.
To avoid this, change the lower bound of **r** in the variables tool panel to be just slightly bigger than 1.5 (1.5001 should work nicely!).

15. Drag the slider for **r** to explore the changing sector shape. You may notice a problem that occurs with our model. Explain this problem in your own words in the space below and explain for what values of **r** this will be a concern.

Teacher Notes

Detailed instructions are provided in the following lessons for constructing the necessary *Geometry Expressions* models. No experience with the software should be necessary **IF** the student begins with the first lesson and moves chronologically through the activities. However, as the lessons proceed, it is assumed the student will gain some facility with the program, and therefore fewer specific instructions and screenshots are given. The teacher may decide to skip some lessons and jump ahead to others, so it is important to understand this and avoid frustration on the part of some students learning their way around the software.

While all teachers utilize different classroom presentation styles for using technology, the author's experience reflects the set-up of these materials. They are written so the students will be basically self-sufficient with the teacher serving only as a resource and for an occasional "nudge" in the correct direction. Ideally, students should work with partners. One student should read the instructions while the other student executes them. Students should switch roles every now and then to insure they are both using the software. Less motivated students may need help staying on task. This is the reason for including occasional "teacher checks" at certain points during the activities. Teachers may choose to utilize, ignore, or modify these instructions depending on the level and/or motivation of their students.

Lesson One

Objectives

The student will be able to define and calculate aspect ratio for a rectangle.

The students will be able to calculate area and density algebraically.

The students will explore the optimum density and configurations for a complete circle and a sector inscribed inside a rectangle of varying aspect ratios.

Lesson Notes

This lesson starts with an introduction to the concept and calculation of aspect ratio. Many farmers with rectangular fields having aspect ratios between 1.5 and 2 have used circle sectors, or "pacmen" shapes to irrigate their fields. The lesson has the students explore the various aspect ratios interactively with *Gx*. It finishes with a look at the algebra describing an aspect ratio of 1.5.

Interesting Investigation

4.

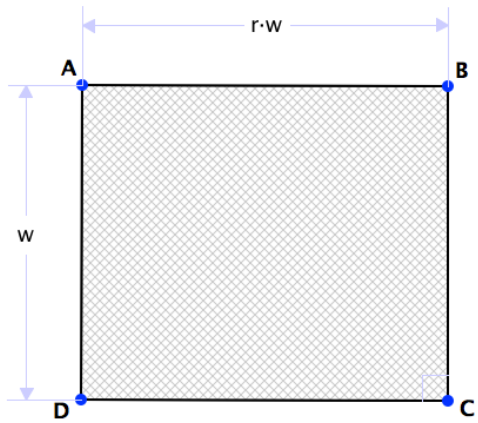
	Length	Width	Aspect Ratio
1a	**.85 miles**	**.5 miles**	**1.7**
1b	**1.3 miles**	**.8 miles**	**1.625**
1c	**.75 miles**	**.5 miles**	**1.5**
Your location	**Answers will vary**		

Engaging Exploration

1. It will be a semi-circle.

3. a. It is a parallelogram.

b. We need a right angle. Students should select two adjacent sides and choose  to create the right angles.



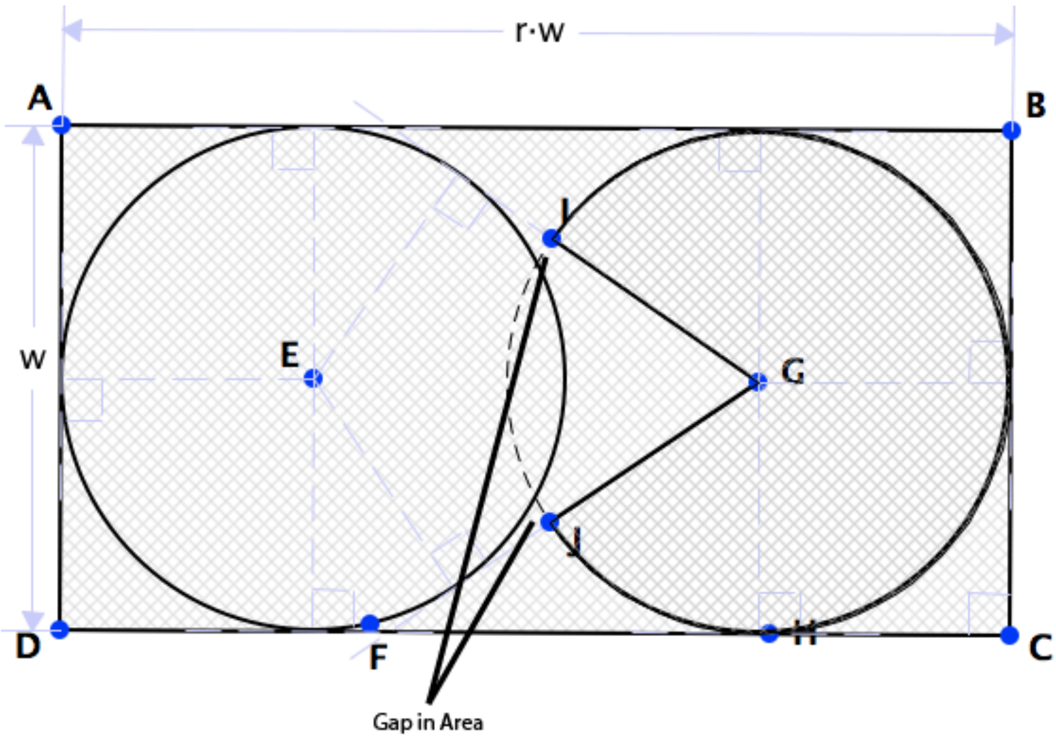
4. r is the aspect ratio of the rectangle.

8. If the aspect ratio were set to 1, the field would be a square and we wouldn't use a second circle at all.

9. If the aspect ratio were set to 2, we could use 2 complete circles and we wouldn't need to make it a sector at all.

11. Note about the **Note** – Students may try to draw the radii tangent to the full circle and accidentally make the point incident to the wrong circle or to the intersection of the two circles which, of course would yield the wrong result. Click the point to see an incidence mark (bowtie). If a <u>double</u> bowtie appears (indicating incidence with <u>both</u> circles), delete the point and radius and try again.

15. If r is bigger than something around 1.7, keeping the lines tangent to the circle leaves us with gaps between the sector and the full circle. For r-values larger than this, it appears that we will need to make the radii just touch the circle, not be tangent to it.



Algebraic Aspects

1.

Object	Exact area in terms of w
Rectangle ABCD	$1.5w^2 = \frac{3}{2}w^2$
Circle E	$\pi\left(\frac{w}{2}\right)^2 = \frac{\pi w^2}{4}$
Semicircle G	$\frac{1}{2}\cdot\frac{\pi w^2}{4} = \frac{\pi w^2}{8}$

2. $\dfrac{\frac{\pi w^2}{4}+\frac{\pi w^2}{8}}{\frac{3}{2}w^2} = \dfrac{\frac{3\pi w^2}{8}}{\frac{3w^2}{2}} = \dfrac{\frac{\pi}{8}}{\frac{1}{2}} = \dfrac{2\pi}{8} = \dfrac{\pi}{4} \approx 0.785$